

# Spherical geometry: isometries

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# Isometries of $S^2$

An  $\mathbb{E}^3$  isometry  $T$  can be restricted to a function  $\underline{T}$  acting on  $S^2$ .

An  $S^2$  isometry  $T$  can be extended to a function

$$\bar{T}(\mathbf{x}) := \begin{cases} |\mathbf{x}|T(\mathbf{x}/|\mathbf{x}|) \\ \mathbf{0} \text{ if } \mathbf{x} = \mathbf{0} \end{cases}$$

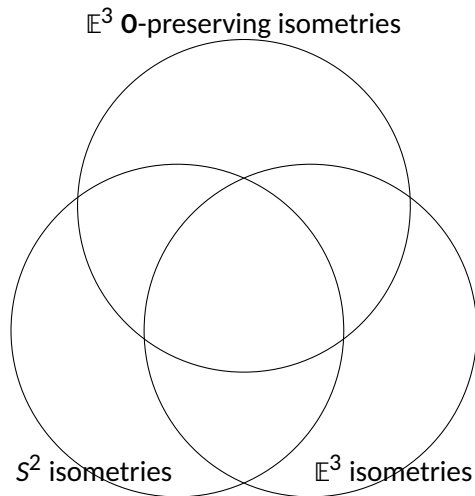
acting on  $\mathbb{E}^3$ .

❓ With this correspondence, what can we say about the Venn diagram?

**3.4.1.**  $T$  is an  $S^2$  isometry  $\implies \bar{T}$  is an  $\mathbb{E}^3$  isometry

**3.4.2.**  $T$  is an  $\mathbb{E}^3$  isometry  $\implies$

$$\underline{T} \text{ is an } S^2 \text{ isometry} \iff T(\mathbf{0}) = \mathbf{0}$$



## Proof of 3.4.1

Given that  $T$  is an  $S^2$  isometry, prove that  $\bar{T}(\mathbf{x}) := \begin{cases} |\mathbf{x}| T(\mathbf{x}/|\mathbf{x}|) \\ \mathbf{0} \text{ if } \mathbf{x} = \mathbf{0} \end{cases}$  is an  $\mathbb{E}^3$  isometry.

- ▶  $\bar{T}$  is distance-preserving:  $d(\bar{T}(\mathbf{x}), \bar{T}(\mathbf{y})) = d(\mathbf{x}, \mathbf{y})$ , for all  $\mathbf{x}, \mathbf{y} \in \mathbb{E}^3$ .

Proof idea: re-express  $d(\mathbf{x}, \mathbf{y})$  in terms of things centered at the origin, since that fits well with  $T$ . In fact, by SAS triangle congruence,  $d(\mathbf{x}, \mathbf{y})$  is preserved if  $\angle(\mathbf{x}, \mathbf{y})$  and  $d(\mathbf{0}, \mathbf{x})$  and  $d(\mathbf{0}, \mathbf{y})$  are preserved. This can be verified from the definition of  $\bar{T}$ :

- ▶ Angles at origin preserved:

$$\begin{aligned} \angle(\bar{T}(\mathbf{x}), \bar{T}(\mathbf{y})) &\stackrel{*}{=} \angle(|\mathbf{x}| T(\mathbf{x}/|\mathbf{x}|), |\mathbf{y}| T(\mathbf{y}/|\mathbf{y}|)) \\ &= \angle(T(\mathbf{x}/|\mathbf{x}|), T(\mathbf{y}/|\mathbf{y}|)) \\ &= \angle(\mathbf{x}/|\mathbf{x}|, \mathbf{y}/|\mathbf{y}|) \\ &= \angle(\mathbf{x}, \mathbf{y}) \end{aligned}$$

\* Unless  $\mathbf{x} = \mathbf{0}$  or  $\mathbf{y} = \mathbf{0}$ : verify these cases separately.

- ▶ Lengths from origin preserved:

$$\begin{aligned} d(\mathbf{0}, \bar{T}(\mathbf{x})) &\stackrel{*}{=} d(\mathbf{0}, |\mathbf{x}| T(\mathbf{x}/|\mathbf{x}|)) \\ &= \left| |\mathbf{x}| T(\mathbf{x}/|\mathbf{x}|) \right| \\ &= |\mathbf{x}| |T(\mathbf{x}/|\mathbf{x}|)| \\ &= |\mathbf{x}| \cdot 1 \\ &= |\mathbf{x}| \\ &= d(\mathbf{0}, \mathbf{x}) \end{aligned}$$

- ▶ Left to prove:  $\bar{T}$  is surjective.

