Hyperbolic geometry

Viktor Blåsjö

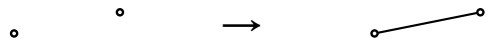
Utrecht University

v.n.e.blasjo@uu.nl
uu.nl/staff/VNEBlasjo
@viktorblasjo

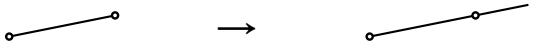
Comparative overview

	spherical	Euclidean	hyperbolic
△ angle sum	$> \pi$	$= \pi$	$< \pi$
parallels to ℓ through P	0	1	∞
metric space	✓	✓	✓
cosine rule	✓	✓	✓
$d(A,B)$ unbounded	✗	✓	✓
line(A,B) unique	✗	✓	✓
Euclid's Postulates 1–4	✗	✓	✓
Euclid's Postulate 5	✓	✓	✗

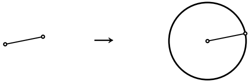
Postulate 1: draw line between points.



Postulate 2: extend line.

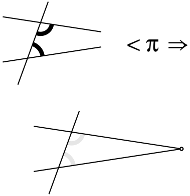


Postulate 3: draw circle.



Postulate 4: identity of right angles.

Postulate 5: condition for crossing.



Hyperbolic geometry proves logical independence of Postulate 5

	Euclidean	hyperbolic
Postulates 1–4	✓	✓
Postulate 5	✓	✗

	3	9
odd	✓	✓
prime	✓	✗

Postulates 1–4 $\not\Rightarrow$ Postulate 5

odd $\not\Rightarrow$ prime

Postulates 1–4 $\not\Rightarrow$ $\neg$Postulate 5

odd $\not\Rightarrow$ $\neg$prime

Before hyperbolic geometry

	Euclidean	hyperbolic
Postulates 1–4	✓	✓
Postulate 5	✓	✗

Many people tried to prove

$$\text{Postulates 1–4} \implies \text{Postulate 5}$$

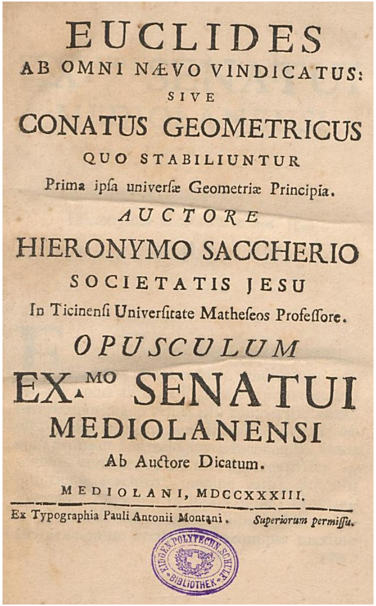
by trying to show

$$\text{Postulates 1–4} \,\&\, \neg\text{Postulate 5}$$
$$\implies \text{contradiction}$$

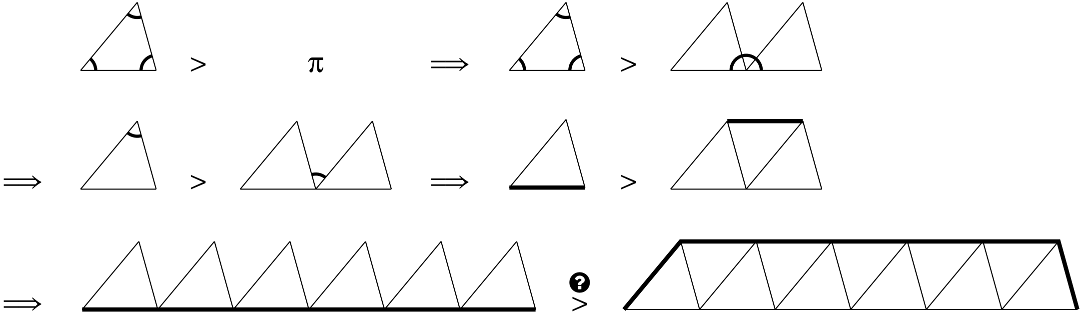
but where only able to show

$$\text{Postulates 1–4} \,\&\, \neg\text{Postulate 5}$$

⟹ theorems “repugnant to the nature of a straight line”

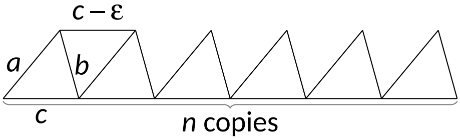

EUCLIDES
AB OMNI NÆVO VINDICATUS:
SIVE
CONATUS GEOMETRICUS
QUO STABILIUNTUR
Prima ipſa univerſæ Geometriæ Principia.
AUCTORE
HIERONYMO SACCHERIO
SOCIETATIS JESU
In Ticinenſi Univerſitate Matheſeos Profeſſore.
OPUSCULUM
EX.^MO SENATUI
MEDIOLANENSI
Ab Auctore Dicatum.
MEDIOLANI, MDCCXXXIII.
Ex Typographia Pauli Antonii Montani. Superiorum permiſſu.

Proof (?) that angle sum of triangle cannot be > π



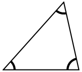
⟹ “shortcuts” exist that violate triangle inequality ⟹ contradiction

❓ Prove algebraically in terms of



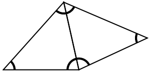
❓ Why not valid in spherical geometry? (Is valid in hyperbolic.)

Proof (?) that angle sum of triangle cannot be $< \pi$



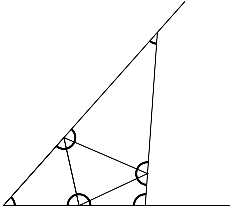
$= \pi - \varepsilon$

$\Longrightarrow$



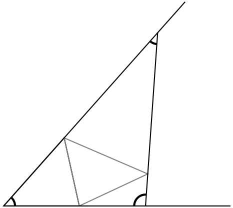
$= 2\pi - 2\varepsilon$

$\Longrightarrow$



$\leq 4\pi - 2\varepsilon$

$\Longrightarrow$



$\leq 4\pi - 2\varepsilon - 3\pi = \pi - 2\varepsilon$

$\exists\triangle$ angular defect ε

$\Longrightarrow \exists\triangle$ angular defect 2ε

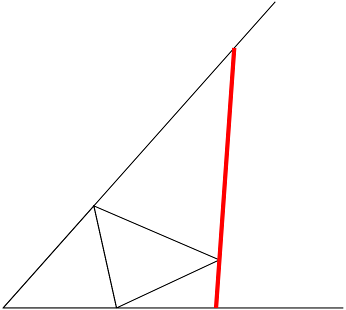
$\Longrightarrow \exists\triangle$ angular defect $2^n\varepsilon$

$\Longrightarrow \exists\triangle$ angular defect $> \pi$

$\Longrightarrow \exists\triangle$ with negative angle sum

$\Longrightarrow$ contradiction

Existence assumption not valid in hyperbolic geometry



Such a line (sometimes) doesn't exist in hyperbolic geometry!

The erroneous proof is due to prominent mathematician Legendre, whose name is engraved in gold on the Eifel Tower.

"Intuition" was embarrassingly wrong about hyperbolic geometry, hence:

"bad" mathematics		"good" mathematics
intuitive		formal
visual		arithmetical

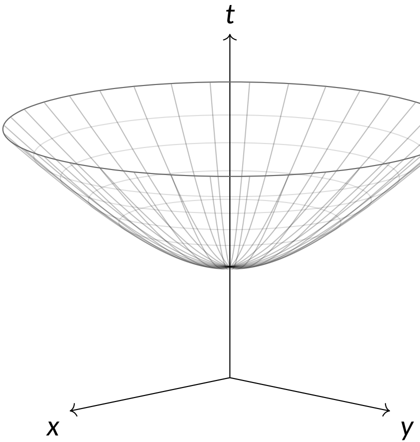
The metric space H^2

$$
\begin{aligned}
H^2 &:= \left\{(t,x,y) \in \mathbb{R}^3_{t>0} : t^2 - x^2 - y^2 = 1\right\} \\
&= \left\{\mathbf{v} \in \mathbb{R}^3_{t>0} : -\mathbf{v} \cdot_L \mathbf{v} = 1\right\} \\
&= \text{hyperboloid}
\end{aligned}
$$

