Affine spaces

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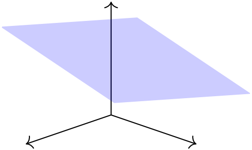
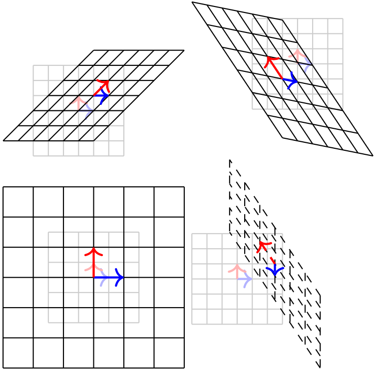
What is affine geometry?

Linear algebra point of view: A plane not through the origin is "almost" a subspace of $\mathbb{R}^n$. It is "enough like a subspace" that one can do some linear algebra in it (dimension, span, basis, ...) but not all (inner product, ...). This is an **affine space**.



Geometry point of view: *Euclidean* geometry studies properties of figures invariant under *orthogonal* matrix transformations (isometries). **Affine geometry** studies properties of figures invariant under *general* ($\det \neq 0$) matrix transformations (collinearity, parallelism, degree of a curve, ...), and ignores properties not preserved by such transformations (lengths, angles, areas, ...).



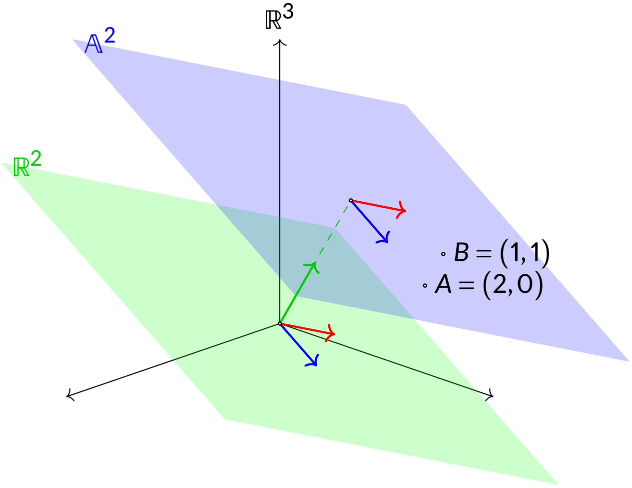
We will start from the linear algebra point of view and eventually come to the conclusion that the geometry point of view is really "the same thing."

Visualisation and motivation

Idea: A plane shifted from the origin is “almost” a linear subspace of $\mathbb{R}^3$.

Wanted: To “do linear algebra” in $\mathbb{A}^2$ intrinsically with points $A, B, C, \ldots \in \mathbb{A}^2$ without knowing its embedding in $\mathbb{R}^3$, but obtaining only results that carry over and remain true in the ambient space $\mathbb{R}^3$.

Work intrinsically. Get results extrinsically.



Cannot (generally) add or multiply by scalar in $\mathbb{A}^n$

Consider for example the embedding:

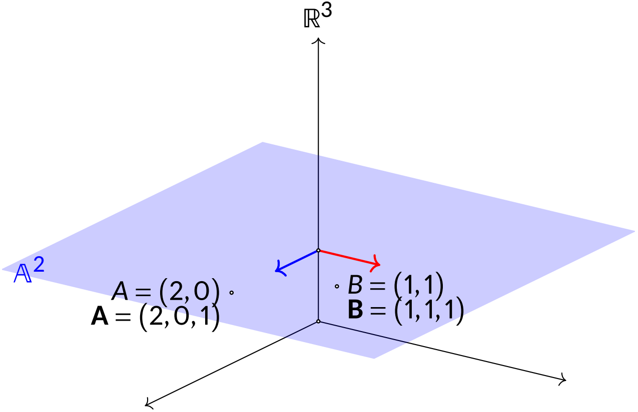
$$\mathbb{A}^2 \ni X = (x_1, x_2) \leftrightarrow (x_1, x_2, 1) = \mathbf{X} \in \mathbb{R}^3$$

