Projective geometry

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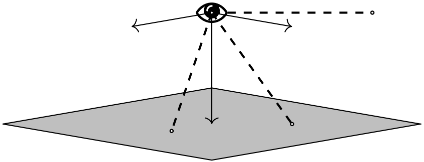
Idea for defining the projective plane $\mathbb{P}^2$

points of projective plane $\mathbb{P}^2$

$\longleftrightarrow$

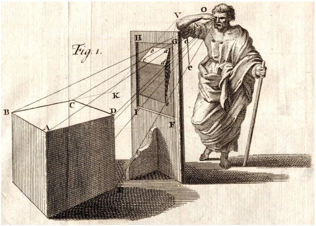
lines through origin in $\mathbb{R}^3$

"lines of sight" through "eye point"

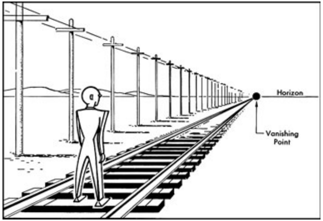


Formalises the intuitive idea of projection:

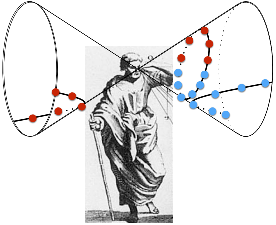
- Point = other point on same line.



- "Points at infinity" are points like any other.



- We "see" equally in all directions.



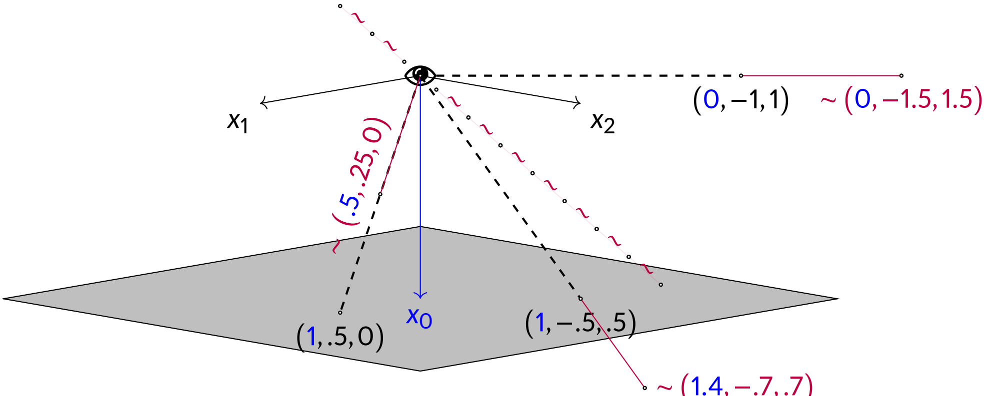
Formal definition of projective space $\mathbb{P}^n$

$\mathbb{P}^n$ is the set of equivalence classes

$$\mathbb{P}^n := (\mathbb{R}^{n+1} - \{\mathbf{0}\}) / \sim$$

of points in $\mathbb{R}^{n+1} - \{\mathbf{0}\}$ under the equivalence relation

$$\mathbf{x} \sim \mathbf{y} \iff \mathbf{x} = \lambda \mathbf{y} \quad \text{for some non-zero } \lambda \in \mathbb{R}$$



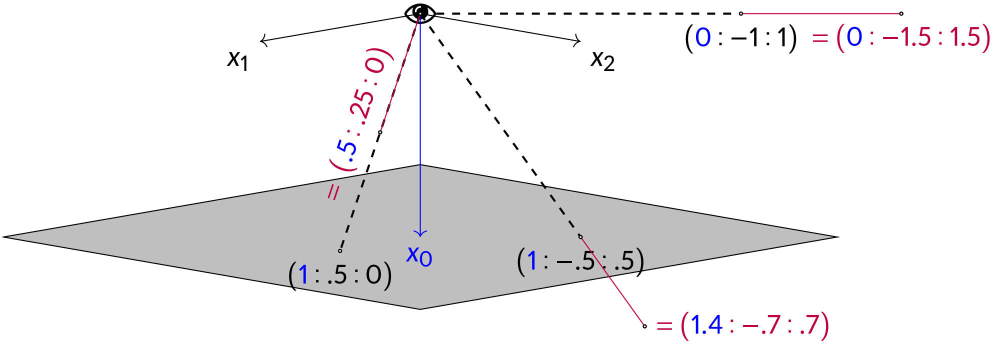
Notation for elements of $\mathbb{P}^n$

The $\sim$-equivalence class of $\mathbf{x} = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} - \{\mathbf{0}\}$ is denoted

$$P_{\mathbf{x}} \in \mathbb{P}^n \quad \text{or} \quad (x_0 : x_1 : \ldots : x_n) \in \mathbb{P}^n \quad \text{("homogenous coordinates")}$$

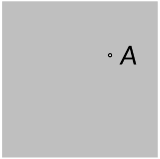
Note:

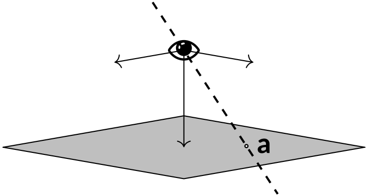
$$P_{\lambda \mathbf{x}} = P_{\mathbf{x}} \qquad (\lambda x_0 : \lambda x_1 : \ldots : \lambda x_n) = (x_0 : x_1 : \ldots : x_n) \qquad P_{\mathbf{0}} \notin \mathbb{P}^n \qquad (0 : 0 : \ldots : 0) \notin \mathbb{P}^n$$



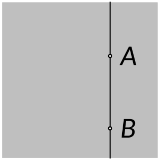
$\mathbb{P}^2$ objects $\leftrightarrow$ $\mathbb{R}^3$ objects

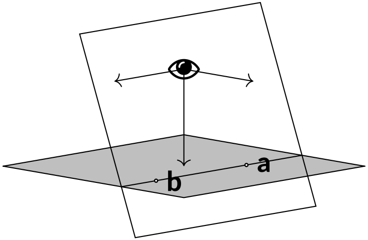
$\mathbb{P}^2$ object	$\mathbb{R}^3$ object (before $\sim$-collapsing)
point $A = P_{\mathbf{a}}$	line $\lambda\mathbf{a} - \{0\}$



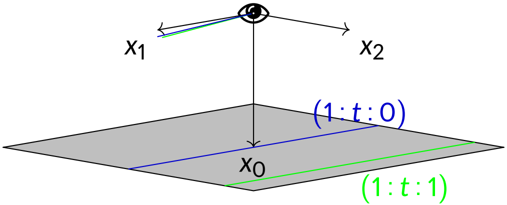


$\mathbb{P}^2$ object	$\mathbb{R}^3$ object (before $\sim$-collapsing)
line AB	plane $\operatorname{span}(\mathbf{a}, \mathbf{b}) - \{0\}$

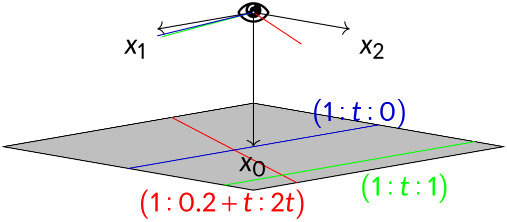




Parallel lines have the same point at infinity: in terms of coordinates

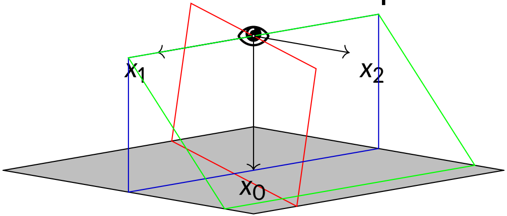


Non-parallel lines have different “points at infinity”:



- $(1:t:1) \xrightarrow{t\to\infty}$?
- $(1:t:0) \xrightarrow{t\to\infty}$?
- $(1:0.2+t:2t) \xrightarrow{t\to\infty}$?

Parallel lines have the same point at infinity: in terms of planes



- Parametrically: $(1 : t : 1)$
  In the "ground plane" $x_0 = 1$: ?
  "Homogenised" eq. for plane: ?
- Parametrically: $(1 : t : 0)$
  In the "ground plane" $x_0 = 1$: ?
  "Homogenised" eq. for plane: ?
- Parametrically: $(1 : 0.2 + t : 2t)$
  In the "ground plane" $x_0 = 1$:
  $x_2/2 = x_1 - 0.2$
  "Homogenised" eq. for plane:
  $x_2/2 = x_1 - 0.2x_0$

Generally, any line can be "homogenised" to the corresponding plane by multiplying constant terms by x_0:

line $ax_1 + bx_2 = c$ in ground plane $x_0 = 1$

$$\leftrightarrow$$

$$\text{plane } ax_1 + bx_2 = cx_0$$

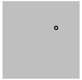
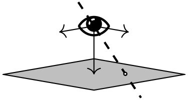
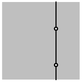
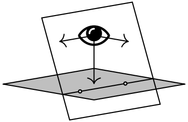
This plane goes through **0** and has the correct line of intersection with ground plane $x_0 = 1$.

