Comparative overview

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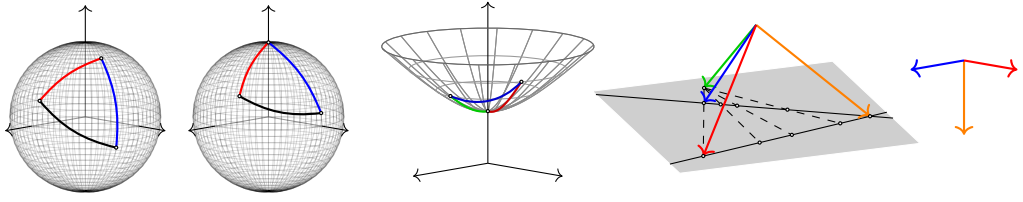
What all transformation groups have in common

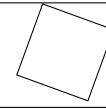
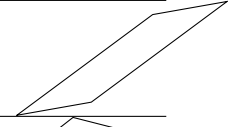
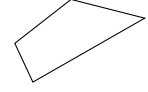
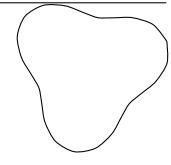
$\mathbb{E}^n$ S^2 H^2 — isometries

$\mathbb{A}^n$ $\mathbb{P}^n$ — transformations

- Equivalence relation of figures.
- Preserve truth values of theorems.
- “The things you can do without loss of generality.”
- Use to place figures conveniently (≅ choose convenient coordinate system).
- “Thereby we may transform the proposed figures, if they are intricate, into others that are more simple.” (Newton, *Principia*, I.V Lemma XXII)



Hierarchy of transformation groups

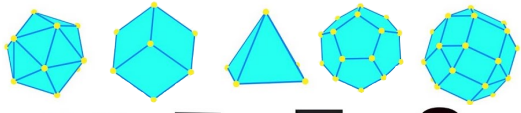
	transformation group	equivalent to
$\mathbb{E}^n$	$d_{\mathbb{E}^n}$-preserving bijections	
$\mathbb{A}^n$	invertible $n \times n$ matrices $+\mathbf{v}$	
$\mathbb{P}^n = \left(\mathbb{R}^{n+1} - \{\mathbf{0}\}\right)/\sim$	invertible $n{+}1 \times n{+}1$ matrices	
topology	continuous bijections $\mathbb{R}^n \to \mathbb{R}^n$	

Using transformations to get general results from simple ones

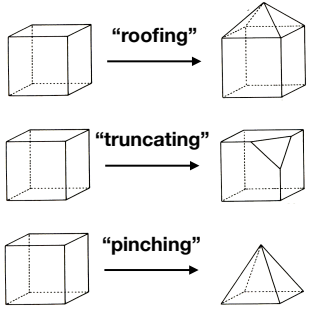
- Property P holds for a simple figure.
- Property P is invariant under a family of transformations.

$\Longrightarrow$ Property P holds for all figures obtainable by such transformations.

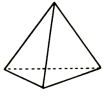
Example:



(image source: https://www.youtube.com/watch?v=80EazC2_0Qo)

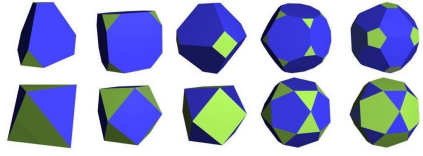


- Check for a simple case:



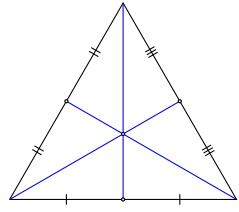
- Check that invariant under certain transformations:

$\Longrightarrow$ Ka-ching! Theorem proved for lots of shapes.