Intrinsic $\mathbb{A}^2$-addition does not correspond to extrinsic $\mathbb{R}^3$-addition:

$$A+B = (3,1) \leftrightarrow (3,1,1) \neq (3,1,2) = \mathbf{A}+\mathbf{B}$$

Intrinsic scalar multiplication does not correspond to extrinsic scalar multiplication:

$$kA = (2k,0) \leftrightarrow (2k,0,1) \neq (2k,0,k) = k\mathbf{A}$$



Cannot take differences, but difference vectors (sort of)

$\mathbb{A}^2 \ni X = (x_1, x_2) \leftrightarrow (x_1, x_2, 2) = \mathbf{X} \in \mathbb{R}^3$

Intrinsic $-$ does not correspond to extrinsic $-$:

$B - A = (-1, 1) \leftrightarrow (-1, 1, 2) \neq (-1, 1, 0) = \mathbf{B} - \mathbf{A}$

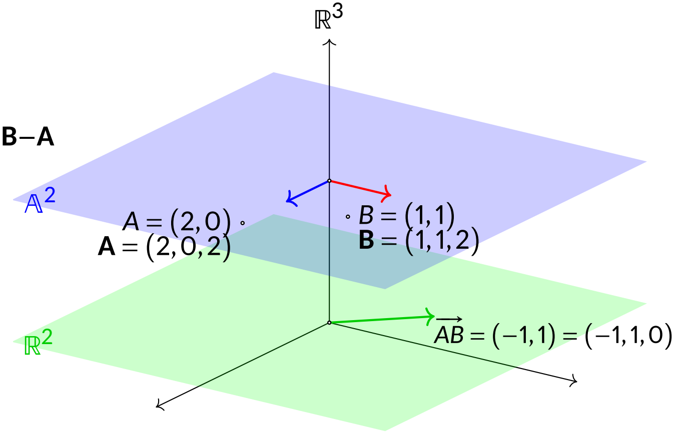
but works with a different embedding

$\mathbb{A}^2 \ni X = (x_1, x_2) \leftrightarrow (x_1, x_2, 0) = \mathbf{X} \in \mathbb{R}^3$

because then

$$\overrightarrow{AB} = (-1, 1) \leftrightarrow (-1, 1, 0) = \mathbf{B} - \mathbf{A}$$

Hence $\overrightarrow{AB}$ can be meaningfully formed but belongs to the base space $\mathbb{R}^2$ rather than to $\mathbb{A}^2$.



So in affine spaces we must distinguish two kinds of objects: points $A, B, \ldots \in \mathbb{A}^n$ and difference vectors $\overrightarrow{AB}, \ldots \in \mathbb{R}^n$, whose interpretation in the ambient world $\mathbb{R}^{n+1}$ is different.

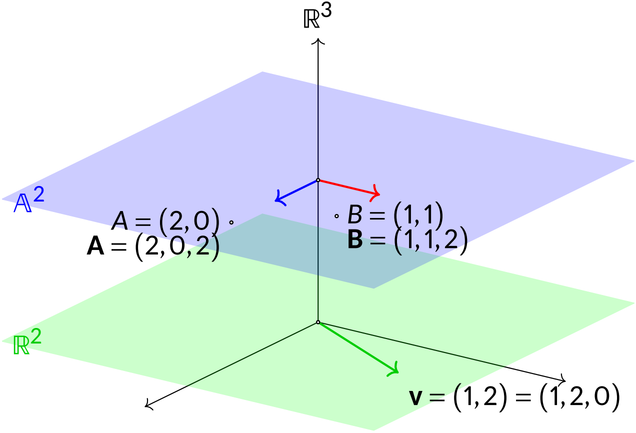
Can add from base space

Consider for example the embedding:

$$\mathbb{A}^2 \ni X = (x_1, x_2) \leftrightarrow (x_1, x_2, 2) = \mathbf{X} \in \mathbb{R}^3$$