Subspaces of $\mathbb{P}^n$

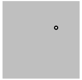
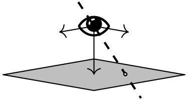
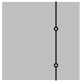
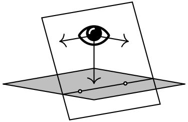
Projective subspaces $\subset \mathbb{P}^n$ are $\sim$-collapsed versions of linear subspaces $\subset \mathbb{R}^{n+1}$:

$$\underbrace{\mathbb{P}(U)}_{\text{subspace } \subset \mathbb{P}^n} := \{P_{\mathbf{u}} : \mathbf{u} \in U - \{\mathbf{0}\}\} \quad \text{where } U \text{ is a linear subspace } \subset \mathbb{R}^{n+1}$$

$$\dim \mathbb{P}(U) := \dim(U) - 1$$

$\mathbb{P}(U)$		U	
point		line	
line		plane	
$\varnothing$		$\{\mathbf{0}\}$	

$\mathbb{P}^n$ objects $\leftrightarrow$ $\mathbb{R}^{n+1}$ objects

$\mathbb{P}^2$ object		$\mathbb{R}^3$ object (before $\sim$-collapsing)	
point		line	
line		plane	

$\mathbb{P}^n$ object	$\mathbb{R}^{n+1}$ object (before $\sim$-collapsing)
projective subspace of dim m	linear subspace of dim $m+1$
hyperplane := {sols. to $\underbrace{c_0x_0+\cdots+c_nx_n}_{\text{not all coefficients 0}} = 0$} dim(hyperplane)= $n-1$	hyperplane: {sols. to $\underbrace{c_0x_0+\cdots+c_nx_n}_{\text{not all coefficients 0}} = 0$} dim(hyperplane)= n

❓ Same equation, different dimension! Paradox?

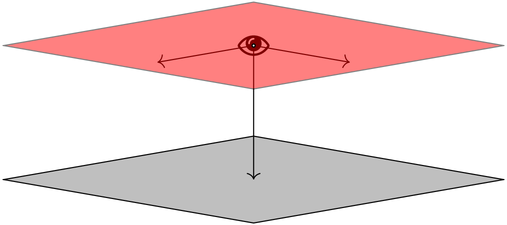
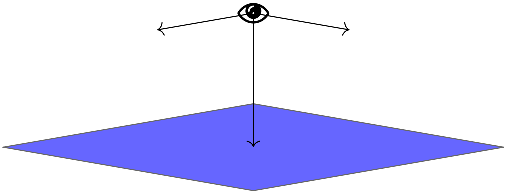
❓ What shape is the horizon?

A pair of parallels (in a plane, “the ground”) intersect in a “point at infinity.” The set of all such points is called the “line at infinity.”

- ❓ Shouldn’t it be “circle at infinity”? Since it “goes all the way around”?
- ❓ What happens if we remove the restriction that the parallels were in a (“ground”) plane? Do the set of all intersections of parallels in space form a “plane at infinity”?

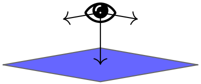
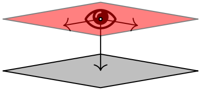
Answers on next slides.

$\mathbb{P}^2$ can be seen as $\mathbb{A}^2 \cup$ horizon (ordinary plane $\cup$ intersections of parallels)

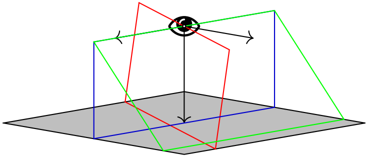
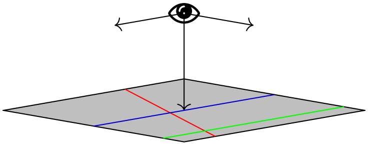
$\mathbb{P}^2$ object	$\mathbb{R}^3$ object (before $\sim$-collapsing)
"points at infinity"; "the horizon"	horizontal plane through eye point $(0,0,0)$
$= \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 : x_0 = 0\}$	
"the rest" = $\mathbb{P}^2 -$ "points at infinity"	every point not in red plane $\sim$ point in blue plane
$= \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 : x_0 = 1\} \cong \mathbb{A}^2$	

$\mathbb{P}^n$ as extended $\mathbb{A}^n$

$$\mathbb{P}^n = \underbrace{\{(x_0 : x_1 : \cdots : x_n) \in \mathbb{P}^n : x_0 = 1\}}_{\cong \mathbb{A}^n} \cup \underbrace{\{(x_0 : x_1 : \cdots : x_n) \in \mathbb{P}^n : x_0 = 0\}}_{\text{“hyperplane at infinity”}}$$

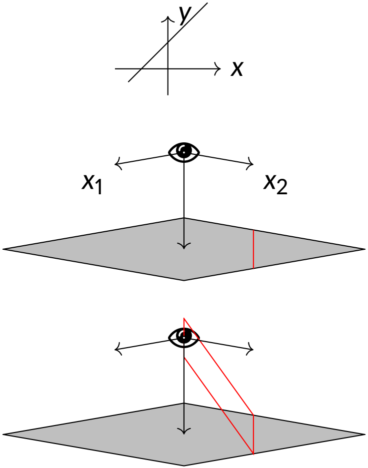

$\cup$


$\mathbb{P}^n$ object		$\mathbb{A}^n$ object
E projective subspace	$\longleftrightarrow$	$E \cap \mathbb{A}^n$ affine subspace
projective hyperplane H	intersect with “ground” $\longrightarrow$	affine hyperplane $H \cap \mathbb{A}^n$
$c_0x_0 + c_1x_1 + \cdots + c_nx_n = 0$	fill in $x_0=1$ $\longrightarrow$	$c_1x_1 + \cdots + c_nx_n = -c_0$
projective hyperplane containing F	“homogenise” $\longleftarrow$	affine hyperplane F
$-dx_0 + c_1x_1 + \cdots + c_nx_n = 0$	multiply constant by x_0 $\longleftarrow$	$c_1x_1 + \cdots + c_nx_n = d$

❓ What is the "point at infinity" of $y = 1 + x$?

$y = 1 + x$



$\rightsquigarrow$ affine line:

$\{(1 : x_1 : x_2) : x_2 = 1 + x_1\}$

(no points at infinity yet)

$\rightsquigarrow$ full projective line:

$\{(x_0 : x_1 : x_2) : x_2 = x_0 + x_1\}$

(homogenised eq.)

(includes points at infinity)

Points at infinity occur when $x_0 = 0$:

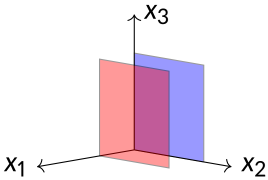
$$\{(0 : x_1 : x_2) : x_2 = x_1\} = \{(0 : 1 : 1)\}$$

❓ What is another line with the same point at infinity?

Intersection at infinity of two parallel planes in $\mathbb{P}^3$

affine coord.	affine eq. ("all points $(1:x_1:x_2:x_3)$ such that …")	homogenous eq. ("all points $(x_0:x_1:x_2:x_3)$ such that …")	projective coord.
$(1:0:*:*)$	$x_1 = 0$	$x_1 = 0$	$(*:0:*:*)$
$(1:1:*:*)$	$x_1 = 1$	$x_1 = x_0$	$(*:x_0:*:*)$

Visualisation of affine part only:
(Cannot draw two intersecting
3-dimensional spaces in $\mathbb{R}^4$!)



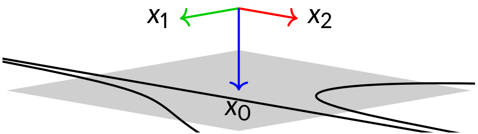
Points at infinity occur when $x_0 = 0$:

$$\{(0:x_1:x_2:x_3) : x_1 = 0\} \cap \{(0:x_1:x_2:x_3) : x_1 = x_0\} = (0:0:*:*) \cap (0:0:*:*) = (0:0:*:*)$$

? $(0:0:*:*)$ is a [?point/?line/?plane/?$\mathbb{R}^4$ hyperplane]. Picture it in the figure!

Finding the full projective curve (homogenous eq.) from an affine curve

Let γ be a curve in the ground plane given by a polynomial equation in x_1, x_2 such as $x_1^3 - 5x_1x_2 + 3x_2 - 8 = 0$.



$$(x_0, x_1, x_2) \sim \mathbf{p} \in \gamma$$

$$\overset{?}{\iff} \left(1, \frac{x_1}{x_0}, \frac{x_2}{x_0}\right) \in \gamma$$

$$\overset{?}{\iff} \left(\frac{x_1}{x_0}\right)^3 - 5\left(\frac{x_1}{x_0}\right)\left(\frac{x_2}{x_0}\right) + 3\left(\frac{x_2}{x_0}\right) - 8 = 0$$

$$\overset{?}{\iff} x_1^3 - 5x_0x_1x_2 + 3x_0^2x_2 - 8x_0^3 = 0$$

These steps can't all be $\iff$'s, because the last eq. includes point(s) at infinity and the first does not, namely

$$(0 : ? : ?)$$

(corresponding to an asymptote of γ).