Example: an affine theorem

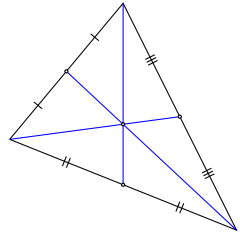
- The three lines (bisection point to opposite vertex) meet in a point:



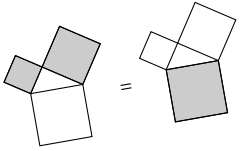
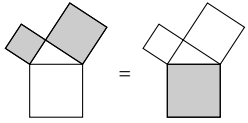
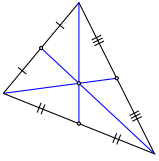
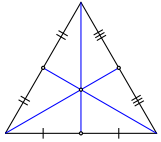
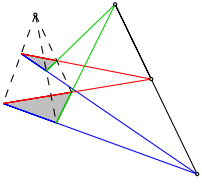
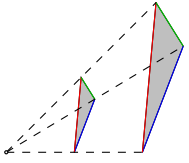
Clear in this simple case.

- ❓ Prove that theorem is invariant under affine transformations.

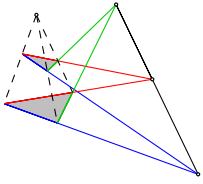
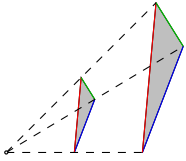
$\Longrightarrow$ ❓ Therefore …



Prove theorems in the largest setting in which they are invariant

theorem	involves	invariant in $\mathbb{E}^2$	$\mathbb{A}^2$	$\mathbb{P}^2$	topology	simplification
	lengths, angles, area	✓	✗	✗	✗	
	midpoints	✓	✓	✗	✗	
	collinearity	✓	✓	✓	✗	

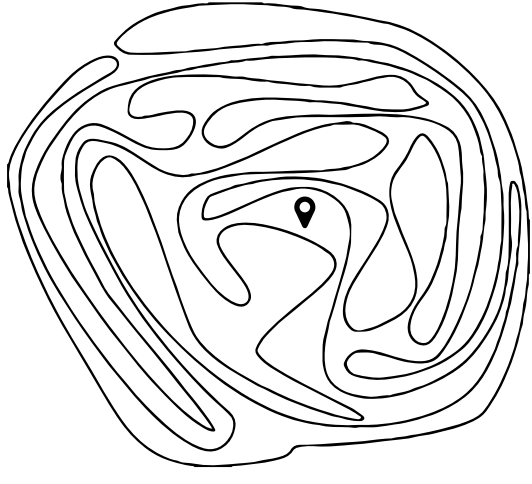
Desargues's Theorem by projective simplification

theorem	involves	invariant in $\mathbb{E}^2$	$\mathbb{A}^2$	$\mathbb{P}^2$	topology	simplification
	collinearity	✓	✓	✓	✗	

Send line($AB \cap A'B'$,$BC \cap B'C'$) to "the horizon." Then

$$
\begin{array}{ccc}
AB \parallel A'B' & \Longrightarrow & OA/AA' = OB/BB' \\
BC \parallel B'C' & \Longrightarrow & OB/BB' = OC/CC' \\
 & & \Downarrow \\
AC \parallel A'C' & \Longleftarrow & OA/AA' = OC/CC' \\
\Downarrow & & \\
AC \cap A'C' & \in & \text{horizon}
\end{array}
$$

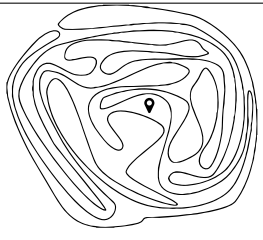
❓ Is the marker inside or outside the region enclosed by the curve?



Prove your answer by means of a property invariant under topological (“rubber-band”) transformations.

(image adapted from https://plus.maths.org/content/winding-numbers-topography-and-topology-ii)

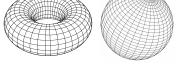
Topological simplification

theorem	involves	invariant in $\mathbb{E}^2$	$\mathbb{A}^2$	$\mathbb{P}^2$	topology	simplification
	inside/outside	✓	✓	✓	✓	

Different equivalence relations are suited to classifying different things

- ✗ The equivalence relation is “too crude”: it “trivialises” or “collapses” all objects into a single class.
- ✗✗ The equivalence relation is “too fine grained”: it counts too few things as equivalent to contribute any insight.
- ✓ The equivalence relation erases insignificant differences but respects qualitative differences; hence reveals structure.