Adding an element $\mathbf{v} \in \mathbb{R}^2$ from the base space does correspond to extrinsic addition by such a vector:

$$A + \mathbf{v} = (3, 2) \leftrightarrow (3, 2, 2) = \mathbf{A} + \mathbf{v}$$

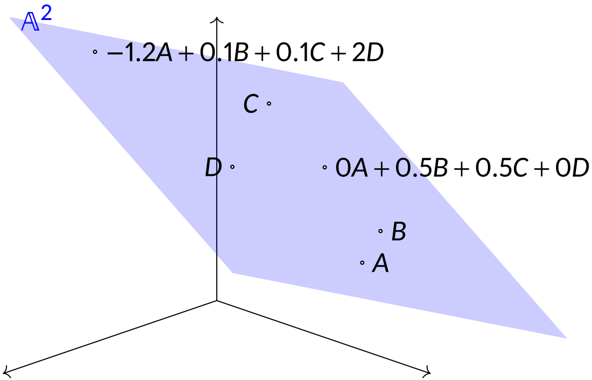


Can take affine-linear combinations

Calculating an affine-linear combination

$$\sum \lambda_i P_i \quad (\Sigma\lambda = 1)$$

always gives the same point regardless of what coordinate system the points are expressed in (whether in $\mathbb{A}^n$ or $\mathbb{R}^{n+1}$).



Two ways of thinking about $\Sigma\lambda = 1$

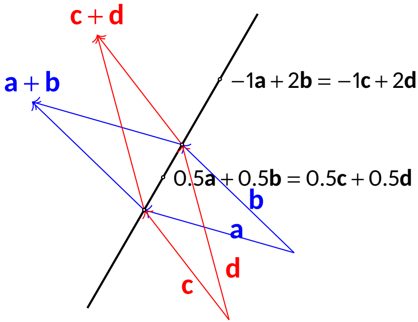
- Affine-linear combinations

$$\sum \lambda_i P_i \quad (\Sigma\lambda = 1)$$

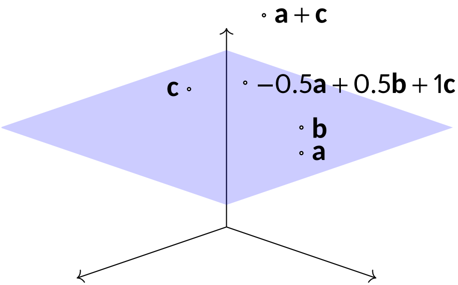
generalise the $\mathbb{E}^n$ definition

$$\text{line}(\mathbf{a},\mathbf{b}) = \{\mathbf{x} \in \mathbb{E}^n : \mathbf{x} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}, \lambda \in \mathbb{R}\}$$

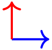
which is invariant under changes of coordinate system, while $\Sigma\lambda \neq 1$ combinations are not.

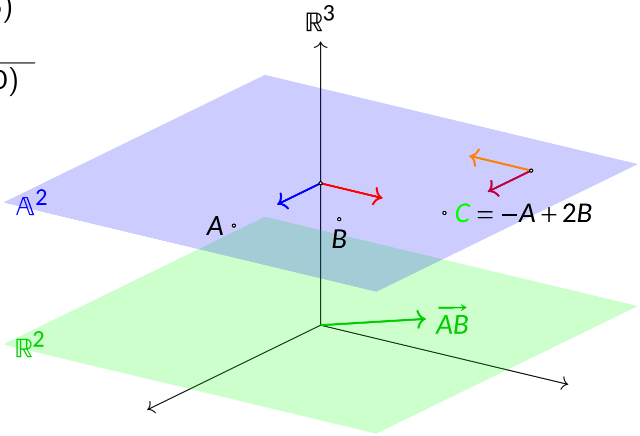


- If $\mathbf{p}_i$ all have z-coordinate Z, the z-coordinate of $\sum \lambda_i \mathbf{p}_i$ is $(\Sigma\lambda)Z$. So $\Sigma\lambda = 1 \iff$ remain in the same plane.

