Spherical geometry: definitions and trigonometry

Viktor Blåsjö

Utrecht University

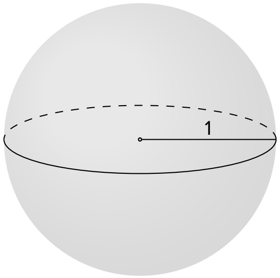
v.n.e.blasjo@uu.nl

uu.nl/staff/VNEBlasjo

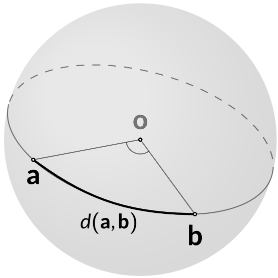
@viktorblasjo

The metric space S^2

$$S^2 := \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\right\}$$
$$= \text{unit sphere}$$



$$d(\mathbf{a},\mathbf{b}) := \angle\mathbf{aob} = \arccos(\mathbf{a}\cdot\mathbf{b})$$
$$= \text{shortest distance along surface}$$
$$\in [0,\pi]$$

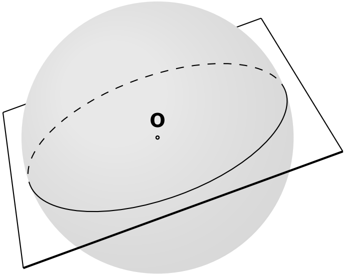


Definition of lines and angles in S^2

line := great circle

:= intersection with plane through $\mathbf{o}$

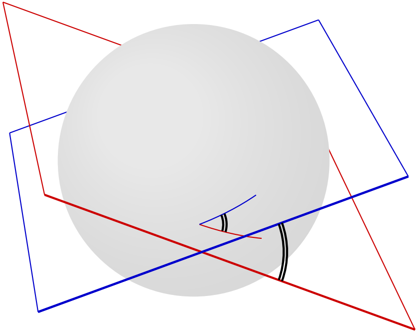
$\supset$ path of shortest distance between two points



line$(\mathbf{a}, \mathbf{b})$:= great circle of plane$(\mathbf{o}, \mathbf{a}, \mathbf{b})$

angle between lines on sphere

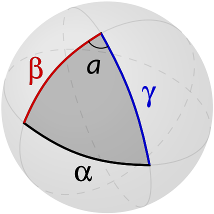
:= angle between corresponding planes



Trigonometry in S^2

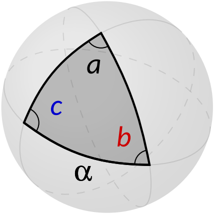
cosine rule:

$$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$$

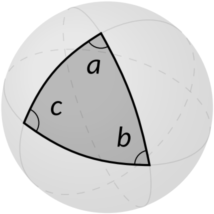


alternative cosine rule:

$$\cos a = \cos\alpha\sin b\sin c - \cos b\cos c$$

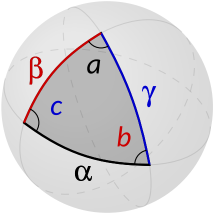


area of triangle $= \underbrace{a+b+c-\pi}_{\text{“angular excess”}}$



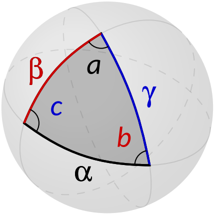
sine rule:

$$\frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b} = \frac{\sin\gamma}{\sin c}$$



$\mathbb{E}^2$ theorems as limit cases of S^2 theorems

$$\frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b} = \frac{\sin\gamma}{\sin c}$$



$$\Big\downarrow \text{ as } \alpha, \beta, \gamma \to 0$$

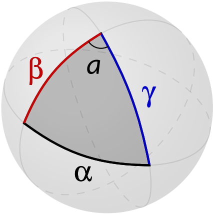
$$\frac{\alpha}{\sin a} = \frac{\beta}{\sin b} = \frac{\gamma}{\sin c}$$

since $\sin x \approx x$ for small x

❓ Cosine rule $\Longrightarrow$ SAS — why?

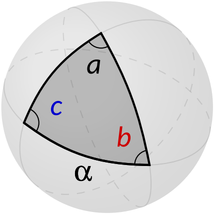
cosine rule:

$$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$$

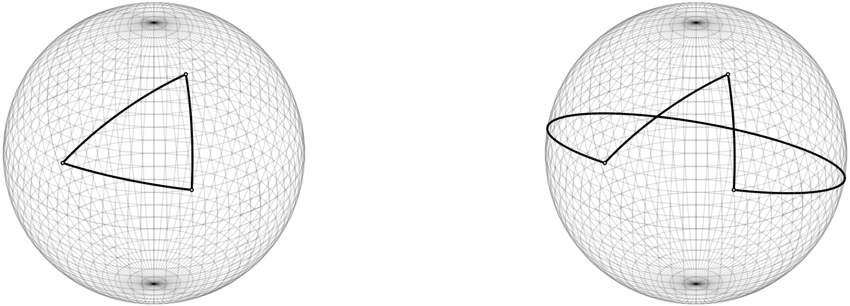


alternative cosine rule:

$$\cos a = \cos\alpha\sin b\sin c - \cos b\cos c$$



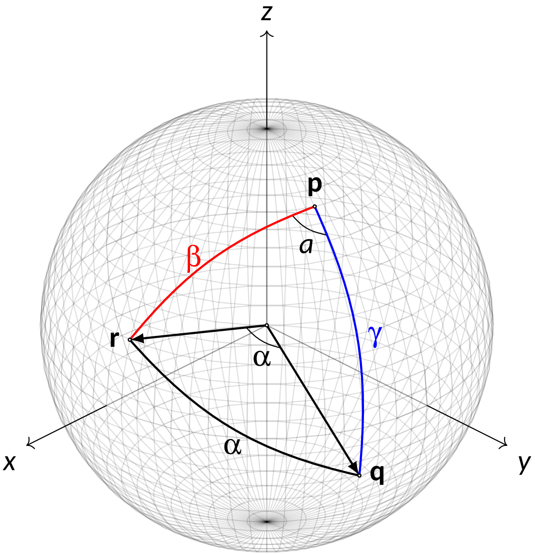
“Counterexample” to SAS



❓ Why does this not contradict (cosine rule $\implies$) SAS?

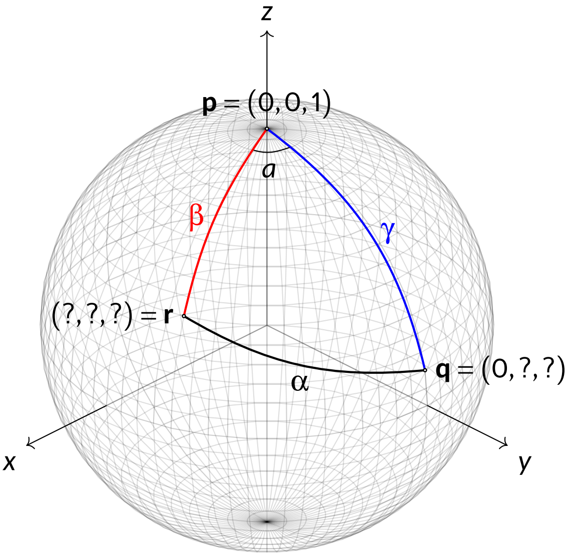
Proof of cosine rule

$$\mathbf{q}\cdot\mathbf{r} = |\mathbf{q}|\,|\mathbf{r}|\cos\alpha = \cos\alpha \stackrel{?}{=} \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$$



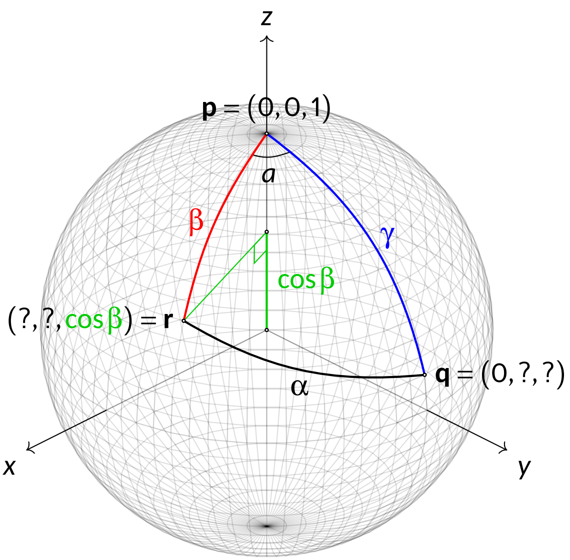
Proof of cosine rule

$$\mathbf{q}\cdot\mathbf{r} = \cos\alpha \stackrel{?}{=} \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$$



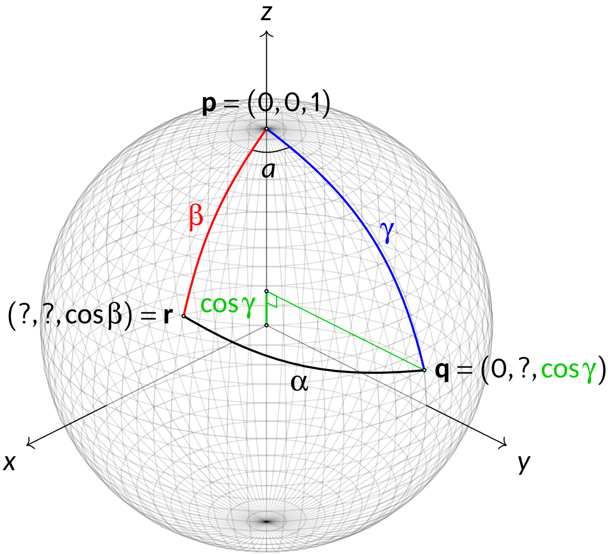
Proof of cosine rule

$$\mathbf{q}\cdot\mathbf{r} = \cos\alpha \stackrel{?}{=} \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$$