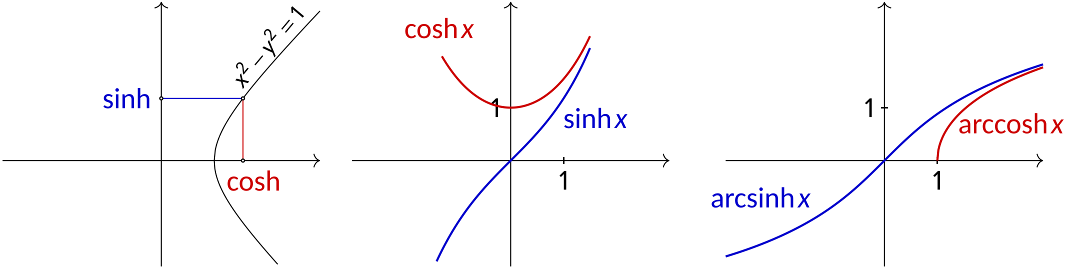
$$d(\mathbf{a},\mathbf{b}) := \operatorname{arccosh}\left(-\mathbf{a} \cdot_L \mathbf{b}\right)$$

Lorentz inner product $\cdot_L$:

$$(t_1,x_1,y_1) \cdot_L (t_2,x_2,y_2) := -t_1t_2 + x_1x_2 + y_1y_2$$



Hyperbolic functions



quasi-Pythagorean identity:	$\cosh^2 x - \sinh^2 x = 1$
sinh odd:	$\sinh(-x) = -\sinh x$
cosh even:	$\cosh(-x) = \cosh x$
addition formulas:	$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
	$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
	$\sinh 2x \ = 2 \sinh x \cosh x$
	$\cosh 2x \ = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$
	$\sinh^2 \frac{x}{2} = \frac{\cosh x - 1}{2}$
	$\cosh^2 \frac{x}{2} = \frac{\cosh x + 1}{2}$
exponential form:	$\sinh(x) = (e^x - e^{-x})/2$
	$\cosh(x) = (e^x + e^{-x})/2$

Ordinary trig. → hyperbolic trig.

$$\sin \mapsto \sinh$$

$$\cos \mapsto \cosh$$

$$\sin^2 \mapsto -\sinh^2$$

$\sin^2\theta + \cos^2\theta = 1$

$\sin(-\theta) = -\sin(\theta)$

$\cos(-\theta) = \cos(\theta)$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\sin^2 \frac{x}{2} = \frac{1-\cos x}{2}$

$\cos^2 \frac{x}{2} = \frac{1+\cos x}{2}$

$\cosh^2 x - \sinh^2 x = 1$

$\sinh(-x) = -\sinh x$

$\cosh(-x) = \cosh x$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

$\sinh 2x = 2 \sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$

$-\sinh^2 \frac{x}{2} = \frac{1-\cosh x}{2}$

$\cosh^2 \frac{x}{2} = \frac{1+\cosh x+1}{2}$

Distance in H^2

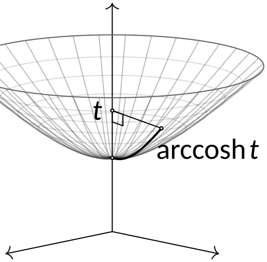
Compared to spherical geometry:

in S^2: $$d(\mathbf{a},\mathbf{b}) := \arccos(\mathbf{a}\cdot\mathbf{b})$$
$$= \mathbb{R}^3\text{-distance along surface}$$

in H^2: $$d(\mathbf{a},\mathbf{b}) := \operatorname{arccosh}(-\mathbf{a}\cdot_L \mathbf{b})$$
$$\neq \mathbb{R}^3\text{-distance along surface}$$

H^2-distance is not easily visualised. Simplest case:

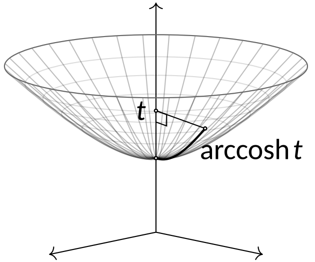
$$d((1,0,0),(t,x,y)) = \operatorname{arccosh} t$$

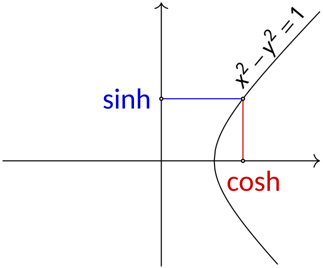


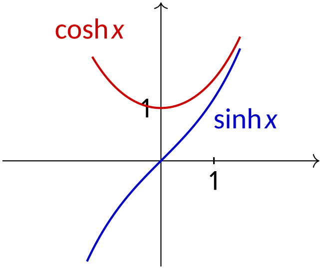
❓ Distance in H^2

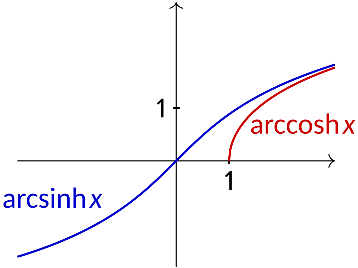
Color H^2 with horizontal bands of equal H^2-thickness. The Euclidean vertical distance Δt for each band …

- ❓ is constant
- ❓ increases as $t \to \infty$
- ❓ decreases as $t \to \infty$



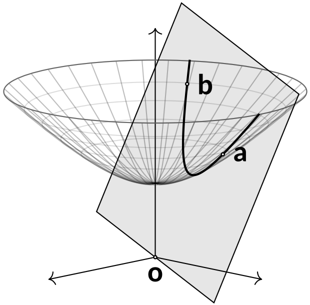






Definition of lines and angles in H^2

$$\begin{aligned} \text{line} &:= \text{great hyperbola} \\ &= H^2 \text{ intersected by plane through } \mathbf{o} \\ &\supset \text{path of shortest distance between two points} \end{aligned}$$

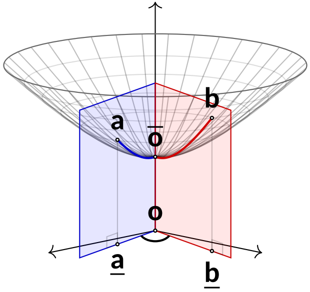


Two points determine a unique line:

$$\text{line}(\mathbf{a}, \mathbf{b}) := \text{ great hyperbola of plane}(\mathbf{o}, \mathbf{a}, \mathbf{b})$$

Angles at apex point $\bar{\mathbf{o}} = (1, 0, 0)$:

$$\begin{aligned} \angle \mathbf{a}\bar{\mathbf{o}}\mathbf{b} &:= \text{angle between corresponding planes} \\ &= \angle \underline{\mathbf{a}}\mathbf{o}\underline{\mathbf{b}} \end{aligned}$$



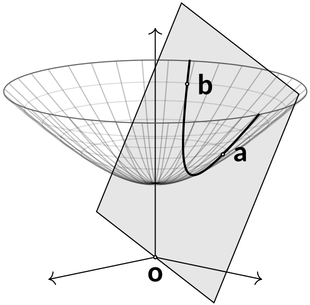
Angles at other points reduce to this case by translation: $\angle \mathbf{apb} := \angle T(\mathbf{a})T(\mathbf{p})T(\mathbf{b}) = \angle T(\mathbf{a})\bar{\mathbf{o}}T(\mathbf{b})$, where T is an H^2-isometry that maps $\mathbf{p}$ to $\bar{\mathbf{o}}$.

❓ Two points determine a line

line := great hyperbola

$= H^2$ intersected by plane through **o**

$\supset$ path of shortest distance between two points



Two points determine a unique line:

$$\text{line}(\mathbf{a},\mathbf{b}) := \text{ great hyperbola of plane}(\mathbf{o},\mathbf{a},\mathbf{b})$$

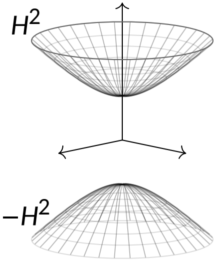
❓ Prove that line$(\mathbf{a},\mathbf{b})$ is uniquely determined for any two different points $\mathbf{a},\mathbf{b} \in H^2$.