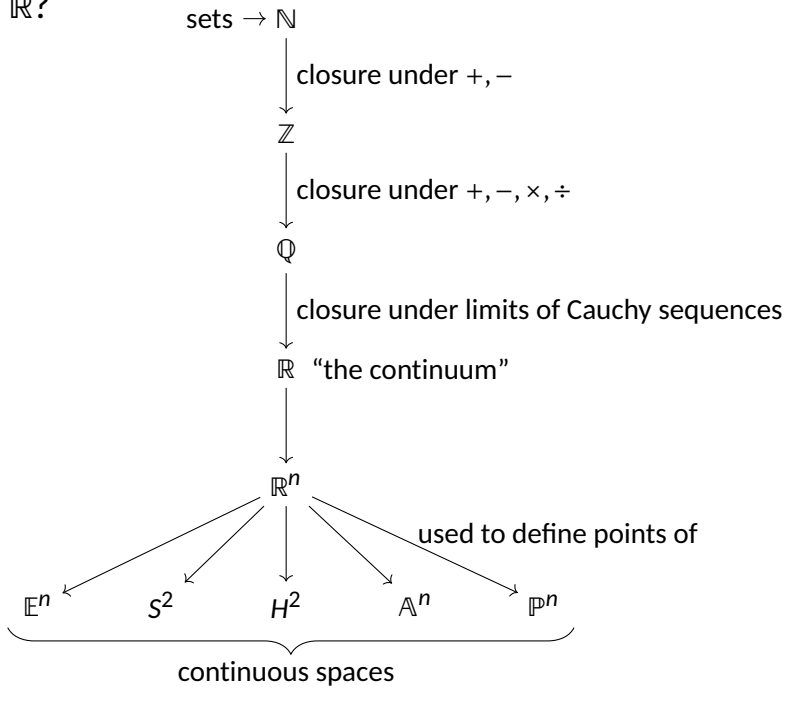
	$\mathbb{E}^n$	$\mathbb{A}^n$	$\mathbb{P}^n$	topology
quadrilaterals	✓	✗	✗	✗
conics	✗✗	✓	✗	✗
cubics	✗✗	✗✗	✓	✗
surfaces	✗✗	✗✗	✗✗	✓



"Linear algebra" in $\mathbb{R}^n$, $\mathbb{A}^n$, $\mathbb{P}^n$

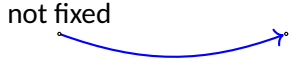
	$\mathbb{R}^n$	$\mathbb{A}^n$	$\mathbb{P}^n = (\mathbb{R}^{n+1} - \{\mathbf{0}\})/\sim$
subspace		point + $\mathbb{R}^n$ subspace	$\mathbb{R}^{n+1}$ subspace $-\{\mathbf{0}\}/\sim$
dimension		$\mathbb{R}^n$ dim	$\mathbb{R}^{n+1}$ dim -1
span	linear comb.	affine-linear comb. = point + $\mathbb{R}^n$ span	($\mathbb{R}^{n+1}$ span)$/\sim$
$\cap$ subspaces = subspace	✓	✓	✓
span = subspace	✓	✓	✓
subspaces closed under span	✓	✓	✓
dimension theorem	✓	✓	✓
independent $\iff$ max. dim	✓	✓	✓
dim(span(k independent))	k	$k-1$	$k-1$
dim n $\iff$ det $\neq 0$	✓	✓	✓
where det =	$\begin{vmatrix} \vert & & \vert \\ \mathbf{x}_1 & \cdots & \mathbf{x}_n \\ \vert & & \vert \end{vmatrix}$	$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \vert & \vert & & \vert \\ \mathbf{x}_0 & \mathbf{x}_1 & \cdots & \mathbf{x}_n \\ \vert & \vert & & \vert \end{vmatrix}$	$\begin{vmatrix} \vert & \vert & & \vert \\ \mathbf{x}_0 & \mathbf{x}_1 & \cdots & \mathbf{x}_n \\ \vert & \vert & & \vert \end{vmatrix}$
# independent to make basis with uniquely determined coord.	n	$n+1$	$n+2$

Why always $\mathbb{R}$?