## Proof of 3.4.1 continued

- ▶ Given:  $T$  is an  $S^2$  isometry.
- ▶ Already proved: Its extension  $\bar{T}(\mathbf{x}) := \begin{cases} |\mathbf{x}| T(\mathbf{x}/|\mathbf{x}|) \\ \mathbf{0} \text{ if } \mathbf{x} = \mathbf{0} \end{cases}$  is a distance-preserving function  $\mathbb{E}^3 \rightarrow \mathbb{E}^3$ .
- ▶ Left to prove:  $\bar{T}$  is surjective.
- ▶ In fact, any distance-preserving  $T : \mathbb{E}^n \rightarrow \mathbb{E}^n$  is automatically (surjective and hence) an isometry, as noted in the dictaat. (Since  $d$ -preserving  $\implies T = M\mathbf{x} + \mathbf{v} \implies \exists T^{-1}$ .)  $\square$

❓ What happens if we try to apply this logic to

$$T : \mathbb{E}^2 \rightarrow \mathbb{E}^3 \quad (x, y) \mapsto (x, y, 0)?$$

❓ How do we know that the case we need is not like that?

## Proof of 3.4.2 ( $\Rightarrow$ )

Given that  $T$  is an  $\mathbb{E}^3$  isometry and its restriction  $\underline{T}$  is an  $S^2$  isometry, show that  $T(\mathbf{0}) = \mathbf{0}$ .

- ▶ We will prove this by contradiction. Suppose  $T$  sends the origin somewhere else,  $T(\mathbf{0}) = \mathbf{m} \neq \mathbf{0}$ .
- ▶ Then  $\mathbf{0}, \mathbf{m}$  define a line  $L := \text{line}_{\mathbb{E}^3}(\mathbf{0}, \mathbf{m})$ .
- ▶ Lines through  $\mathbf{0}$  cut  $S^2$  in two points:  $L \cap S^2 = \{\mathbf{a}, \mathbf{b}\}$ .
- ▶ Since  $T$  is an isometry, it has an inverse  $T^{-1}$  that is also an isometry.
- ▶ Since isometries of  $\mathbb{E}^3$  send lines to lines, the inverse image  $L^{-1} := T^{-1}(L)$  of  $L$  must be a line as well.
- ▶ Furthermore, this line  $L^{-1}$  must go through  $\mathbf{0}$  as well, since it contains  $T^{-1}(\mathbf{l})$  for any  $\mathbf{l} \in L$ , and hence in particular  $T^{-1}(\mathbf{m}) = T^{-1}(T(\mathbf{0})) = \mathbf{0}$ .
- ▶ Hence  $L^{-1}$  too cuts  $S^2$  in two points:  $L^{-1} \cap S^2 = \{\mathbf{a}', \mathbf{b}'\}$ .
- ▶ By definition,  $T$  sends  $L^{-1}$  to  $L$ , so it must send its sphere points  $\mathbf{a}', \mathbf{b}'$  somewhere on  $L$ . And since  $\underline{T}$  is an  $S^2$  isometry,  $T$  must map  $S^2$  to itself. Therefore  $T$  must send  $\mathbf{a}', \mathbf{b}' \mapsto \mathbf{a}, \mathbf{b}$ .
- ▶ Thus  $T$  is an isometry that sends the line segment  $\mathbf{a}'\mathbf{b}'$  to the line segment  $\mathbf{a}\mathbf{b}$ . Hence it must also send the midpoint of  $\mathbf{a}'\mathbf{b}'$  to the midpoint of  $\mathbf{a}\mathbf{b}$ , that is to say,  $\mathbf{0}$  to  $\mathbf{0}$ .
- ▶ Which contradicts the assumption  $T(\mathbf{0}) = \mathbf{m} \neq \mathbf{0}$ . □

## Proof of 3.4.2 ( $\Leftarrow$ )

Given that  $T$  is an  $\mathbb{E}^3$  isometry and  $T(\mathbf{0}) = \mathbf{0}$ , show that its restriction  $\underline{T}$  is an  $S^2$  isometry.

Need to show:  $\underline{T}$  is a surjective, distance-preserving function  $S^2 \rightarrow S^2$ .

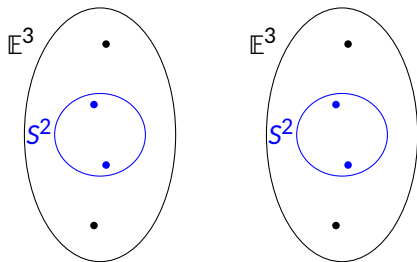
We will show:

- ▶  $\underline{T}$  is a function  $S^2 \rightarrow S^2$ .
- ▶  $\underline{T}$  is an invertible function  $S^2 \rightarrow S^2$ .
- ▶ Hence  $\underline{T}$  is a surjective function  $S^2 \rightarrow S^2$ .

❓ Illustrate the meaning of these steps in terms of the diagram below.

❓ Why can we not argue as follows:

$$\begin{aligned} T \text{ isometry} &\implies T \text{ surjective} \\ &\implies \underline{T} \text{ surjective} \end{aligned}$$



## Proof of 3.4.2 ( $\Leftarrow$ ) continued

Given that  $T$  is an  $\mathbb{E}^3$  isometry and  $T(\mathbf{0}) = \mathbf{0}$ , show that its restriction  $\underline{T}$  is an  $S^2$  isometry.