Hint: recall

$$H^2 = \left\{(t,x,y) \in \mathbb{R}^3_{t>0} : t^2 - x^2 - y^2 = 1\right\}$$

and consider

$$H^2 \cup -H^2 = H^2_2 = \left\{(t,x,y) \in \mathbb{R}^3 : t^2 - x^2 - y^2 = 1\right\}$$

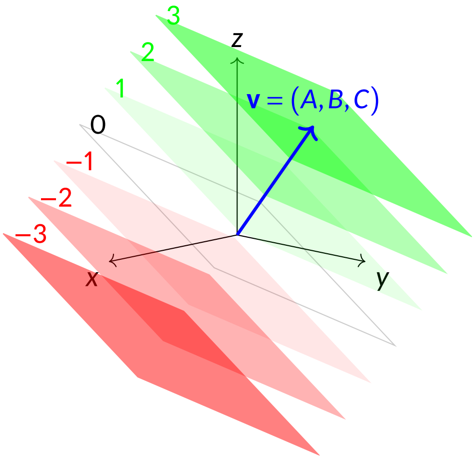


Geometrical interpretation of ordinary inner product

$$\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \mathbf{v} = k\}$$

$$= \{(x, y, z) \in \mathbb{R}^3 : Ax + By + Cz = k\}$$

= plane with $\mathbf{v}$ as normal vector



$\mathbf{x} \cdot \mathbf{v} \sim$ how much $\mathbf{x}$ “agrees with” (points in the same direction as) $\mathbf{v}$

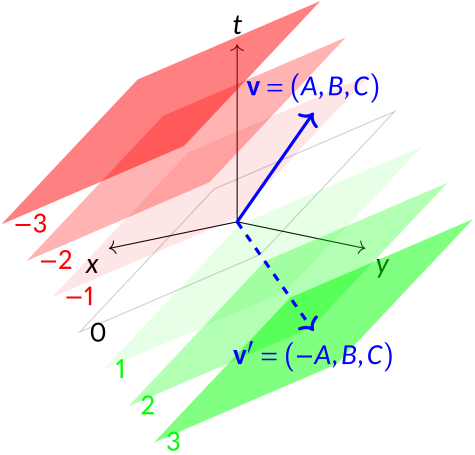
Geometrical interpretation of Lorentz inner product

Lorentz inner product $\cdot_L$:

$$(t_1, x_1, y_1) \cdot_L (t_2, x_2, y_2) = -t_1 t_2 + x_1 x_2 + y_1 y_2$$

$$\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot_L \mathbf{v} = k\}$$
$$= \{(x, y, z) \in \mathbb{R}^3 : -At + Bx + Cy = k\}$$
$$= \text{plane with } \mathbf{v}' \text{ as normal vector}$$

$\mathbf{x} \cdot_L \mathbf{v} \sim$ how much $\mathbf{x}$ "agrees with" (points in the same direction as) $\mathbf{v}'$ (the mirror image of $\mathbf{v}$ in the xy-plane)

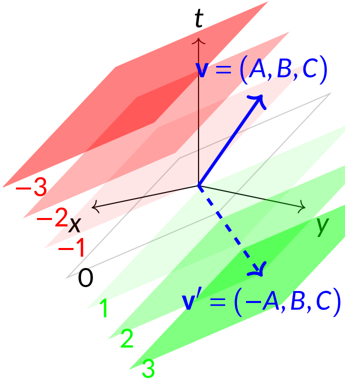


❓ Interpret geometrically

Lorentz inner product $\cdot_L$:

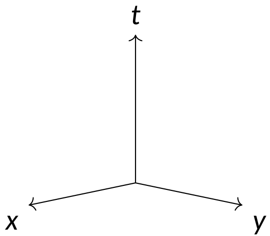
$$(t_1, x_1, y_1) \cdot_L (t_2, x_2, y_2) = -t_1 t_2 + x_1 x_2 + y_1 y_2$$

$\mathbf{x} \cdot_L \mathbf{v} \sim$ how much $\mathbf{x}$ "agrees with" (points in the same direction as) $\mathbf{v}'$ (the mirror image of $\mathbf{v}$ in the xy-plane)

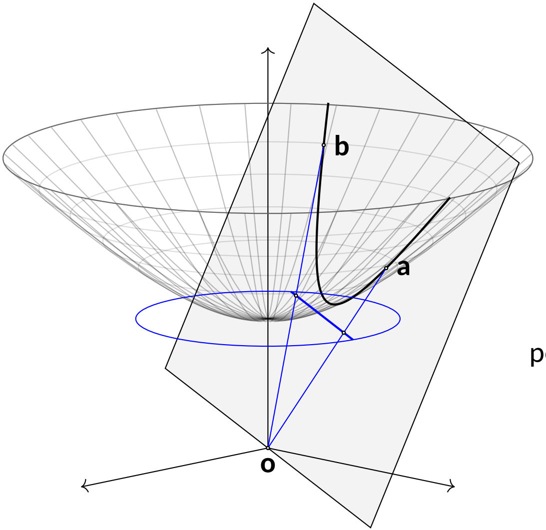


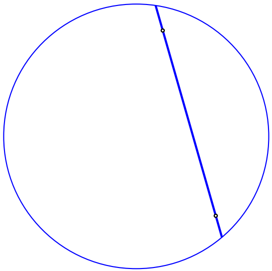
Describe geometrically the set of all points $\mathbf{x} \in \mathbb{R}^3$ such that:

- ❓ $\mathbf{x} \cdot_L \mathbf{x} = 0$
- ❓ $\mathbf{x} \cdot_L \mathbf{x} = 1$
- ❓ $\mathbf{x} \cdot_L \mathbf{x} > 0$
- ❓ $\mathbf{x} \cdot_L \mathbf{x} = -1$
- ❓ $\mathbf{x} \cdot_L \mathbf{x} < 0$



Projective Klein disc view of H^2





point on hyperboloid ↔ point in unit disc

great hyperbola ↔ line in unit disc

intersection of great hyperbolas

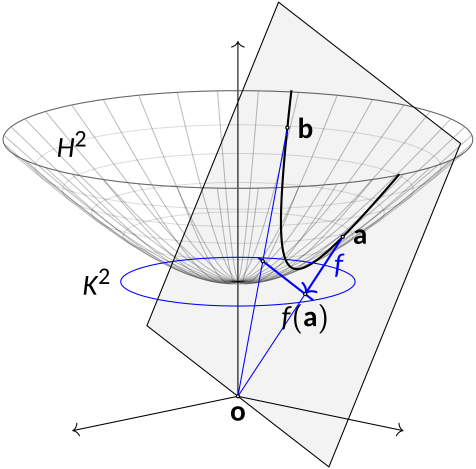
↔ intersection of lines in unit disc

Klein disc projection is a bijection $f : H^2 \to K^2$

$$f((t,x,y)) := \frac{1}{t}(t,x,y) = \left(1, \frac{x}{t}, \frac{y}{t}\right)$$

$$H^2 = \left\{(t,x,y) : t^2 = 1 + x^2 + y^2, t > 0\right\}$$

$$K^2 = \text{ Klein disc } = \left\{(1,X,Y) : X^2 + Y^2 < 1\right\}$$



- $f(H^2) \subseteq K^2$

$$(t,x,y) \in H^2$$

$$\implies f((t,x,y)) = \left(1, \frac{x}{t}, \frac{y}{t}\right)$$

$$= \Big(1, \underbrace{\frac{x}{\sqrt{1+x^2+y^2}}}_{X}, \underbrace{\frac{y}{\sqrt{1+x^2+y^2}}}_{Y}\Big)$$

$$\implies X^2 + Y^2 = \frac{x^2+y^2}{1+x^2+y^2} < 1$$

$$\implies (1,X,Y) \in K^2$$

- f surjective

$$\underbrace{f\left(\frac{1}{\sqrt{1-X^2-Y^2}}(1,X,Y)\right)}_{\in H^2} = (1,X,Y)$$