You can’t fix everything

Call a subset X of a space “fixable” if there is a non-trivial (non-identity) transformation T of that space that “fixes” each point: $T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$.

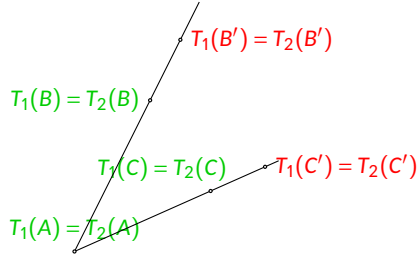


In each of the spaces we have studied ($\mathbb{E}^n$, S^2, H^2, $\mathbb{A}^n$, $\mathbb{P}^n$), what is the largest fixable set?

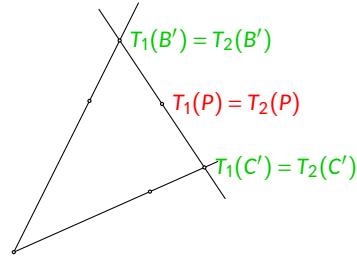
Largest fixable sets in $\mathbb{E}^n$

Call a subset X of a space "fixable" if there is a non-trivial (non-identity) transformation T of that space that "fixes" each point: $T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$.

space	transf.	largest fixable set
$\mathbb{E}^1$	isometries	❓
$\mathbb{E}^2$	isometries	❓
$\mathbb{E}^3$	isometries	❓
$\mathbb{E}^n$	isometries	❓



$$\text{isometry} = \underbrace{M}_{\text{orth.}}\mathbf{x} + \mathbf{v}$$



Two $\mathbb{E}^n$-isometries equal on two points $\Longrightarrow$ equal on entire line:

Largest fixable sets in S^2

Call a subset X of a space "fixable" if there is a non-trivial (non-identity) transformation T of that space that "fixes" each point: $T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$.

space	transf.	largest fixable set
S^2	isometries	❓

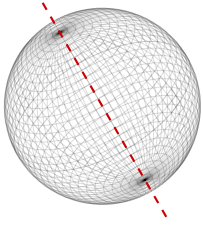
S^2 isometry $=$ **0**-preserving $\mathbb{E}^3$ isometry

$=$ orthogonal matrix

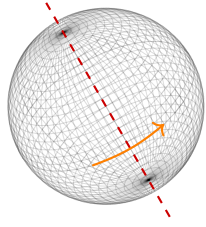
$$= \begin{pmatrix} \pm 1 & 0 \\ 0 & R_\varphi \end{pmatrix}$$

Classification theorem:

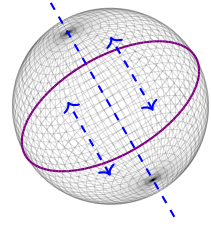
identity



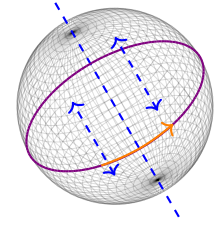
rotation



reflection



composite



Largest fixable sets in H^2

Call a subset X of a space "fixable" if there is a non-trivial (non-identity) transformation T of that space that "fixes" each point: $T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$.

space	transf.	largest fixable set
H^2	isometries	❓

Classification theorem:

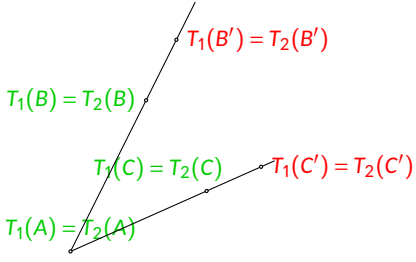
- identity
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
- reflection
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
- rotation
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$
- translation
$$\begin{pmatrix} \cosh s & \sinh s & 0 \\ \sinh s & \cosh s & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
- glide reflection
$$\begin{pmatrix} \cosh s & \sinh s & 0 \\ \sinh s & \cosh s & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
- rotation about ideal point
$$\begin{pmatrix} \frac{p^2+2}{2} & \frac{-p^2}{2} & p \\ \frac{p^2}{2} & \frac{2-p^2}{2} & p \\ p & -p & 1 \end{pmatrix}$$