- ❓ Which step introduced the point(s) at infinity?
- ❓ Instead of going through the above calculations, what is the quick recipe (corresponding to the name "homogenous equation") for going from things like

$$x_1^3 - 5x_1x_2 + 3x_2 - 8 = 0$$

to things like

$$x_1^3 - 5x_0x_1x_2 + 3x_0^2x_2 - 8x_0^3 = 0?$$

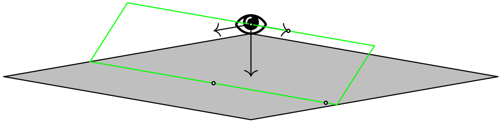
$\mathbb{P}^n$ span $\leftrightarrow \mathbb{R}^{n+1}$ span

$$\begin{aligned}\langle P_{\mathbf{a}}, P_{\mathbf{b}}, P_{\mathbf{c}}, \ldots\rangle &= \text{projective span}\\ &:= \text{smallest } \mathbb{P}^n \text{ subspace containing } P_{\mathbf{a}}, P_{\mathbf{b}}, P_{\mathbf{c}}, \ldots\\ &= \mathbb{P}(U) \text{ for some subspace } U \subset \mathbb{R}^{n+1} \text{ that:}\end{aligned}$$

- contains $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$
- is the smallest such U

$$= \mathbb{P}(\text{span}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots))$$

Example:



$$\begin{aligned}\langle(1:1:0),(1:1:3)\rangle &= \langle P_{(1,1,0)}, P_{(1,1,3)}\rangle\\ &= \mathbb{P}\left(\text{span}\left((1,1,0),(1,1,3)\right)\right)\\ = \langle(1:1:0),(0:0:1)\rangle &= \mathbb{P}\left(\text{span}\left((1,1,0),(0,0,1)\right)\right)\end{aligned}$$

subspace ∩ subspace = subspace

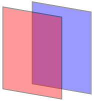
Recall: $\underbrace{\mathbb{P}(U)}_{\text{subspace } \subset \mathbb{P}^n} := \{P_{\mathbf{u}} : \mathbf{u} \in U - \{\mathbf{0}\}\}$ where U is a linear subspace $\subset \mathbb{R}^{n+1}$

So:

$$\mathbb{P}(U) \cap \mathbb{P}(V) = \{P_{\mathbf{x}} : \mathbf{x} \in U - \{\mathbf{0}\}\} \cap \{P_{\mathbf{x}} : \mathbf{x} \in V - \{\mathbf{0}\}\}$$
$$= \{P_{\mathbf{x}} : \mathbf{x} \in U \cap V - \{\mathbf{0}\}\}$$
$$= \mathbb{P}(\underbrace{U \cap V}_{\text{subspace } \subset \mathbb{R}^{n+1}}) = \text{subspace} \subset \mathbb{P}^n \quad \square$$

Example (recall the parallel planes):

aff. coord.	aff. eq.	hom. eq.	proj. coord.	$\mathbb{R}^{3+1}$ subspace	= span of
$(1:0:*:*)$	$x_1 = 0$	$x_1 = 0$	$(*:0:*:*)$	$U = (*,0,*,*)$	(1,0,0,0) (0,0,1,0) (0,0,0,1)
$(1:1:*:*)$	$x_1 = 1$	$x_1 = x_0$	$(*:x_0:*:*)$	$V = (*,x_0,*,*)$	(1,1,0,0) (0,0,1,0) (0,0,0,1)



$$U \cap V = \text{span}\begin{pmatrix}(0,0,1,0)\\(0,0,0,1)\end{pmatrix} = (0,0,*,*) \qquad \mathbb{P}(U \cap V) = (0:0:*:*)$$

Dimension theorem for $\mathbb{P}^n$

$$\dim \underbrace{\underbrace{\mathbb{P}(U) \cap \mathbb{P}(V)}_{=\mathbb{P}(U \cap V)}}_{:=\dim(U \cap V)-1} + \dim \underbrace{\underbrace{\langle \mathbb{P}(U), \mathbb{P}(V) \rangle}_{\mathbb{P}(\mathrm{span}(U \cup V))}}_{:=\dim \mathrm{span}(U \cup V)-1} = \underbrace{\dim \mathbb{P}(U)}_{:=\dim(U)-1} + \underbrace{\dim \mathbb{P}(V)}_{:=\dim(V)-1}$$

$$\Updownarrow$$

Dimension theorem for $\mathbb{R}^{n+1}$

$$\dim(U \cap V) + \dim \mathrm{span}(U \cup V) = \dim U + \dim V$$

Applications of dimension theorem

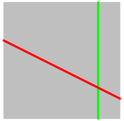
Simplest types of subspaces of $\mathbb{P}^n$:

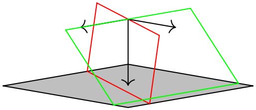
U	$\dim U$	$\mathbb{P}(U)$	$\dim \mathbb{P}(U) := \dim U - 1$
$\{\mathbf{0}\}$	0	$\emptyset$	-1
$\mathrm{span}(\mathbf{a})$	1	$P_{\mathbf{a}}$	0
$\mathrm{span}(\mathbf{a}, \mathbf{b})$	2	$\langle P_{\mathbf{a}}, P_{\mathbf{b}} \rangle$	1

More "intrinsic" interpretation of last line by dimension theorem:

$$\dim \underbrace{\langle A, B \rangle}_{A \neq B} = \dim\{A\} + \dim\{B\} - \dim\{A\} \cap \{B\}$$

$$= 0 + 0 - (-1) = 1$$





"All lines in $\mathbb{P}^2$ intersect" reflected in dimension theorem:

$$\dim \ell_1 \cap \ell_2 = \dim \ell_1 + \dim \ell_2 - \dim\langle \ell_1 \cup \ell_2 \rangle$$

$$= 1 + 1 - \begin{cases} 1 & \text{if } \ell_1 = \ell_2 \\ 2 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } \ell_1 = \ell_2 \\ 0 & \text{otherwise} \end{cases}$$

❓ In $\mathbb{P}^3$, what are the possible $\dim \ell_1 \cap \ell_2$?

Application of dimension theorem

If projective space is decomposed into its "ground" and "horizon" parts

$$\mathbb{P}^n = \underbrace{\{(x_0 : x_1 : \cdots : x_n) \in \mathbb{P}^n : x_0 = 1\}}_{\cong \mathbb{A}^n} \cup \underbrace{\{(x_0 : x_1 : \cdots : x_n) \in \mathbb{P}^n : x_0 = 0\}}_{:= H \text{ ("hyperplane at infinity")}}$$

then lines $\ell \in \mathbb{P}^n$ that "touch the ground" (contain points with $x_0 \neq 0$; $\ell \cap \mathbb{A}^n \neq \emptyset$) have precisely 1 point "at infinity":

$$\dim \ell \cap H = \dim \ell + \dim H - \dim \langle H, \ell \rangle = 1 + (n-1) - n = 0$$

since

$$\langle H, \ell \rangle \supseteq \mathbb{P}\left(\text{span}\left(\underbrace{\{(0, x_1, \ldots, x_n) : x_i \in \mathbb{R}\}}_{\substack{\text{incl. } (0,1,0,\ldots,0), \\ \ldots, (0,0,0,\ldots,1)}} \cup \left\{ (\underbrace{x_0}_{\neq 0}, x_1, \ldots, x_n) \right\}\right)\right) = \mathbb{P}(\mathbb{R}^{n+1}) = \mathbb{P}^n$$

(In)dependence in $\mathbb{P}^n$

Points $\subset \mathbb{P}^n$ are independent if "every point is outside the span of the previous ones":

$$P_{\mathbf{x}_0}, P_{\mathbf{x}_1}, \ldots, P_{\mathbf{x}_k} \text{ independent in } \mathbb{P}^n$$

$$\overset{\text{def.}}{\iff} \dim\langle P_{\mathbf{x}_0}, P_{\mathbf{x}_1}, \ldots, P_{\mathbf{x}_k}\rangle = k$$

(maximal dimension)

$$\iff \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k \text{ independent in } \mathbb{R}^{n+1}$$

Determinant condition for dependence:

$$P_{\mathbf{x}_0}, P_{\mathbf{x}_1}, \ldots, P_{\mathbf{x}_k} \text{ proj. dependent}$$
$$\iff \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k \text{ linearly dependent}$$
$$\iff \det(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k) = 0$$

Determinant expression for line:

$$\text{line}(P_{\mathbf{a}}, P_{\mathbf{b}}) \subset \mathbb{P}^2$$
$$= \{P_{\mathbf{x}} \in \mathbb{P}^2 : P_{\mathbf{a}}, P_{\mathbf{b}}, P_{\mathbf{x}} \text{ proj. dep.}\}$$
$$= \left\{ (x_0 : x_1 : x_2) \in \mathbb{P}^2 : \begin{vmatrix} | & | & x_0 \\ \mathbf{a} & \mathbf{b} & x_1 \\ | & | & x_2 \end{vmatrix} = 0 \right\}$$

Example: Line through $(1:1:0)$ and $(1:1:1)$:

$$\begin{vmatrix} 1 & 1 & x_0 \\ 1 & 1 & x_1 \\ 0 & 1 & x_2 \end{vmatrix} = 0 \implies x_2 - x_1 - x_2 + x_0 = 0$$
$$\implies x_0 = x_1$$

The "naive" way to define coordinates with respect to a spanning set in $\mathbb{P}^n$ is ill-defined