Klein disc projection is a bijection $f : H^2 \to K^2$

$$f((t,x,y)) := \frac{1}{t}(t,x,y) = \left(1, \frac{x}{t}, \frac{y}{t}\right)$$

$$H^2 = \left\{(t,x,y) : t^2 = 1 + x^2 + y^2, t > 0\right\}$$

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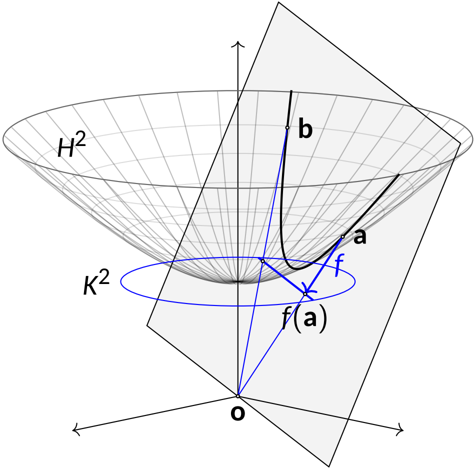


$$H_2^2 = \left\{(t,x,y) : t^2 = 1 + x^2 + y^2\right\}$$

- f injective
  - By symmetry of H_2^2:

    $$\mathbf{x} \in H^2 \implies -\mathbf{x} \in H_2^2 \implies \left|\text{line}(\mathbf{0},\mathbf{x}) \cap H_2^2\right| \geq 2$$

  - By Fundamental Theorem of Algebra:

    $$\left|\underbrace{\text{any line}}_{\text{dim } 1} \cap \underbrace{H_2^2}_{\text{degree } 2}\right| \leq 2$$

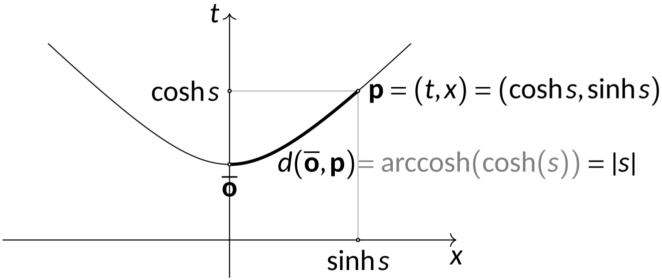
  - Hence line$(\mathbf{0},\mathbf{x})$ does not intersect H^2 in any other points.
  - Since f is a scaling, $f(\mathbf{x}) \in \text{line}(\mathbf{0},\mathbf{x})$.
  - So for any $\mathbf{p},\mathbf{q} \in H^2$:

    $$f(\mathbf{p}) = f(\mathbf{q}) \implies \mathbf{q} \in \text{line}(\mathbf{0},\mathbf{p})$$

    $$\implies \mathbf{q} \in \text{line}(\mathbf{0},\mathbf{p}) \cap H^2 \implies \mathbf{p} = \mathbf{q} \quad \square$$

The metric space $H^1 \subset H^2$

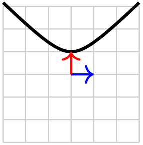
H^2 restricted to the tx-plane:



Isometries of H^1:

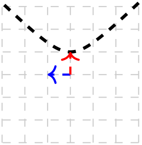
identity

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



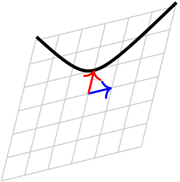
reflection

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



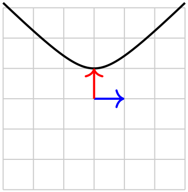
translation

$\begin{pmatrix} \cosh(s) & \sinh(s) \\ \sinh(s) & \cosh(s) \end{pmatrix}$



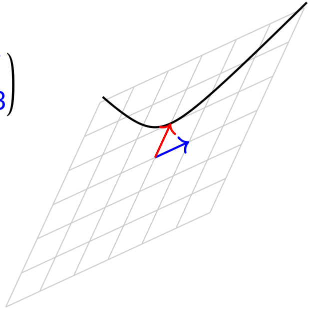
Visualisation of H^1 translation $\begin{pmatrix} \cosh(s) & \sinh(s) \\ \sinh(s) & \cosh(s) \end{pmatrix}$

$s = 0.5$



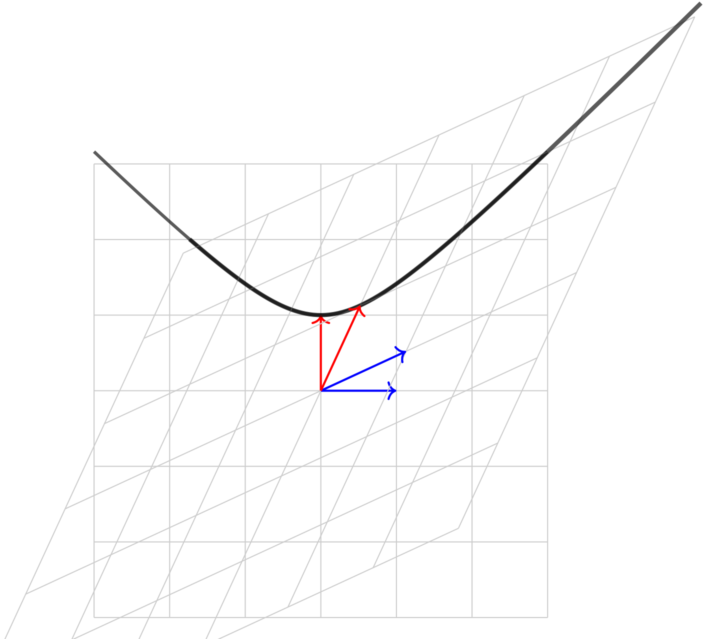
$$\begin{pmatrix} 0.52103 & 1.12767 \\ 1.12767 & 0.52103 \end{pmatrix}$$

$$\longrightarrow$$



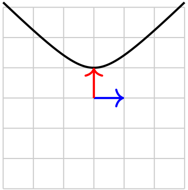
Visualisation of H^1 translation $\begin{pmatrix} \cosh(s) & \sinh(s) \\ \sinh(s) & \cosh(s) \end{pmatrix}$

$s = 0.5$



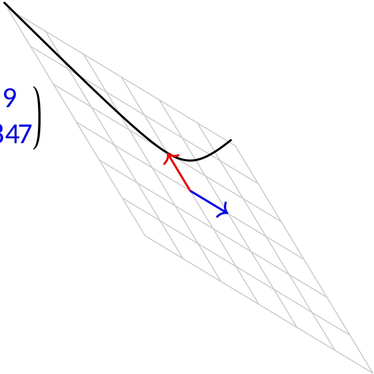
Visualisation of H^1 translation $\begin{pmatrix} \cosh(s) & \sinh(s) \\ \sinh(s) & \cosh(s) \end{pmatrix}$

$s = -0.7$

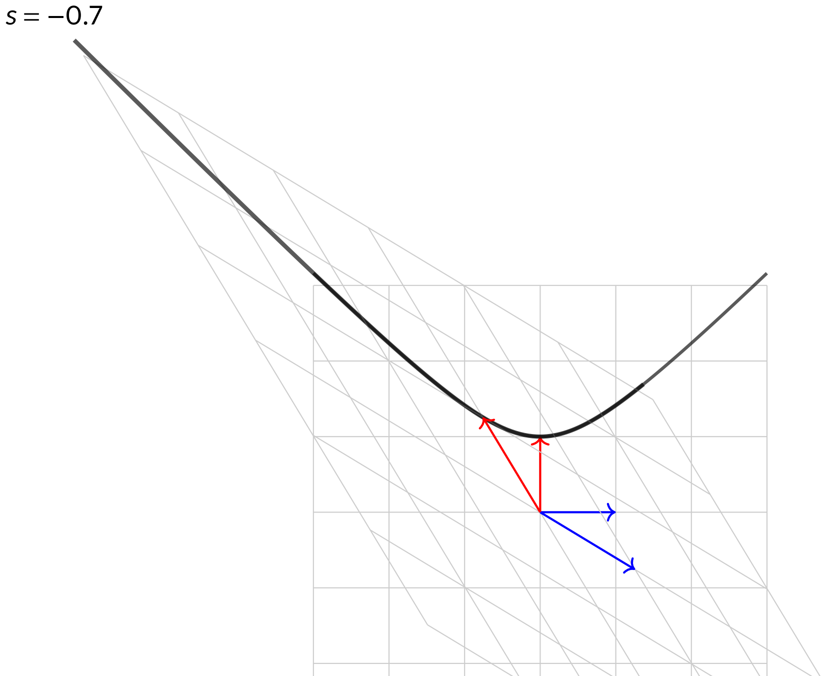


$$\begin{pmatrix} -0.75847 & 1.25519 \\ 1.25519 & -0.75847 \end{pmatrix}$$

$$\longrightarrow$$



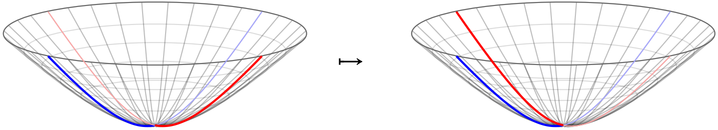
Visualisation of H^1 translation $\begin{pmatrix} \cosh(s) & \sinh(s) \\ \sinh(s) & \cosh(s) \end{pmatrix}$