Largest fixable sets in $\mathbb{A}^n$

Call a subset X of a space "fixable" if there is a non-trivial (non-identity) transformation T of that space that "fixes" each point: $T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$.

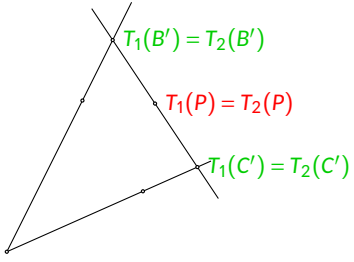
space	transf.	largest fixable set
$\mathbb{A}^2$	affine transf.	❓
$\mathbb{A}^n$	affine transf.	❓

$$\mathbb{A}^n \text{ transf.} = \underbrace{M}_{\det \neq 0} \mathbf{x} + \mathbf{v}$$



$$T_1(A_1) = T_2(A_1), \ldots, T_1(A_k) = T_2(A_k)$$

$$\Longrightarrow T_1(\mathbf{x}) = T_2(\mathbf{x}) \text{ for all } \mathbf{x} \in \langle A_1, \ldots, A_k \rangle$$



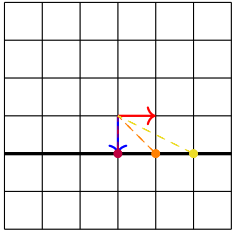
Largest fixable sets in $\mathbb{P}^1$

Call a subset X of a space "fixable" if there is a non-trivial (non-identity) transformation T of that space that "fixes" each point: $T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$.

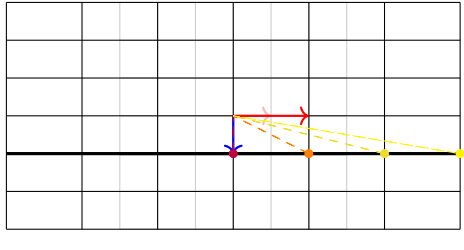
space	transf.	largest fixable set
$\mathbb{P}^1$	proj. transf.	❓

$T : \mathbb{P}^1 \to \mathbb{P}^1$ determined by 3 points.

$(ABCD) = (A'B'C'D')$



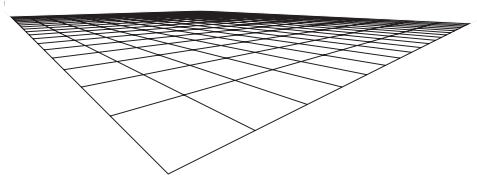
$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mapsto$$



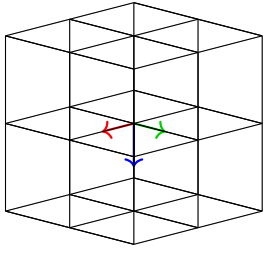
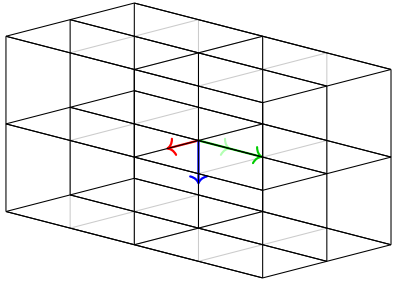
Largest fixable sets in $\mathbb{P}^2$

Call a subset X of a space "fixable" if there is a non-trivial (non-identity) transformation T of that space that "fixes" each point: $T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$.

space	transf.	largest fixable set
$\mathbb{P}^2$	proj. transf.	❓



$T : \mathbb{P}^2 \to \mathbb{P}^2$ determined by 4 points (no 3 collinear).



$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mapsto$$