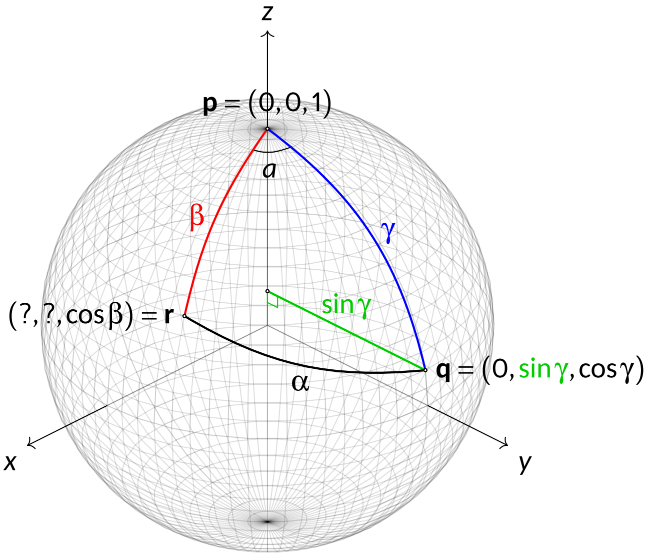
Proof of cosine rule

$$\mathbf{q}\cdot\mathbf{r} = \cos\alpha \stackrel{?}{=} \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$$



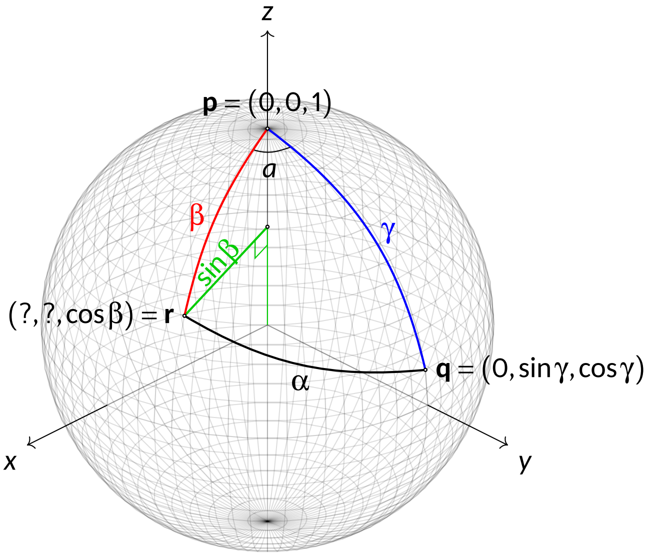
Proof of cosine rule

$$\mathbf{q}\cdot\mathbf{r} = \cos\alpha \stackrel{?}{=} \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$$



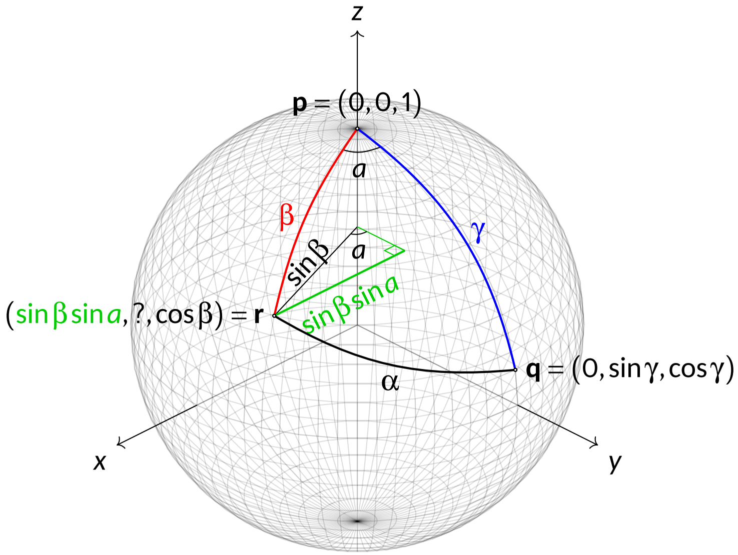
Proof of cosine rule

$$\mathbf{q}\cdot\mathbf{r} = \cos\alpha \overset{?}{=} \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$$