Some isometries of H^2

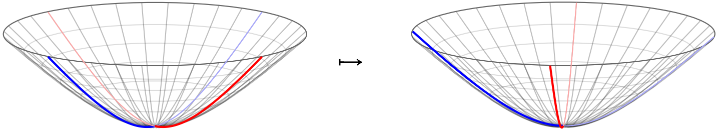
reflection:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$



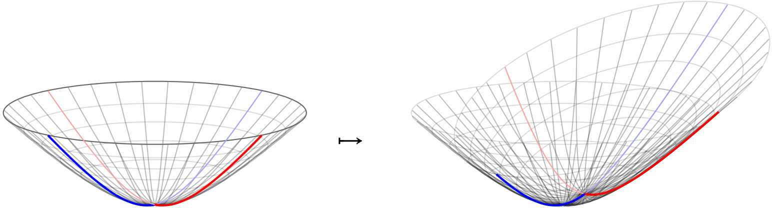
rotation:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$



translation:

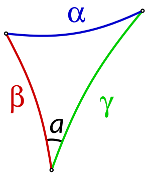
$$\begin{pmatrix} \cosh s & \sinh s & 0 \\ \sinh s & \cosh s & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Trigonometry in H^2

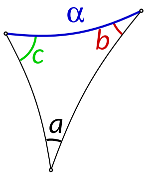
Cosine rule:

$$\cosh\alpha = \cosh\beta\cosh\gamma - \sinh\beta\sinh\gamma\cos a$$

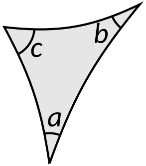


Alternate cosine rule:

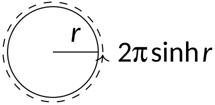
$$\cos a = \cosh\alpha\sin b\sin c - \cos b\cos c$$



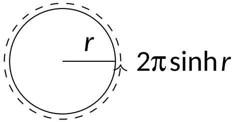
$$\text{area of triangle} = \underbrace{\pi - (\text{angle sum})}_{\text{“angular defect”}} = \pi - (a+b+c)$$



$$\text{circumference of circle} = 2\pi\sinh r$$



❓ Do circumferences grow faster or slower in hyperbolic geometry compared to Euclidean geometry?

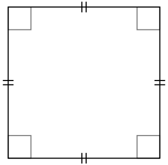


Hint: Recall: $\sinh(x) = (e^x - e^{-x})/2$.

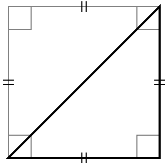
❓ Is this good or bad if you are a criminal trying to escape the cops?

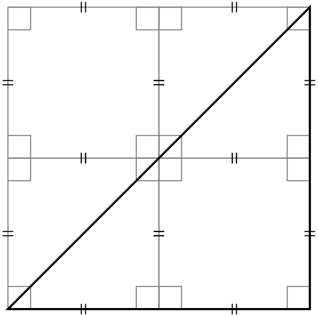
❓ How does this relate to the behaviour of parallel lines?

❓ Do squares exist in hyperbolic geometry?

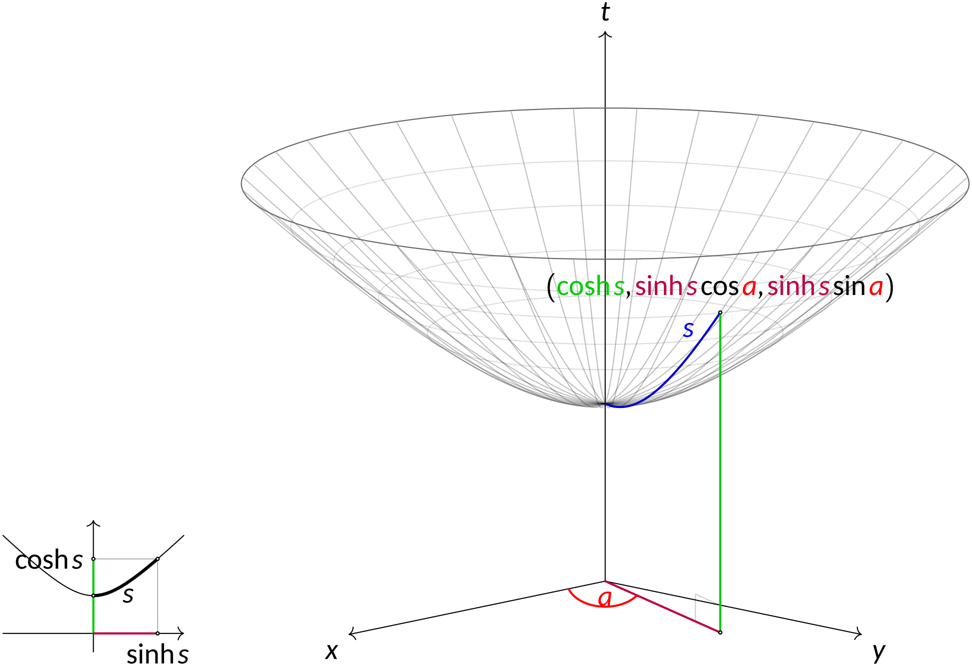


Hints:



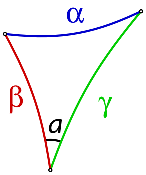


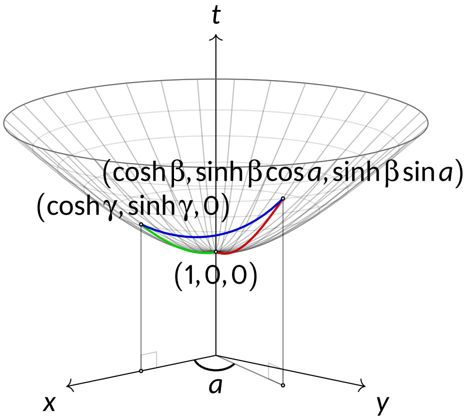
H^2 "polar coordinates"



Proof of cosine rule

$$\cosh \alpha = \cosh \beta \cosh \gamma - \sinh \beta \sinh \gamma \cos a$$





Put △ in simplified position using isometries, and then calculate:

$$\alpha = d\left(\begin{pmatrix}\cosh\gamma\\ \sinh\gamma\\ 0\end{pmatrix}, \begin{pmatrix}\cosh\beta\\ \sinh\beta\cos a\\ \sinh\beta\sin a\end{pmatrix}\right)$$

$$= \operatorname{arccosh}\left(-\begin{pmatrix}\cosh\gamma\\ \sinh\gamma\\ 0\end{pmatrix} \cdot_L \begin{pmatrix}\cosh\beta\\ \sinh\beta\cos a\\ \sinh\beta\sin a\end{pmatrix}\right)$$

$$\cosh\alpha = -\begin{pmatrix}\cosh\gamma\\ \sinh\gamma\\ 0\end{pmatrix} \cdot_L \begin{pmatrix}\cosh\beta\\ \sinh\beta\cos a\\ \sinh\beta\sin a\end{pmatrix}$$

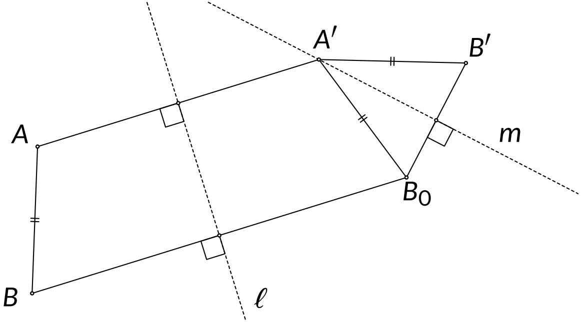
$$= \cosh\gamma\cosh\beta - \sinh\gamma\sinh\beta\cos a \qquad \square$$

Still valid in hyperbolic geometry?