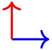


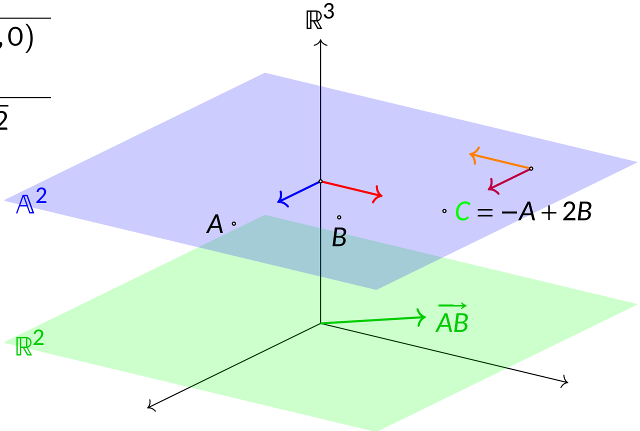
Example

			$\mathbb{R}^3$
A	$(2,0)$	$(4,2)$	$(2,0,2)$
B	$(1,1)$	$(3,1)$	$(1,1,2)$
$-A+2B$ $(\Sigma\lambda = 1)$	$(0,2)$ $= C$	$(2,0)$ $= C$	$(0,2,2)$ $= C$
$A+2B$ $(\Sigma\lambda \neq 1)$	$(4,2)$ $\neq$	$(10,4)$ $\neq$	$(4,2,6)$ $\neq$
$-A+B$ $= \overrightarrow{AB}$	$(-1,1)$ $\cong$	$(-1,-1)$ $\cong$	$(-1,1,0)$ $\cong$

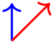


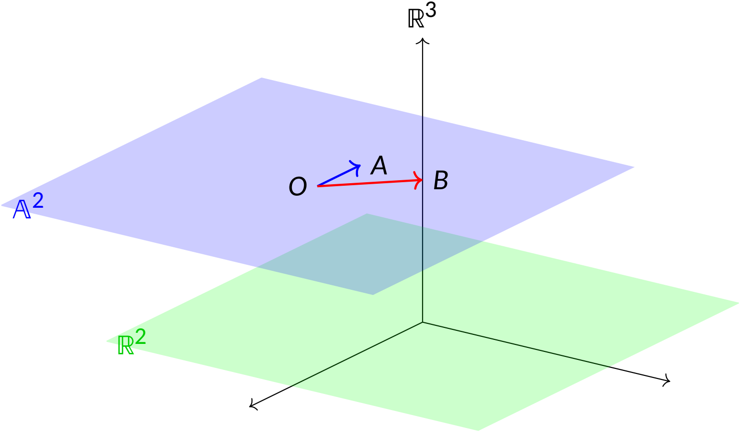
Example with different lengths of unit vectors in $\mathbb{A}^2$ compared to $\mathbb{R}^3$

			$\mathbb{R}^3$
A	$(2,0)$	$(4,2)$	$(1,0,1)$
B	$(1,1)$	$(3,1)$	$(\frac{1}{2},\frac{1}{2},1)$
$-A+2B$ $(\Sigma\lambda=1)$	$(0,2)$ $=C$	$(2,0)$ $=C$	$(0,1,1)$ $=C$
$A+2B$ $(\Sigma\lambda\neq 1)$	$(4,2)$ $\neq$	$(10,4)$ $\neq$	$(2,1,3)$ $\neq$
$-A+B$ $=\overrightarrow{AB}$	$(-1,1)$ $\cong$	$(-1,-1)$ $\cong$	$(-\frac{1}{2},\frac{1}{2},0)$ $\cong$
$\left\lvert\overrightarrow{AB}\right\rvert$	$\sqrt{2}$ $\neq$	$\sqrt{2}$ $\neq$	$1/\sqrt{2}$ $\neq$



Example with extrinsically non-orthogonal basis vectors

		$\mathbb{R}^3$
O	$(0,0)$	$(1,-1,1)$
A	$(1,0)$	$(0,-1,1)$
B	$(0,1)$	$(0,0,1)$
$2B-A$	?	?
$A+B$	?	?
$\angle AOB$	?	?
$\overrightarrow{OA}\cdot\overrightarrow{OB}$	?	?