Given a set of independent points that span $\mathbb{P}^n$

$$\langle P_{\mathbf{x}_0}, P_{\mathbf{x}_1}, \ldots, P_{\mathbf{x}_n} \rangle = \mathbb{P}^n$$

Ex.: $$\langle P_{(1,0,0)}, P_{(0,2,0)}, P_{(0,0,2)} \rangle = \mathbb{P}^2$$

the "naive" way to define coordinates of any $P_{\mathbf{a}} \in \mathbb{P}^n$ with respect to this "basis" would be

$$P_{\mathbf{a}} = (a_0 : a_1 : \cdots : a_n)$$

where $(a_0, a_1, \ldots, a_n)$ are the coordinates of $\mathbf{a}$ with respect to the basis $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ in $\mathbb{R}^{n+1}$. Ex.:

$$(1,2,4) = 1 \cdot (1,0,0) + 1 \cdot (0,2,0) + 2 \cdot (0,0,2)$$

$$\implies P_{(1,2,4)} = (1:1:2)$$

But these coordinates are ill-defined since they are dependent on the choice of representative (the definition "doesn't respect" $\sim$). For example:

$$P_{(0,2,0)} = P_{(0,1,0)}$$

so the same basis also gives

$$(1,2,4) = 1 \cdot (1,0,0) + 2 \cdot (0,1,0) + 2 \cdot (0,0,2)$$

$$\implies P_{(1,2,4)} = (1:2:2) \neq (1:1:2)$$

The problem comes from the freedom to scale each "basis vector" independently.

Solution: add one more point as a “scaling lock”

Given a set of $n+2$ mutually independent points each $n+1$ of which span $\mathbb{P}^n$

$$P_{\mathbf{x}_0}, P_{\mathbf{x}_1}, \ldots, P_{\mathbf{x}_n}, P_{\mathbf{x}_{n+1}}$$

define the coordinates of any $P_{\mathbf{a}} \in \mathbb{P}^n$ in the “naive” way

$$P_{\mathbf{a}} = (a_0 : a_1 : \cdots : a_n)$$

where $(a_0, a_1, \ldots, a_n)$ are the coordinates of $\mathbf{a}$ with respect to the basis $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ in $\mathbb{R}^{n+1}$, except with the additional demand that the representatives $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ are chosen so that

$$\sum_{i=0}^{n} \mathbf{x}_i \sim \mathbf{x}_{n+1}$$

Ex: a basis of $\mathbb{P}^2$ is

$$P_{(1,0,0)}, P_{(0,2,0)}, P_{(0,0,2)}, P_{(1,2,2)}$$

The condition of the “scaling lock” point $P_{(1,2,2)}$ means that the basis vectors can only be scaled together:

$$\begin{aligned} (2,0,0) &\sim (1,0,0) \\ (0,4,0) &\sim (0,2,0) \\ (0,0,4) &\sim (0,0,2) \end{aligned} \qquad \begin{array}{r} (2,0,0) \\ (0,4,0) \\ +(0,0,4) \\ \hline (2,4,4) \end{array} \sim (1,2,2)$$

$$\begin{aligned} (1,0,0) &\sim (1,0,0) \\ (0,4,0) &\sim (0,2,0) \\ (0,0,2) &\sim (0,0,2) \end{aligned} \qquad \begin{array}{r} (1,0,0) \\ (0,4,0) \\ +(0,0,2) \\ \hline (1,4,2) \end{array} \not\sim (1,2,2)$$

Ex.: with a "scaling lock" point, coordinates are well-defined

$$P_{(1,0,0)}, P_{(0,2,0)}, P_{(0,0,2)}, P_{(1,2,2)}$$

$$(1,2,4) = 1\cdot(1,0,0) + 1\cdot(0,2,0) + 2\cdot(0,0,2)$$

$\Longrightarrow P_{(1,2,4)} = (1:1:2)$ legitimate since

$$\begin{array}{lll} (1,0,0)\sim(1,0,0) & & (1,0,0) \\ (0,2,0)\sim(0,2,0) & & (0,2,0) \\ (0,0,2)\sim(0,0,2) & & +(0,0,2) \\ \hline & & (1,2,2) \quad \sim (1,2,2) \end{array}$$

$$(1,2,4) = \frac{1}{2}\cdot(2,0,0) + \frac{1}{2}\cdot(0,4,0) + 1\cdot(0,0,4)$$

$\Longrightarrow P_{(1,2,4)} = \left(\frac{1}{2} : \frac{1}{2} : 1\right)$ legitimate since

$$\begin{array}{lll} (2,0,0)\sim(1,0,0) & & (2,0,0) \\ (0,4,0)\sim(0,2,0) & & (0,4,0) \\ (0,0,4)\sim(0,0,2) & & +(0,0,4) \\ \hline & & (2,4,4) \quad \sim (1,2,2) \end{array}$$

$$(1,2,4) = 1\cdot(1,0,0) + 2\cdot(0,1,0) + 2\cdot(0,0,2)$$

$\Longrightarrow P_{(1,2,4)} = (1:2:2)$ not legitimate since

$$\begin{array}{lll} (1,0,0)\sim(1,0,0) & & (1,0,0) \\ (0,1,0)\sim(0,2,0) & & (0,1,0) \\ (0,0,2)\sim(0,0,2) & & +(0,0,2) \\ \hline & & (1,1,2) \quad \not\sim (1,2,2) \end{array}$$

All legitimate choices of representatives give $\sim$-equivalent results.

Projective transformations

A projective transformation $\mathbb{P}^n \to \mathbb{P}^n$ is an invertible matrix M transformation $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$:

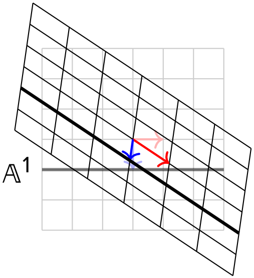
$$P_{\mathbf{x}} \mapsto P_{M\mathbf{x}}$$

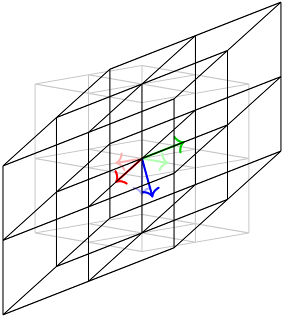
Check that well-defined:

- Independent of the choice of representative:

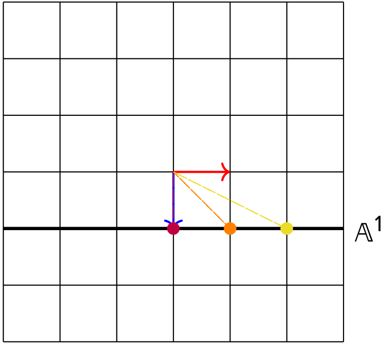
$$P_{\mathbf{x}} = P_{\lambda\mathbf{x}} \mapsto P_{M\lambda\mathbf{x}} = P_{\lambda M\mathbf{x}} = P_{M\mathbf{x}}$$

- Always lands in $\mathbb{P}^n$: $\mathbf{0} \in \mathbb{R}^{n+1}$ is the only $\mathbf{x}$ for which $P_{\mathbf{x}} \notin \mathbb{P}^n$, but this is not hit by $\mapsto$ since $M\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0} \implies P_{\mathbf{x}}$ not among the inputs $\mathbb{P}^n$.

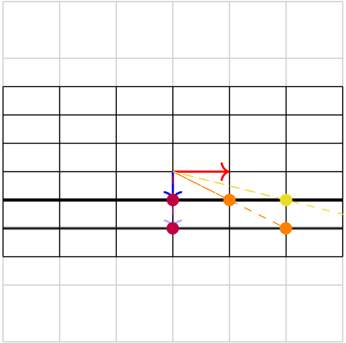




$\mathbb{P}^1 \to \mathbb{P}^1$ "dilation": make lengths bigger by bringing plane closer to the eye

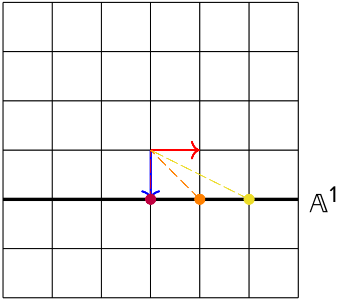


$$\begin{pmatrix} .5 & 0 \\ 0 & 1 \end{pmatrix} \quad \mapsto$$



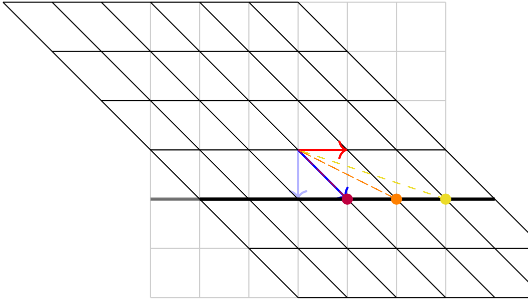
$$\begin{pmatrix} .5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\mathbb{P}^1 \to \mathbb{P}^1$ "translation"