Given $A, B, A', B' \in \mathbb{E}^2$, $d(A,B) = d(A',B')$ construct isometry that sends $A \mapsto A'$ and $B \mapsto B'$. Let

$$\ell := \text{perpendicular bisector of } AA'$$
$$B_0 := \mathrm{Rfl}_\ell(B)$$
$$m := \text{perpendicular bisector of } B_0B' \overset{?}{=} \{\text{points equidistant to } B_0,\, B'\} \ni A'$$

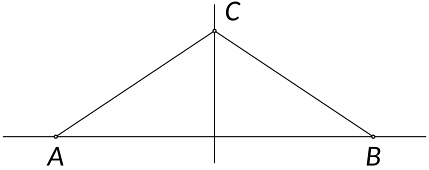
Possible choices for T are $\mathrm{Rfl}_m \circ \mathrm{Rfl}_\ell$ and $\mathrm{Rfl}_{A'B'} \circ \mathrm{Rfl}_m \circ \mathrm{Rfl}_\ell$.



❓ Still valid in hyperbolic geometry?

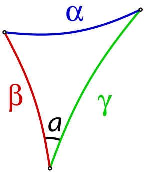
$C \in$ perpendicular bisector of AB

$$\overset{?}{\iff} d(A,C) = d(B,C)$$

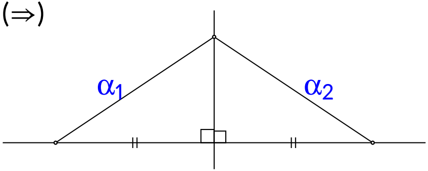


Cosine rule:

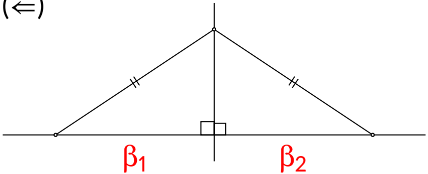
$$\cosh \alpha = \cosh \beta \cosh \gamma - \sinh \beta \sinh \gamma \cos a$$



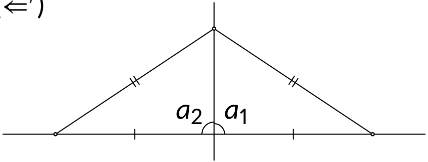
(⇒)



(⇐)

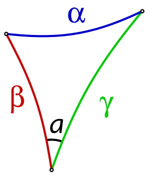


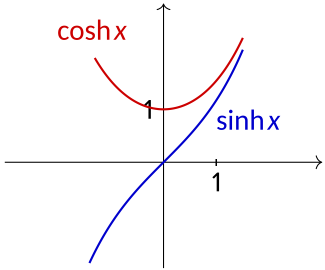
(⇐′)



Triangle inequality in H^2

$$\cosh\alpha = \cosh\beta\cosh\gamma - \sinh\beta\sinh\gamma\cos a$$





$$\cosh\alpha = \cosh\beta\cosh\gamma - \sinh\beta\sinh\gamma\cos a$$

cosine rule

$$\leq \cosh\beta\cosh\gamma - \sinh\beta\sinh\gamma(-1)$$

replace $\cos a$ with $\min(\cos a)$

$\beta, \gamma > 0 \implies \sinh\beta\sinh\gamma > 0$

$$= \cosh(\beta+\gamma)$$

addition formula for cosh

From the graph of cosh we see that

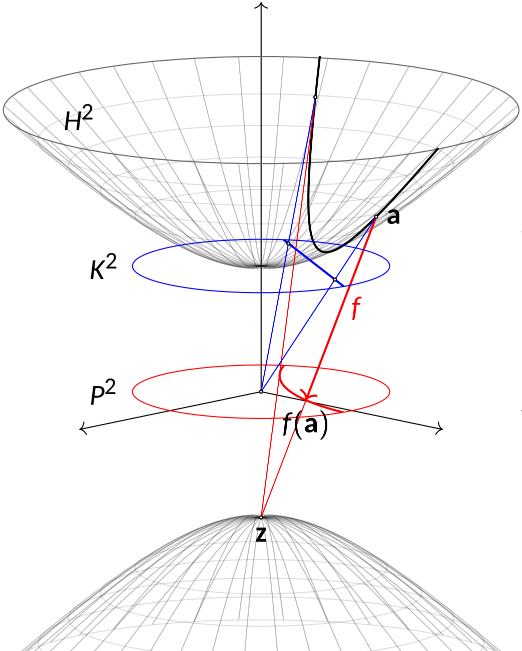
$$\cosh \underbrace{\alpha}_{>0} \leq \cosh(\underbrace{\beta+\gamma}_{>0}) \iff \alpha \leq \beta+\gamma \quad \square$$

Equality occurs when

$$\min(\cos a) = \cos a \iff a = \pi \iff$$

the "triangle" is a line

Projections between models



$$\text{line}(\mathbf{z},\mathbf{a}) = L(\lambda) = \mathbf{z}+\lambda(\mathbf{a}-\mathbf{z}) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} t+1 \\ x \\ y \end{pmatrix}$$

L intersects plane $t = 0$ when:

$$-1+\lambda(t+1) = 0 \implies \lambda = \frac{1}{t+1}$$

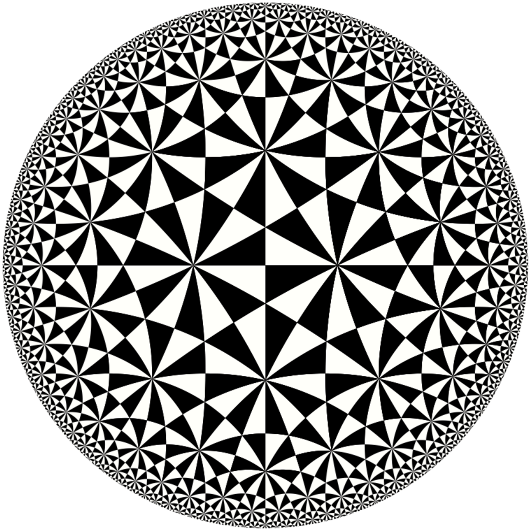
So

$$f((t,x,y)) = L\left(\frac{1}{t+1}\right) = \left(0, \frac{x}{t+1}, \frac{y}{t+1}\right)$$

The conformal Poincaré model of hyperbolic geometry P^2

points $= \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\}$

$=$ interior of unit disc



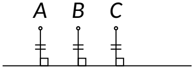
lines $=$ circular arcs perpendicular to edge

including lines through origin (circles with "$r = \infty$")

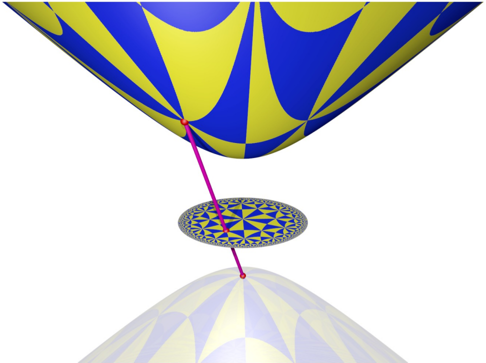
angles $=$ Euclidean angles ("conformal")

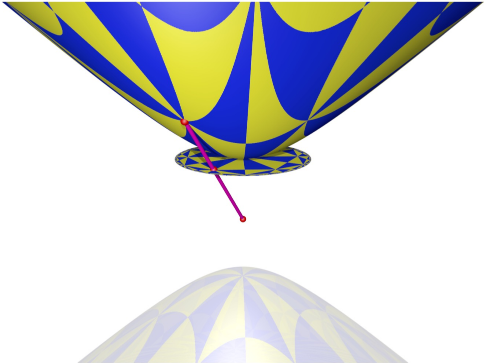
$$d(\mathbf{o},\mathbf{a}) = \log \frac{1+r}{1-r}$$ where $r = d_{\mathbb{E}^2}(\mathbf{o},\mathbf{a})$

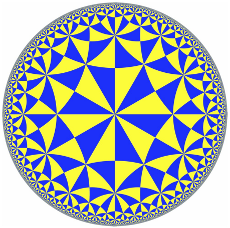
❓ Are A, B, C (three points on one side of a given line, equidistant from that line) collinear in hyperbolic geometry?