Formal definition of $\mathbb{A}^n$

Elements of $\mathbb{A}^n$ are n-tuples of real numbers: $A = (a_1, \ldots, a_n)$. Operations in $\mathbb{A}^n$:

$$\oplus : \mathbb{A}^n \times \mathbb{R}^n \to \mathbb{A}^n \qquad A \oplus \mathbf{v} = (a_1 + v_1, \ldots, a_n + v_n) \in \mathbb{A}^n$$

$$\to : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{R}^n \qquad \overrightarrow{AB} = (b_1 - a_1, \ldots, b_n - a_n) \in \mathbb{R}^n$$

Subject to the relations:

(a) $A \oplus \mathbf{0} = A$

(b) $(A \oplus \mathbf{x}) \oplus \mathbf{y} = A \oplus (\mathbf{x} + \mathbf{y})$

(c) $A \oplus \overrightarrow{AB} = B$

(d) $\overrightarrow{A(A \oplus \mathbf{x})} = \mathbf{x}$

(e) $\overrightarrow{AA} = \mathbf{0}$

(f) $\overrightarrow{AB} = -\overrightarrow{BA}$

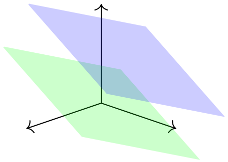
(g) $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

(h) $\overrightarrow{A(B \oplus \mathbf{x})} = \overrightarrow{AB} + \mathbf{x}$

(j) $\overrightarrow{(A \oplus \mathbf{x})(B \oplus \mathbf{y})} = \overrightarrow{AB} + \mathbf{y} - \mathbf{x}$

We can **see** this in the case of $\mathbb{A}^2$ embedded in $\mathbb{R}^3$. We can take this to **define** $\mathbb{A}^n$ for any n.

❓ Which of the above are “blue objects” and which are “green objects” in terms of our picture?



The list of $\mathbb{A}^n$ axioms can be reduced

(a) $A \oplus \mathbf{0} = A$

(b) $(A \oplus \mathbf{x}) \oplus \mathbf{y} = A \oplus (\mathbf{x} + \mathbf{y})$

(c) $A \oplus \overrightarrow{AB} = B$

(d) $\overrightarrow{A(A \oplus \mathbf{x})} = \mathbf{x}$

(e) $\overrightarrow{AA} = \mathbf{0}$

(f) $\overrightarrow{AB} = -\overrightarrow{BA}$

(g) $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

(h) $\overrightarrow{A(B \oplus \mathbf{x})} = \overrightarrow{AB} + \mathbf{x}$

(j) $\overrightarrow{(A \oplus \mathbf{x})(B \oplus \mathbf{y})} = \overrightarrow{AB} + \mathbf{y} - \mathbf{x}$

Proof of (j):

$$\begin{aligned}
\overrightarrow{(A \oplus \mathbf{x})(B \oplus \mathbf{y})} &= \overrightarrow{(A \oplus \mathbf{x})B} + \mathbf{y} && (h) \\
&= -\overrightarrow{B(A \oplus \mathbf{x})} + \mathbf{y} && (f) \\
&= -(\overrightarrow{BA} + \mathbf{x}) + \mathbf{y} && (h) \\
&= \overrightarrow{AB} + \mathbf{y} - \mathbf{x} && (f)
\end{aligned}$$

Proof of (h):

$$\begin{aligned}
\overrightarrow{A(B \oplus \mathbf{x})} &= \overrightarrow{A((\quad ? \quad) \oplus \mathbf{x})} && (c) \\
&= ? && ? \\
&= \overrightarrow{AB} + \mathbf{x} && ?
\end{aligned}$$

Proof of (e):

$$\begin{aligned}
\overrightarrow{AA} &= ? && ? \\
&= \mathbf{0} && ?
\end{aligned}$$

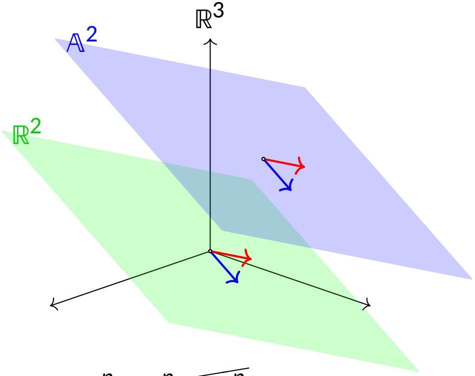
Affine space as solution space

Solutions of $M\mathbf{x} = \mathbf{0}$ make a linear space

Ex.: plane ($\sim \mathbb{R}^2$) through origin if $M = 1 \times 3$ matrix and $\mathbf{x} \in \mathbb{R}^3$