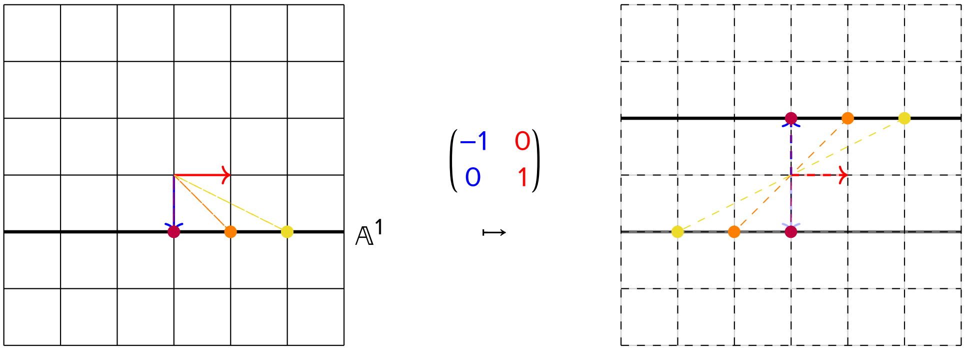


$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

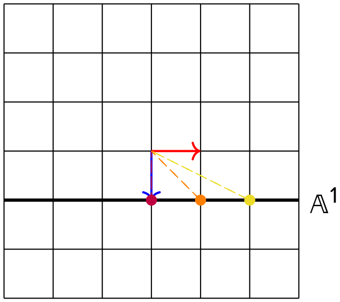
$$\longmapsto$$



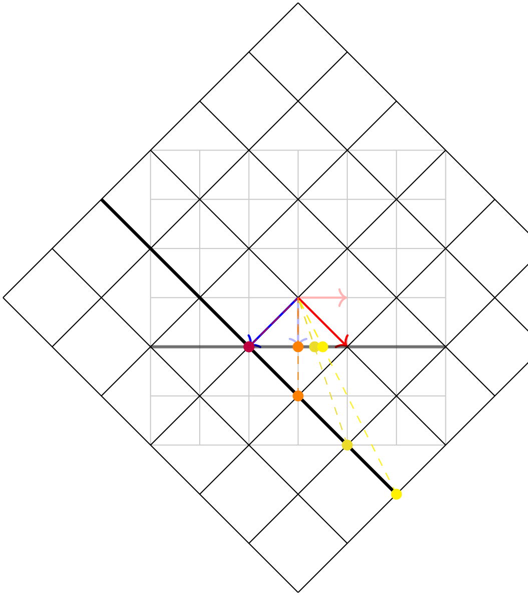
$\mathbb{P}^1 \to \mathbb{P}^1$ “reflection”: put “ground” and “canvas” on opposite sides of the eye



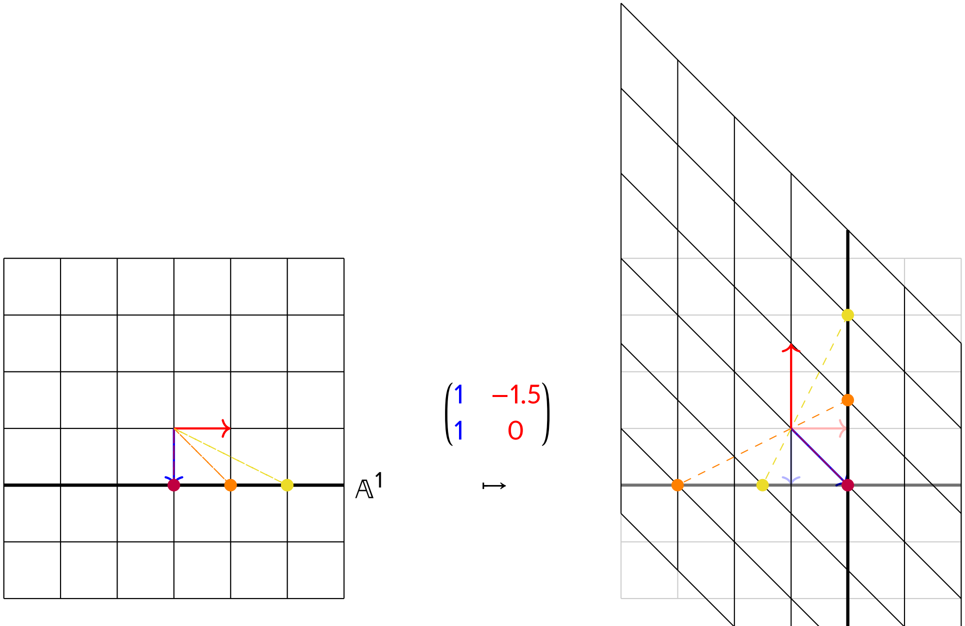
$\mathbb{P}^1 \to \mathbb{P}^1$: distances shrinking when approaching horizon point



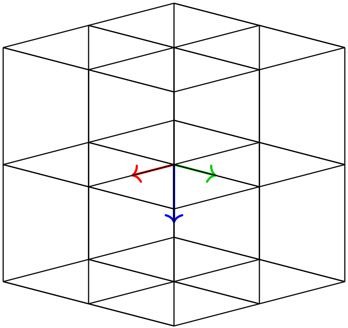
$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \mapsto$$



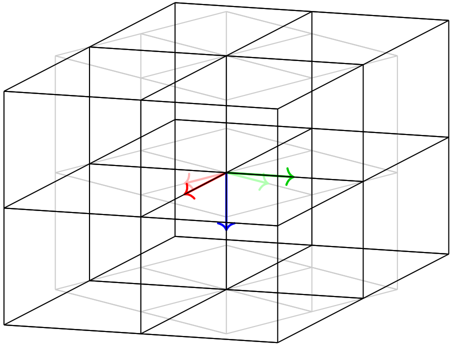
$\mathbb{P}^1 \to \mathbb{P}^1$ "permutation" of point order (points moved "past the horizon" "come back on the other side")



$\mathbb{P}^2 \to \mathbb{P}^2$: horizon preserved ↔ affine transformation in ground plane

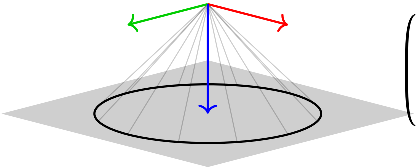


$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \quad \mapsto$$



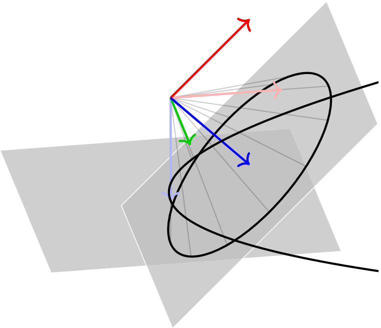
preserved by …	straightness	parallelism	angles	orientation	congruence	similarity	d	d-ratios	d-ratios ⊂ line	squares	triangles	circles	ellipses	conics	degrees
Euclidean transf.	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
affine transf.	✓	✓	✗	✗	✗	✗	✗	✗	✓	✗	✓	✗	✓	✓	✓
projective transf.	✓	✗	✗	✗	✗	✗	✗	✗	✗	✗	✓	✗	✗	✓	✓

$\mathbb{P}^2 \to \mathbb{P}^2$: parabola = circle with one point on the horizon



$$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

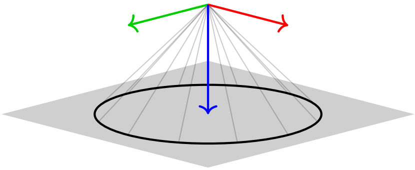
$$\mapsto$$



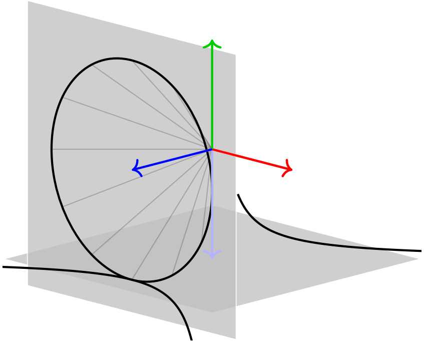
Ground point sent to infinity:

$$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ ? \\ ? \end{pmatrix} = \begin{pmatrix} 0 \\ ? \\ ? \end{pmatrix}$$

$\mathbb{P}^2 \to \mathbb{P}^2$: hyperbola = circle with two points on the horizon



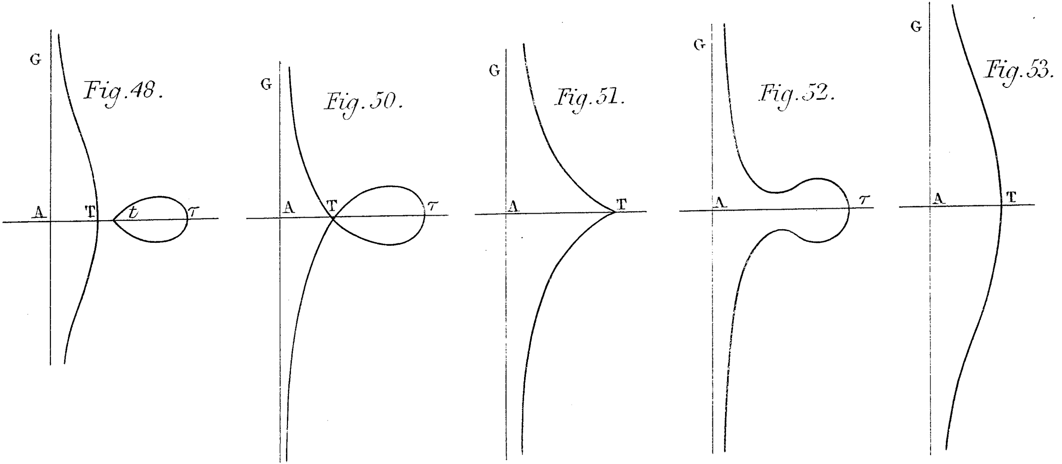
$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \mapsto$$