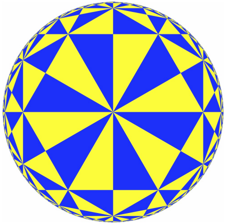


Equivalence of H^2 and P^2 (source: http://mediatum.ub.tum.de/doc/1210572/document.pdf)





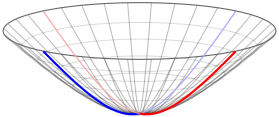


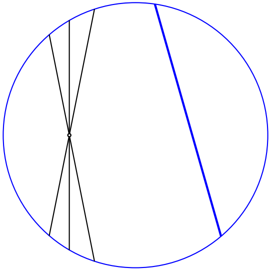


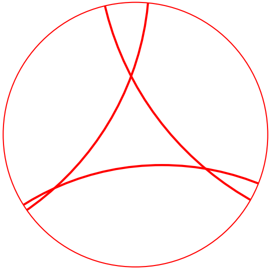
Interactive visualisation of these projections: https://www.geogebra.org/m/T8S9ctS7

Comparative overview of models

	hyperboloid	Klein disc	Poincaré disc
points	$\{(t,x,y) \in \mathbb{R}^3_{t>0} : t^2 - x^2 - y^2 = 1\}$	$\{(1,X,Y) \in \mathbb{R}^3 : X^2 + Y^2 < 1\}$	$\{(0,X,Y) \in \mathbb{R}^3 : X^2 + Y^2 < 1\}$
lines	$\cap$ planes through $\mathbf{0}$ = great hyperbolas	$\cap$ planes through $\mathbf{0}$ = line segments = chords of the disc	projections of lines in H^2 = circular arcs $\perp$ boundary
angles	Euclidean at $(1,0,0)$	complicated	Euclidean
best for	lines, d, angles $\sim S^2$ isometries $\sim \mathbb{E}^n$	lines, intersections	angles





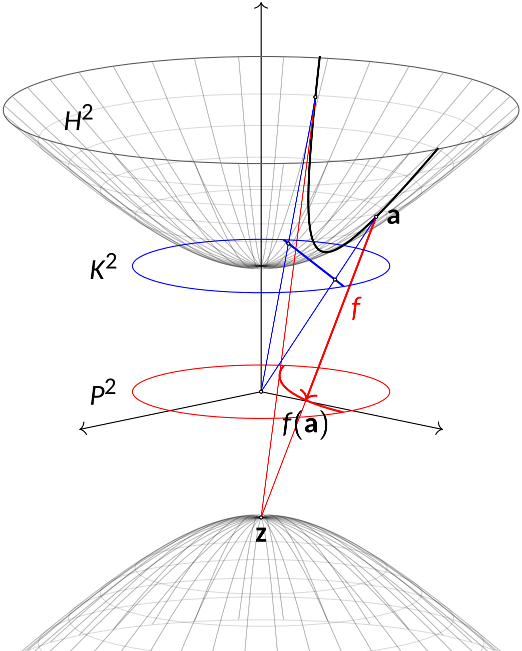


❓ In which model(s) are hyperbolic circles represented by Euclidean circles?

The name “hyperbolic” does not refer to the hyperboloid

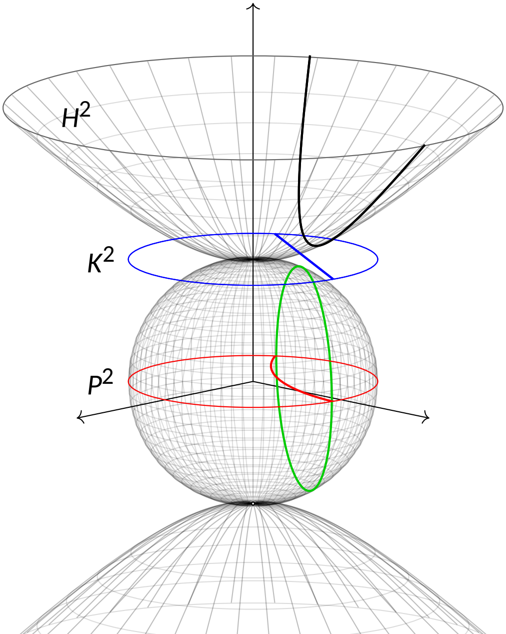
	spherical ↕ elliptic	Euclidean ↕ parabolic	hyperbolic ↕ hyperbolic
etymology	“too little”	“just right”	“too much”
	ellipsis = …	parable = parallel case	hyperbole = exaggeration
conic sections	$x^2 = py - ky^2$	$x^2 = py$	$x^2 = py + ky^2$
$\pi -$ △ angle sum	$-$	0	$+$
parallels to ℓ through P	0	1	∞

Projections between models



How do we know that ∅ = circle perpendicular to boundary?

Equivalence $K^2 \leftrightarrow P^2$ via hemisphere intermediate



- projection from origin: ◡ ↦ \
- \ = plane ∩ plane = line
- vertical projection: \ ↦ 0
- 0 = sphere ∩ plane = circle
- stereographic projection from south pole onto equatorial plane: 0 ↦ ⊘
- ⊘ = circle perpendicular to boundary since stereographic projection preserves circles and angles

The pseudo-sphere (constant negative curvature) is “locally hyperbolic”


SAGGIO DI INTERPETRAZIONE

DELLA

GEOMETRIA NON-EUCLIDEA

DEL PROFESSORE

EUGENIO BELTRAMI

Napoli 1868

La formola

$$ds^2 = R^2 \frac{(a^2-v^2)du^2+2uvdudv+(a^2-u^2)dv^2}{(a^2-u^2-v^2)^2} \tag{1}$$

rappresenta il quadrato dell'elemento lineare di una superficie la cui curvatura sferica è dovunque costante, negativa ed eguale a $-\frac{1}{R^2}$. La forma di quest'espres-

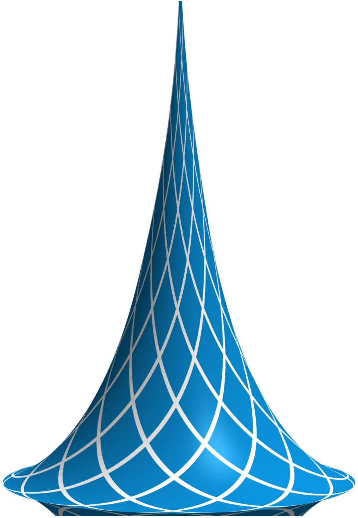


Image source: Brander & Markvorsen, Surfaces with Natural Ridges

Gaussian curvature $K = \kappa_1 \kappa_2$

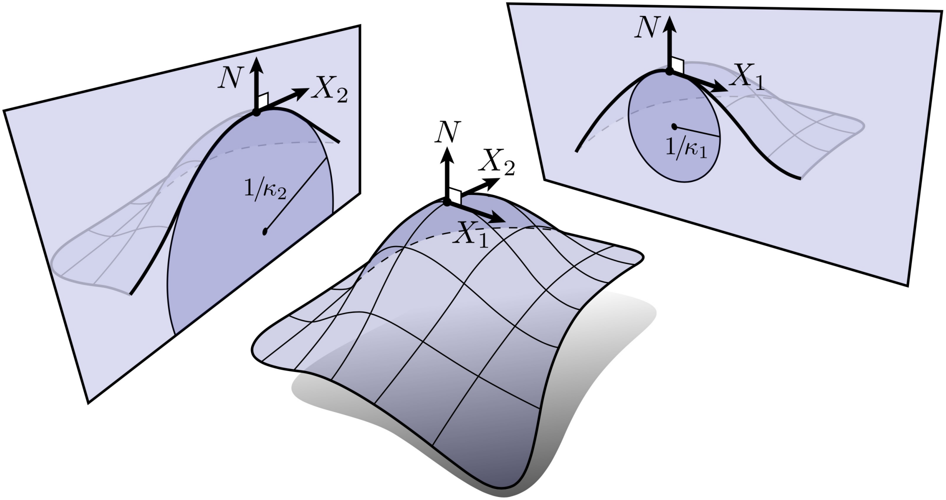


Image source: Keenan Crane, A Quick and Dirty Introduction to the Curvature of Surfaces