- ▶  $\underline{T}$  is a function  $S^2 \rightarrow S^2$ :

$$\begin{aligned} \mathbf{s} \in S^2 &\implies d_{\mathbb{E}^3}(\mathbf{0}, \mathbf{s}) = 1 \\ &\implies d_{\mathbb{E}^3}(T(\mathbf{0}), T(\mathbf{s})) = 1 \\ &\implies d_{\mathbb{E}^3}(\mathbf{0}, T(\mathbf{s})) = 1 \\ &\implies T(\mathbf{s}) \in S^2 \end{aligned}$$

- ▶  $\underline{T}$  is an invertible function  $S^2 \rightarrow S^2$ :

$$T \text{ isometry of } \mathbb{E}^3 \implies T^{-1} \text{ isometry of } \mathbb{E}^3$$

and this inverse reduces to a function  $S^2 \rightarrow S^2$ :

$$\begin{aligned} \mathbf{s} \in S^2 &\implies d_{\mathbb{E}^3}(\mathbf{0}, \mathbf{s}) = 1 \\ &\implies d_{\mathbb{E}^3}(T^{-1}(\mathbf{0}), T^{-1}(\mathbf{s})) = 1 \\ &\implies d_{\mathbb{E}^3}(\mathbf{0}, T^{-1}(\mathbf{s})) = 1 \\ &\implies T^{-1}(\mathbf{s}) \in S^2 \end{aligned}$$

- ▶ Hence  $\underline{T}$  is a surjective function: For any  $\mathbf{s} \in S^2$  there is a  $T^{-1}(\mathbf{s}) \in S^2$  such that  $T(T^{-1}(\mathbf{s})) = \mathbf{s}$ .
- ▶  $\underline{T}$  is distance-preserving.

$$\begin{aligned} d_{S^2}(T(\mathbf{x}), T(\mathbf{y})) &= \angle T(\mathbf{x})\mathbf{0}T(\mathbf{y}) \\ &= \angle \mathbf{x}\mathbf{0}\mathbf{y} \\ &= d_{S^2}(\mathbf{x}, \mathbf{y}) \quad \square \end{aligned}$$

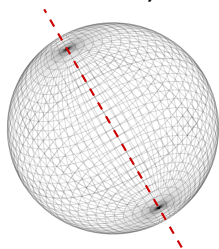
# Classification of isometries of $S^2$

isometries of  $S^2 \leftrightarrow \mathbf{O}$ -preserving isometries of  $\mathbb{E}^3$

= orthogonal  $3 \times 3$  matrices  $M$

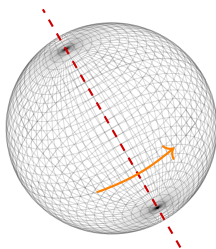
$$= \begin{pmatrix} \pm 1 & 0 \\ 0 & R_\varphi \end{pmatrix}$$

identity



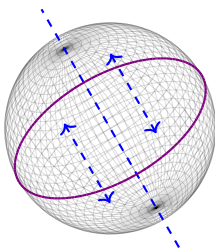
$$\begin{pmatrix} 1 & 0 \\ 0 & R_0 \end{pmatrix}$$

rotation



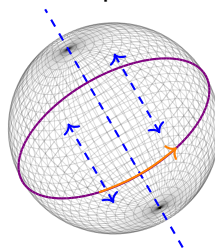
$$\begin{pmatrix} 1 & 0 \\ 0 & R_\varphi \end{pmatrix}$$

reflection



$$\begin{pmatrix} -1 & 0 \\ 0 & R_0 \end{pmatrix}$$

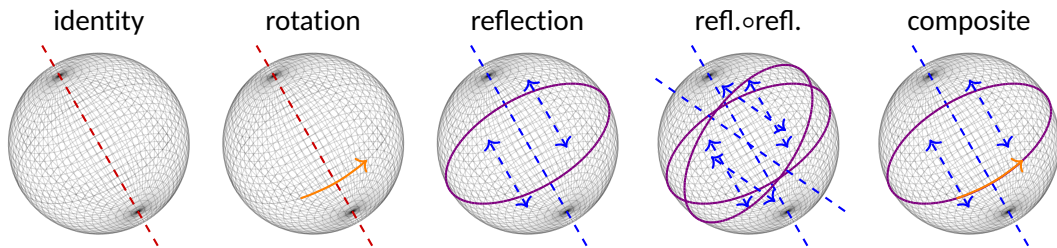
composite



$$\begin{pmatrix} -1 & 0 \\ 0 & R_\varphi \end{pmatrix}$$



❓ Prove that not every isometry of  $S^2$  can be expressed as a composition of one or two reflections



Hint: Consider fixed points.

## ❓ Isometry definition of lines

Playfair's definition of straight lines: A curve  $L \subset \mathbb{E}^n$  is called a line if and only if

$$|L \cap T(L)| \geq 2 \implies L = T(L)$$

for any isometry  $T$  of  $\mathbb{E}^n$ .

- ❓ Does this definition correctly identify straight lines and only straight lines in  $\mathbb{E}^n$ ?
- ❓ In  $S^2$ ?