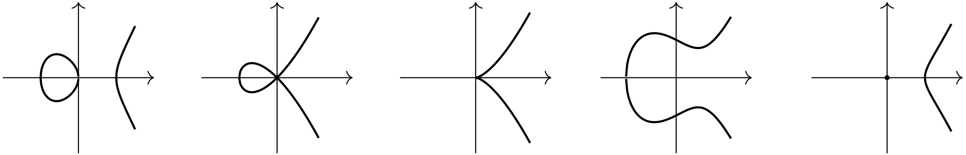
Ground points sent to infinity:

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ ? \\ ? \end{pmatrix} = \begin{pmatrix} 0 \\ ? \\ ? \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ ? \\ ? \end{pmatrix} = \begin{pmatrix} 0 \\ ? \\ ? \end{pmatrix}$$

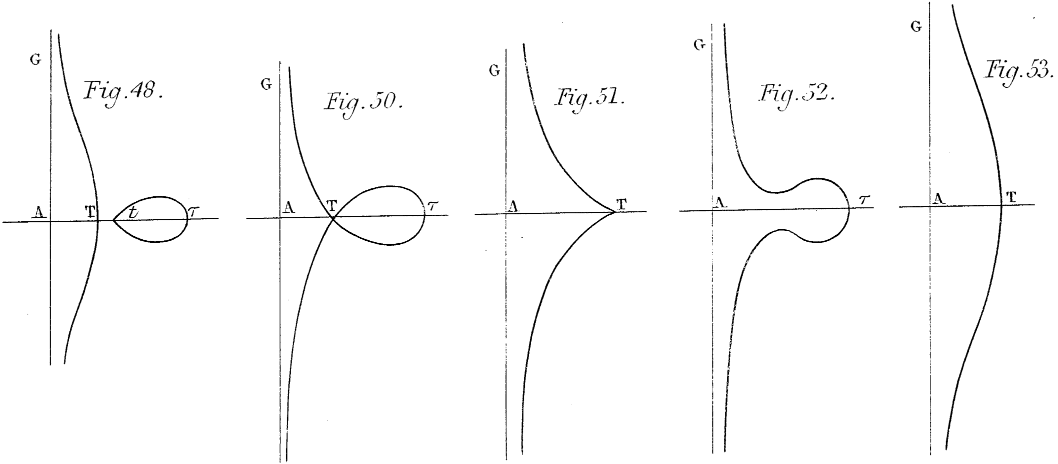
Newton’s classification “by shadows” of cubic curves into five “species”



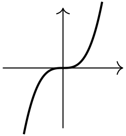
$y^2 = x^3 - x$ $\quad$ $y^2 = x^2(x+1)$ $\quad$ $y^2 = x^3$ $\quad$ $y^2 = x^3 - x + 1$ $\quad$ $y^2 = x^2(x-1)$



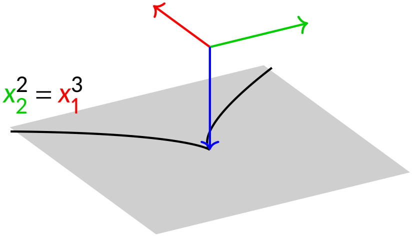
Newton's classification "by shadows" of cubic curves into five "species"



Which one is $y = x^3$?

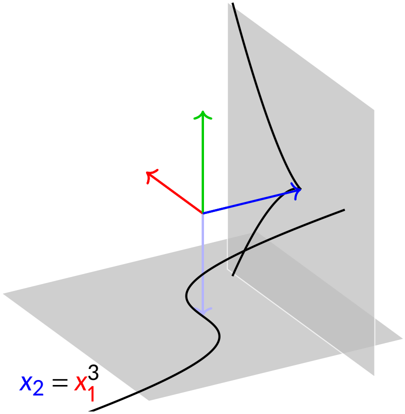


Projective equivalence of $y = x^3$ and $y^2 = x^3$



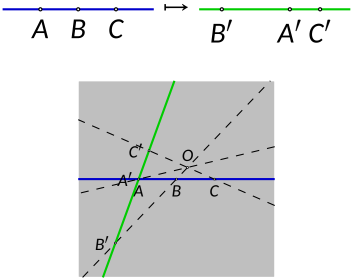
$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\longmapsto$$

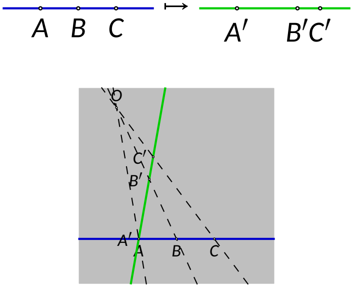


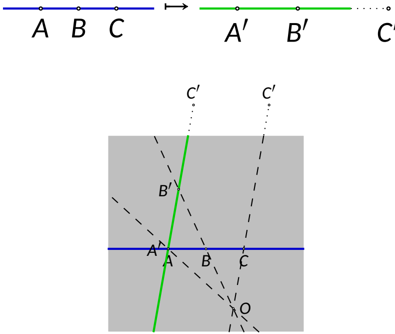
In $\mathbb{P}^1$, any* 3 points can be mapped to any* 3 points (* non-identical)

$\mathbb{P}^1$ version: Given* $A, B, C \in \ell \cong \mathbb{P}^1$ and $A', B', C' \in \ell' \cong \mathbb{P}^1$, there is a way to place these lines in $\mathbb{P}^2$ so that $A'B'C'$ is the perspective view of ABC from O.



Possible strategy: Put A' on top of A. Draw BB' and CC', and let their intersection be the projection point O.

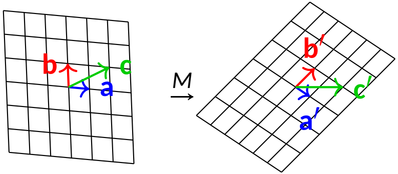


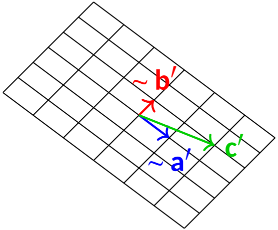


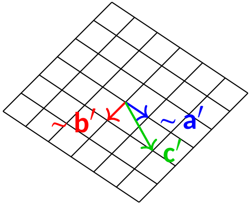
In $\mathbb{P}^1$, any* 3 points can be mapped to any* 3 points (* no 2 $\sim$-equivalent)

$\mathbb{R}^2$ version (before $\sim$-collapsing): Given $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{a}', \mathbf{b}', \mathbf{c}' \in \mathbb{R}^2$, $\exists$ a 2×2-matrix M such that $M\mathbf{a} \sim \mathbf{a}'$, $M\mathbf{b} \sim \mathbf{b}'$, $M\mathbf{c} \sim \mathbf{c}'$. Proof:

- $\exists$ 2×2-matrix M such that $M\mathbf{a} = \mathbf{a}'$, $M\mathbf{b} = \mathbf{b}'$.
- Express $\mathbf{c}$ using $\mathbf{a}, \mathbf{b}$ as a basis: $\mathbf{c} = c_{\mathbf{a}}\mathbf{a} + c_{\mathbf{b}}\mathbf{b}$.
- $M\mathbf{c}$ has the same coordinates in the new basis: $M\mathbf{c} = c_{\mathbf{a}}M\mathbf{a} + c_{\mathbf{b}}M\mathbf{b} = c_{\mathbf{a}}\mathbf{a}' + c_{\mathbf{b}}\mathbf{b}'$.
- Since $\mathbf{x} \sim \lambda\mathbf{x}$, it would have been the same projective transformation if we had taken $M\mathbf{a} = \lambda_1\mathbf{a}'$, $M\mathbf{b} = \lambda_2\mathbf{b}'$, in which case $M\mathbf{c} = c_{\mathbf{a}}\lambda_1\mathbf{a}' + c_{\mathbf{b}}\lambda_2\mathbf{b}'$.
- So by choosing λ_1, λ_2 we can ensure that $M\mathbf{c} = \mathbf{c}'$ without disturbing $M\mathbf{a} \sim \mathbf{a}'$, $M\mathbf{b} \sim \mathbf{b}'$. $\square$



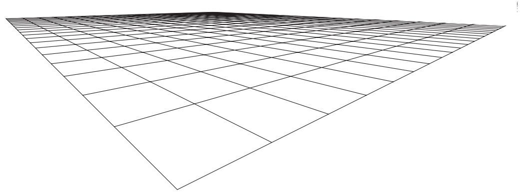




In $\mathbb{P}^2$, any* 4 points can be mapped to any* 4 points (* no 3 collinear)

$\mathbb{P}^2$ version	$\mathbb{R}^3$ version (before $\sim$-collapsing)
Any non-degenerate quadrilateral can be mapped to any non-degenerate quadrilateral.	Given $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{a}', \mathbf{b}', \mathbf{c}', \mathbf{d}' \in \mathbb{R}^3$, $\exists$ a 3×3-matrix M such that $M\mathbf{a} \sim \mathbf{a}'$, $M\mathbf{b} \sim \mathbf{b}'$, $M\mathbf{c} \sim \mathbf{c}'$, $M\mathbf{d} \sim \mathbf{d}'$.
(No 3 of $ABCD$, no 3 of $A'B'C'D'$ collinear.)	(No 3 of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, no 3 of $\mathbf{a}', \mathbf{b}', \mathbf{c}', \mathbf{d}'$ coplanar.)