Points of curvature zero on Apollo Belvedere

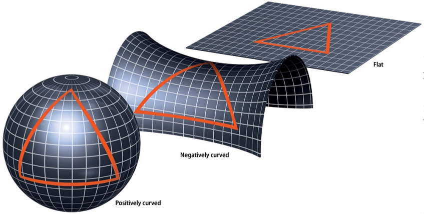
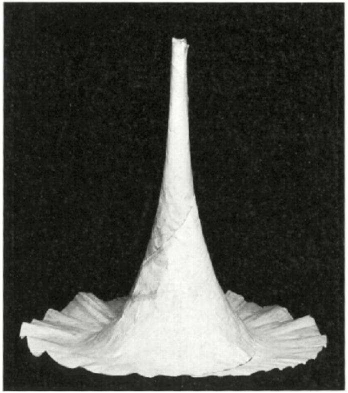
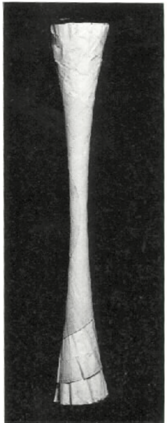


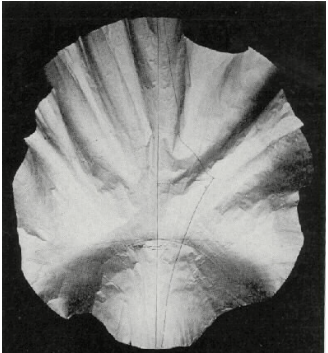
Image source: https://astronomy.com/magazine/ask-astro/2012/02/cosmic-shape-and-size

Beltrami’s paper model of a pseudosphere http://www-dimat.unipv.it/cornalba/lezioni/beltrami.pdf

e un modello in carta
di superficie di curvatura
costante negativa
costruito da Beltrami
intorno al 1869:
la “cuffia di Beltrami”

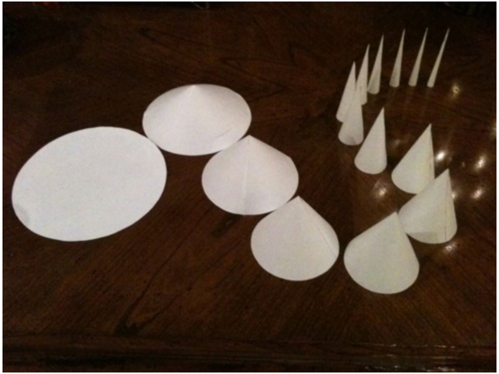


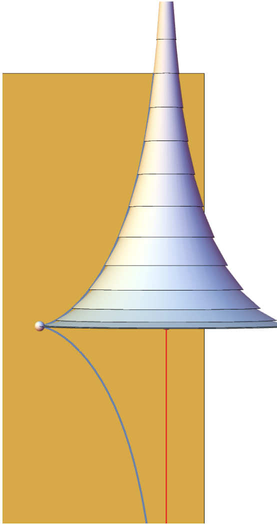




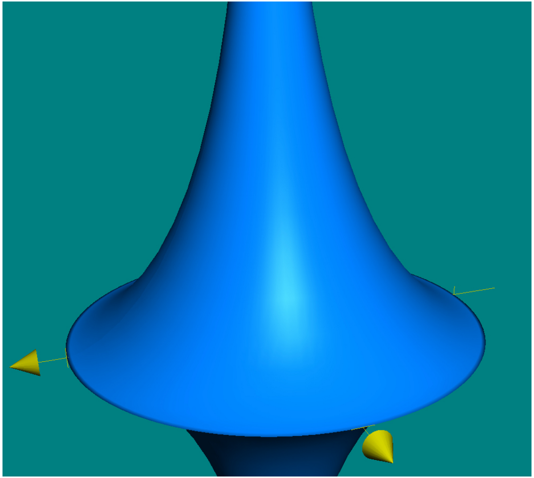
Make your own pseudosphere

Cut $n\theta$ degrees out discs for a range of *n*s; fold into cones; stack.





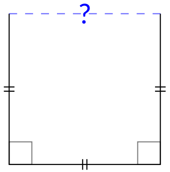
Locally, distances along the pseudosphere surface represent hyperbolic distances



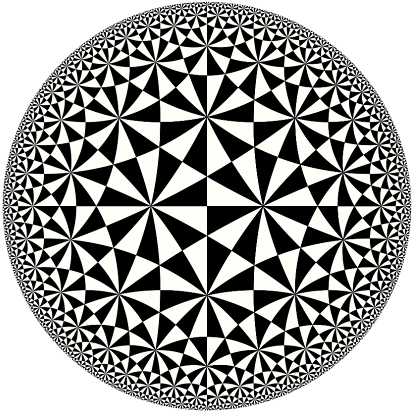
A (small-ish) circle on the pseudosphere has . . . circumference compared to a Euclidean circle of the same radius.

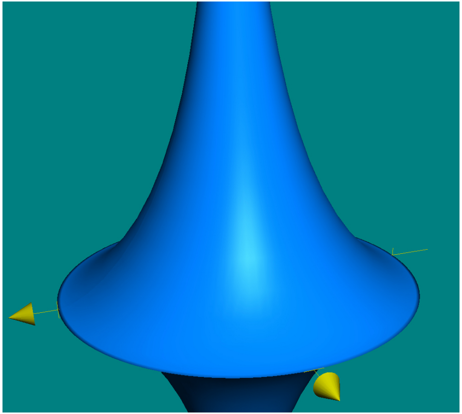
- equal
- greater
- smaller

❓ What can you say about the figure that results when connecting the two endpoints?

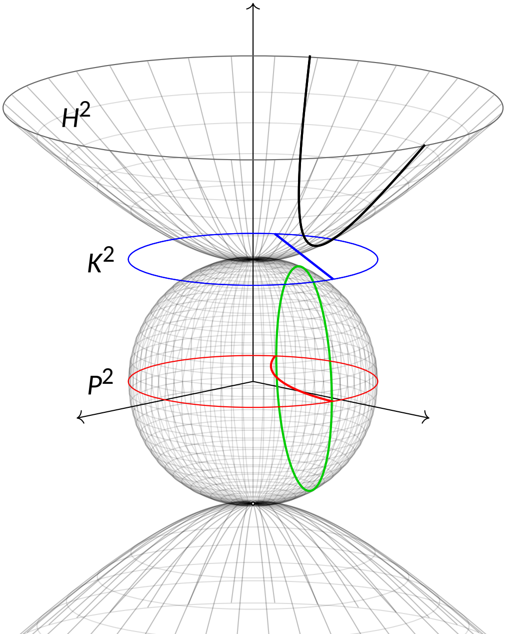


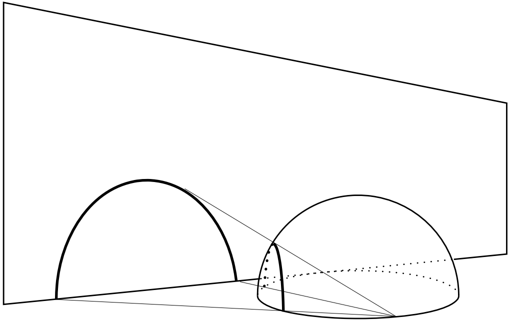
Hints:





Half-plane model

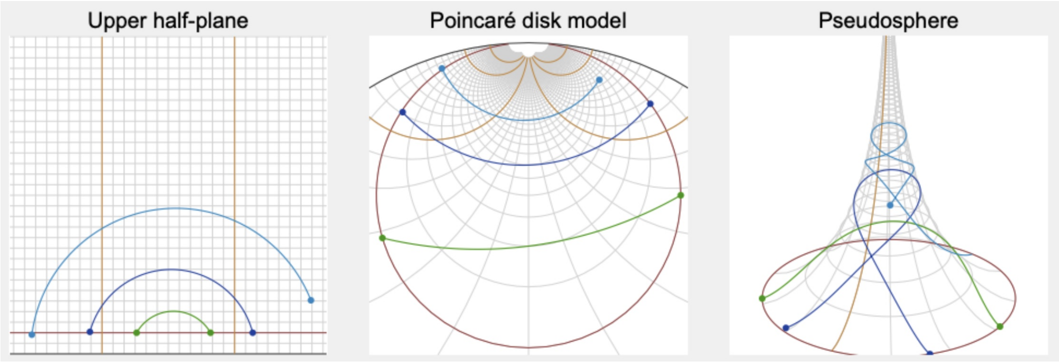




(image source: Stillwell, *Mathematics and Its History*)

lines ⟼ semi-circles perpendicular to boundary

Half-plane as unrolled pseudosphere



https://timhutton.github.io/PseudosphereGeodesics/