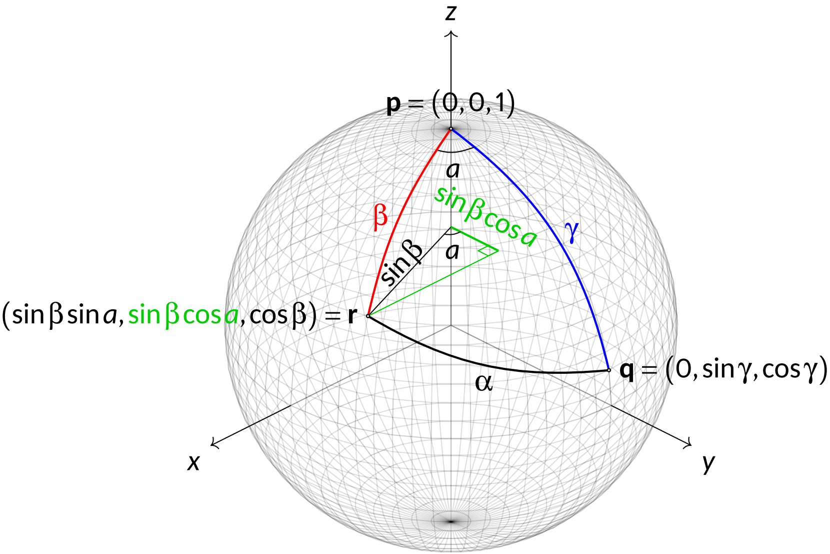
Proof of cosine rule

$$\mathbf{q} \cdot \mathbf{r} = \cos\alpha \stackrel{?}{=} \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$$



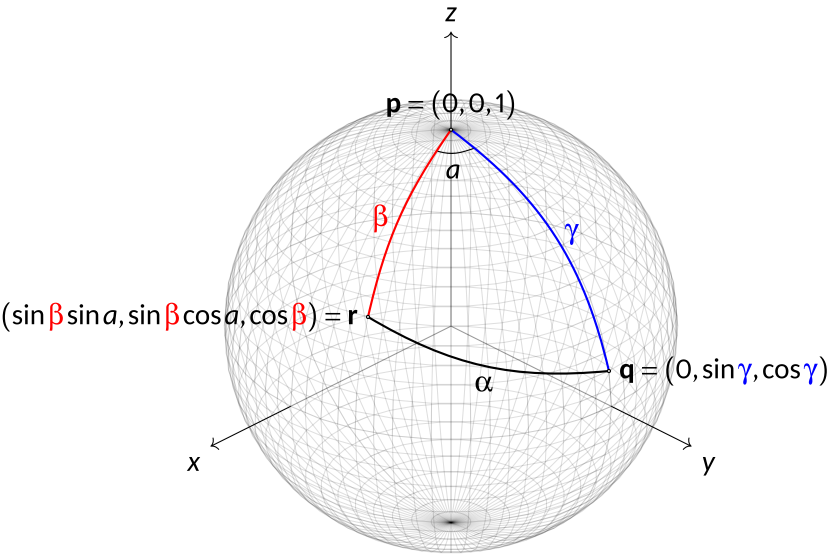
Proof of cosine rule

$$\mathbf{q}\cdot\mathbf{r} = \cos\alpha \stackrel{?}{=} \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$$

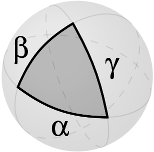


Proof of cosine rule

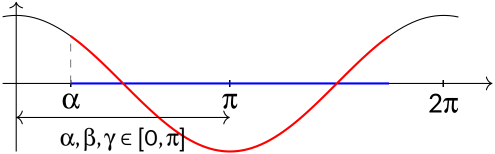
$$\cos\alpha = \mathbf{q}\cdot\mathbf{r} = (0,\sin\gamma,\cos\gamma)\cdot(\sin\beta\sin a,\sin\beta\cos a,\cos\beta) = \sin\gamma\sin\beta\cos a + \cos\gamma\cos\beta$$ ✓



Proof of triangle inequality in S^2



$$
\begin{aligned}
\cos\alpha &= \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a && \text{cosine rule} \\
&\geq \cos\beta\cos\gamma + \underbrace{\sin\beta\sin\gamma}_{\substack{\beta,\gamma\in[0,\pi] \implies \\ \sin\beta\sin\gamma\geq 0}}(-1) && \min(\cos a) = -1 \\
&= \cos(\beta+\gamma) && \text{trig. addition formula}
\end{aligned}
$$



$\cos\alpha \geq \cos(\beta+\gamma)$

$\implies \cos(\beta+\gamma) \in$ 

$\implies \beta+\gamma \in$

$\implies \alpha \leq \beta+\gamma \quad \square$