Intuitive in terms of paintings:



(image source: Stillwell, *The Four Pillars of Geometry*)

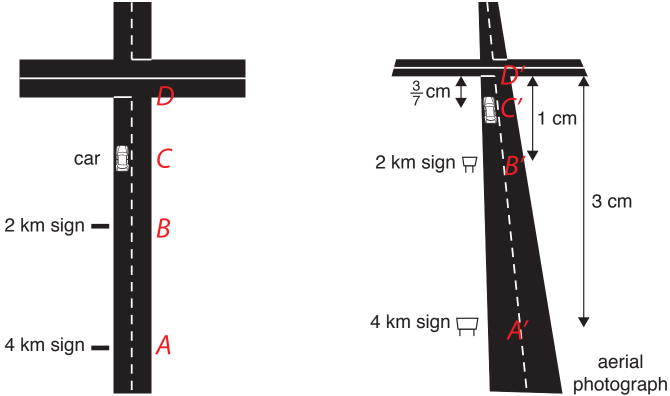
- The 3×3 entries of the matrix are enough degrees of freedom to send 3 vectors to 3 vectors, say $\mathbf{a}, \mathbf{b}, \mathbf{c}$ to $\lambda_a \mathbf{a}', \lambda_b \mathbf{b}', \lambda_c \mathbf{c}'$.
- The three scaling degrees of freedom $\lambda_a, \lambda_b, \lambda_c$ are enough to then also send $\mathbf{d}$ to $\mathbf{d}'$.

Cross-ratio

3 points on a line can be mapped to any 3, but that determines where any fourth point goes. The cross-ratio $(ABCD)$ is a projective invariant that expresses the condition on the fourth point.

$\mathbb{P}^n$ version	$\mathbb{R}^{n+1}$ version (before $\sim$-collapsing)
4 collinear points $ABCD$. $(ABCD) = \frac{AC}{BC}\frac{BD}{AD}$ (signed lengths).	4 coplanar vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. $(ABCD) = \frac{c_\mathbf{b}}{c_\mathbf{a}}\frac{d_\mathbf{a}}{d_\mathbf{b}}$ where $c_\mathbf{b}, c_\mathbf{a}, d_\mathbf{a}, d_\mathbf{b}$ are the coordinates of $\mathbf{c}$ and $\mathbf{d}$ in the basis $\mathbf{a}, \mathbf{b}$.
Check that these two expressions for $(ABCD)$ are equivalent (in the generic case of $\mathbb{P}^1$ with no points at infinity): $$\mathbf{a} = (1, a) \quad \mathbf{b} = (1, b) \quad \mathbf{c} = (1, c) \quad \mathbf{d} = (1, d)$$ $$\implies \mathbf{c} = \frac{c-b}{a-b}\mathbf{a} + \frac{a-c}{a-b}\mathbf{b} \quad \mathbf{d} = \frac{d-b}{a-b}\mathbf{a} + \frac{a-d}{a-b}\mathbf{b}$$ $$\implies (ABCD) = \frac{c_\mathbf{b}}{c_\mathbf{a}}\frac{d_\mathbf{a}}{d_\mathbf{b}} = \frac{a-c}{c-b}\frac{d-b}{a-d} = \frac{AC}{BC}\frac{BD}{AD} \text{ (signed)}$$	Check that definition does not depend on choice of representatives. If $\mathbf{a}$ is replaced by $\lambda\mathbf{a}$: $$(ABCD) = \frac{c_\mathbf{b}}{c_\mathbf{a}/\lambda}\frac{d_\mathbf{a}/\lambda}{d_\mathbf{b}}$$ If $\mathbf{c}$ is replaced by $\lambda\mathbf{c}$: $$(ABCD) = \frac{\lambda c_\mathbf{b}}{\lambda c_\mathbf{a}}\frac{d_\mathbf{a}}{d_\mathbf{b}}$$ ...

❓ Application of the cross-ratio: How far from the intersection is the car?



(image source: Brannan, Esplen, Gray, *Geometry*)

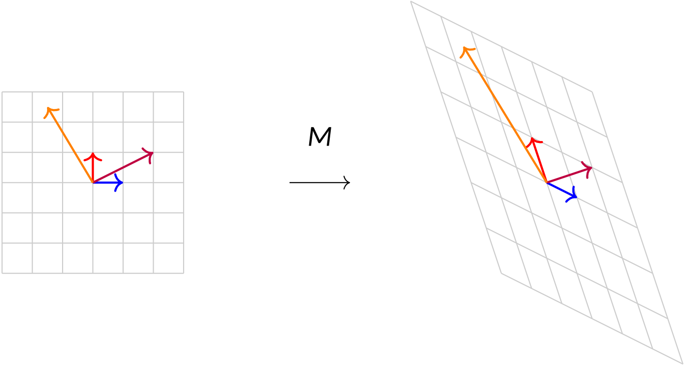
$$(ABCD) = \frac{AC}{BC}\frac{BD}{AD} = \frac{4-x}{2-x}\frac{2}{4}$$

$$= (A'B'C'D') = \frac{A'C'}{B'C'}\frac{B'D'}{A'D'} = \frac{3-\frac{3}{7}}{1-\frac{3}{7}}\frac{1}{3} = \frac{3-\frac{3}{7}}{3-\frac{9}{7}} = \frac{21-3}{21-9} = \frac{18}{12} = \frac{3}{2}$$

The cross-ratio is invariant under projective transformations

The coordinates of $\mathbf{c}$ and $\mathbf{d}$ in the basis $\mathbf{a}, \mathbf{b}$ are also the coordinates of $\mathbf{c}' := M\mathbf{c}$ and $\mathbf{d}' := M\mathbf{d}$ in the basis $\mathbf{a}' := M\mathbf{a}, \mathbf{b}' := M\mathbf{b}$. Hence

$$(ABCD) = \frac{c_{\mathbf{b}}}{c_{\mathbf{a}}} \frac{d_{\mathbf{a}}}{d_{\mathbf{b}}} = \frac{c'_{\mathbf{b}'}}{c'_{\mathbf{a}'}} \frac{d'_{\mathbf{a}'}}{d'_{\mathbf{b}'}} = (A'B'C'D') \quad \square$$

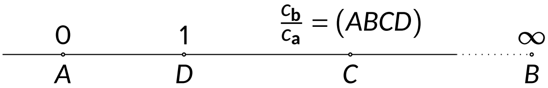


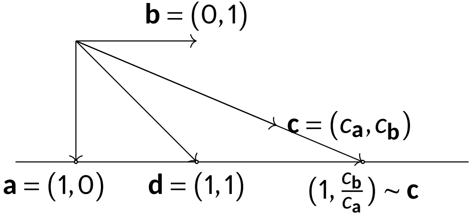
❓ Why doesn't the same reasoning prove that $\frac{c_{\mathbf{b}}}{c_{\mathbf{a}}}$ is invariant?

Cross-ratio in simplified configuration

We can use "any 3 $\mapsto$ any 3" to choose a simplified (equivalent) configuration:

$\mathbb{P}^1$ version	$\mathbb{R}^3$ version (before $\sim$-collapsing)



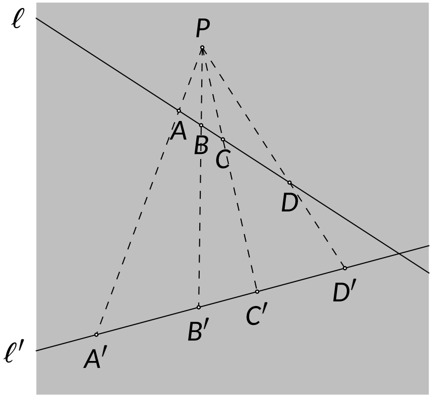


$$(ABCD) = \frac{c_\mathbf{b}}{c_\mathbf{a}} \frac{d_\mathbf{a}}{d_\mathbf{b}} = \frac{c_\mathbf{b}}{c_\mathbf{a}} \frac{1}{1} = \frac{c_\mathbf{b}}{c_\mathbf{a}}$$

The "length form" of the cross-ratio also works if we allow "common-sensical" rules for calculating with ∞:

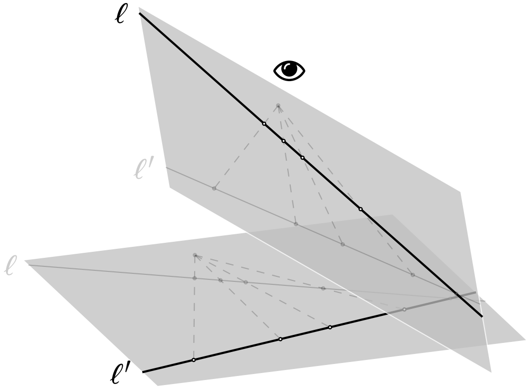
$$(ABCD) = \frac{AC}{BC} \frac{BD}{AD} = \frac{AC}{-\infty} \frac{-\infty}{1} = AC$$

Cross-ratio is invariant under $\mathbb{P}^n$-internal projection from a point

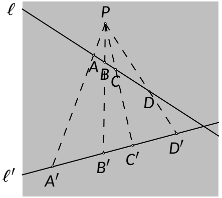


$$(ABCD) = (A'B'C'D')$$

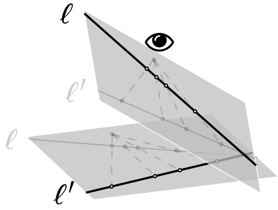
We already know that $(ABCD)$ is preserved by projective transformations $\mathbb{P}^n \to \mathbb{P}^n$. So in other words we need to show: If two lines are perspectively related "within the painting" (as in the above figure) then they are also projections of each other as seen "from without":



Cross-ratio is invariant under $\mathbb{P}^n$-internal projection from a point

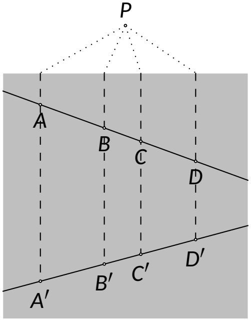


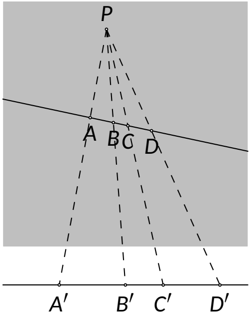
$$(ABCD) = (A'B'C'D')$$



Intuitive in the case of $\mathbb{P}^2$, but we will give a proof that:

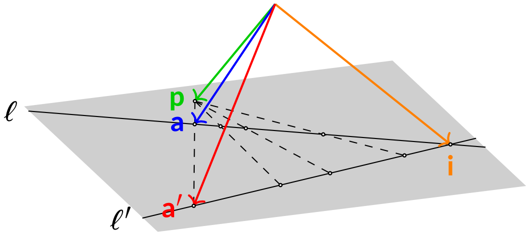
- Generalises to any $\mathbb{P}^n$.
- Works only with the $\mathbb{R}^{n+1}$ representation of $\mathbb{P}^n$, and hence applies equally to cases involving points at infinity, such as:





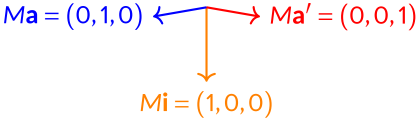
Simplification of projective configuration in $\mathbb{R}^3$

In $\mathbb{R}^3$, the $\mathbb{P}^2$ configuration on the previous slide becomes:



Goal: Simplify the configuration by applying suitable matrices*. Since matrices* preserve collinearity and cross-ratios, any matrix* sends this configuration to another configuration with the same cross-ratios and the same collinearity and intersection relationships. (* invertible)

- $\exists\ 3 \times 3$ matrix that sends $\mathbf{i}, \mathbf{a}, \mathbf{a}'$ to the standard basis

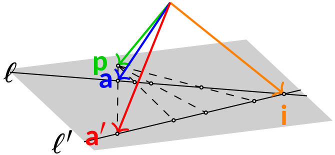


- In this basis, ℓ is $x_2 = 0$ and ℓ' is $x_1 = 0$.
- This still holds if we change the scaling to $M\mathbf{a} = \lambda_1(0,1,0)$ and $M\mathbf{b} = \lambda_2(0,0,1)$.
- By choosing λ_1, λ_2, we can make $M\mathbf{p}$ go anywhere in the plane $\mathrm{span}(M\mathbf{a}, M\mathbf{a}')$. (Same principle as in "any 3 $\mapsto$ any 3" proof.)
- Hence altogether we can choose M so that:

$$M\mathbf{i} = (1,0,0) \quad M\mathbf{a} = (0,\lambda_1,0) \quad M\mathbf{a}' = (0,0,\lambda_2)$$

$$M\mathbf{p} = (0,-1,1) \quad \ell : x_2 = 0 \quad \ell' : x_1 = 0$$

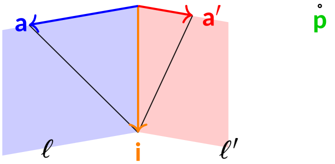
Projection $\ell \mapsto \ell'$ from P in simplified coordinate system



We obtained the simplified configuration:

$$M\mathbf{i} = (1,0,0) \quad M\mathbf{a} = (0,\lambda_1,0) \quad M\mathbf{a}' = (0,0,\lambda_2)$$

$$M\mathbf{p} = (0,-1,1) \quad \ell : x_2 = 0 \quad \ell' : x_1 = 0$$



In this simplified configuration, the projection of $\ell \mapsto \ell'$ from P takes a simple algebraic form.

- A point $L \in \ell$ is represented in $\mathbb{R}^3$ by $\mathbf{l} = (L_1, L_2, 0)$.

- The line $PL \subset \mathbb{P}^2$ corresponds to the plane $\text{span}(\mathbf{p},\mathbf{l}) = \text{span}((0,-1,1),(L_1,L_2,0)) \subset \mathbb{R}^3$.
- We need to find $L' := PL \cap \ell'$ which in $\mathbb{R}^3$ corresponds to $\{\text{span}(\mathbf{p},\mathbf{l})\} \cap \{x_1 = 0\}$.
- $L_2\mathbf{p} + \mathbf{l} = (L_1, 0, L_2)$ is in this intersection, so it is a representative of L'.
- So the projection $T : P_L \mapsto P_{L'}$ can be represented in coordinates by $(L_1, L_2, 0) \mapsto (L_1, 0, L_2)$.
- This is realised by the projective transformation $T : \mathbb{P}^2 \to \mathbb{P}^2$

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and is hence cross-ratio-preserving.

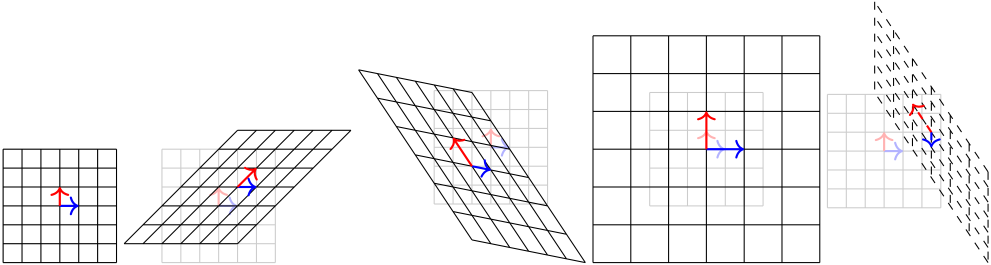
Ratio invariance in $\mathbb{A}^n$

Since affine space permits fewer transformations than projective space (leaves points at infinity), it has a simpler invariant than the cross-ratio (special case of one point at infinity) for 3 collinear points ABC:

$$(ABC) = -\frac{c_{\mathbf{b}}}{c_{\mathbf{a}}} \qquad \text{("coordinates" of } \mathbf{c} = c_{\mathbf{a}}\mathbf{a} + c_{\mathbf{b}}\mathbf{b} \text{ as linear combination of } \mathbf{a}, \mathbf{b})$$

$$\iff \qquad \overrightarrow{CA} = (ABC)\overrightarrow{CB}$$

❓ Why is (ABC) invariant under affine transformations?



Example

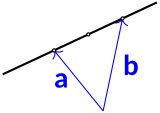
Recall: (ABC) is defined by

$$(ABC) = -\frac{c_\mathbf{b}}{c_\mathbf{a}} \qquad (\text{"coordinates" of } \mathbf{c} = c_\mathbf{a}\mathbf{a} + c_\mathbf{b}\mathbf{b} \text{ as linear combination of } \mathbf{a}, \mathbf{b})$$

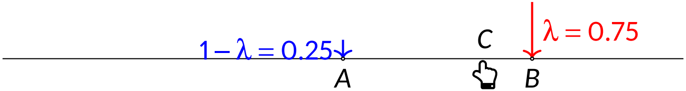
or

$$\overrightarrow{CA} = (ABC)\overrightarrow{CB}$$

Determine (ABC) in the case where C is the midpoint of AB:



- ❓ $(ABC) =$
- ❓ Express the meaning of (ABC) in words when C is not the midpoint of AB.



Overview of invariants

preserved by …	straightness	parallelism	angles	orientation	congruence	similarity	d	d-ratios	d-ratios $\subset$ line	squares	triangles	circles	ellipses	conics	degrees	cross-ratio
Euclidean transf.	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
affine transf.	✓	✓	✗	✗	✗	✗	✗	✗	✓	✗	✓	✗	✓	✓	✓	✓
projective transf.	✓	✗	✗	✗	✗	✗	✗	✗	✗	✗	✓	✗	✗	✓	✓	✓