Solutions of $M\mathbf{x} = \mathbf{c}$ make an affine space

Ex.: plane ($\sim \mathbb{A}^2$) not through origin if $M = 1 \times 3$ matrix and $\mathbf{x} \in \mathbb{R}^3$ (and $\mathbf{c}$ a 1×1 "vector")



solution + solution ≠ solution ~~$+ : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n$~~

solution + solution = solution $\oplus : \mathbb{A}^n \times \mathbb{R}^n \to \mathbb{A}^n$ $A \oplus \mathbf{v} \in \mathbb{A}^n$

solution − solution = solution $\rightarrow : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{R}^n$ $\overrightarrow{AB} \in \mathbb{R}^n$

Example of a non-geometrical affine space

$$y'' + y = 0 \quad \Longrightarrow \quad y(x) = A\sin x + B\cos x =$$

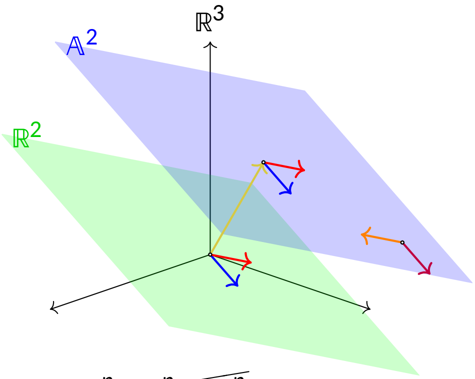
linear space $\sim \mathbb{R}^2$

$$y'' + y = x^2 \quad \Longrightarrow$$

$$y = y_p + y_h = x^2 - 2 + A\sin x + B\cos x$$

$$= \text{ affine space } \sim \mathbb{A}^2$$

$$= x^2 - 2 + (A-2)(\sin x) + (B-2)(-\cos x)$$



solution + solution ≠ solution

solution + homogenous solution = solution

solution − solution = homogenous solution

~~$+ : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n$~~

$\oplus : \mathbb{A}^n \times \mathbb{R}^n \to \mathbb{A}^n \quad A \oplus \mathbf{v} \in \mathbb{A}^n$

$\to : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{R}^n \quad \overrightarrow{AB} \in \mathbb{R}^n$

Affine-linear combination defined in terms of ⊕ and →

$$\sum_{(\Sigma\lambda=1)} : \mathbb{A}^n \times \cdots \times \mathbb{A}^n \to \mathbb{A}^n \qquad \lambda_a A + \cdots + \lambda_k K = (\lambda_a a_1 + \cdots + \lambda_k k_1, \ldots, \lambda_a a_n + \cdots + \lambda_k k_n) \in \mathbb{A}^n$$

reduces (by calculation rule (c) $A \oplus \overrightarrow{AB} = B$) to the other two operations

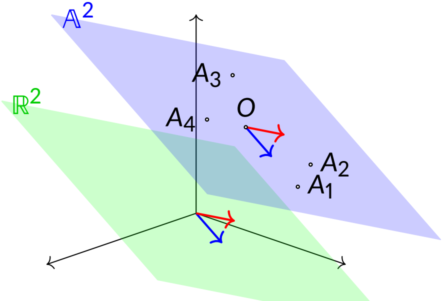
$$\oplus : \mathbb{A}^n \times \mathbb{R}^n \to \mathbb{A}^n \qquad A \oplus \mathbf{v} = (a_1 + v_1, \ldots, a_n + v_n) \in \mathbb{A}^n$$

$$\to : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{R}^n \qquad \overrightarrow{AB} = (b_1 - a_1, \ldots, b_n - a_n) \in \mathbb{R}^n$$

by

$$\lambda_0 A_0 + \lambda_1 A_1 + \cdots + \lambda_k A_k := O \oplus \left(\lambda_0 \overrightarrow{OA_0} + \cdots + \lambda_k \overrightarrow{OA_k}\right) \qquad (\Sigma\lambda = 1)$$

❓ What makes this definition valuable but $A + B := O \oplus \left(\overrightarrow{OA} + \overrightarrow{OB}\right)$ not? Answer on next slide!



Affine-linear combination does not depend on choice of origin

Consider two different choices of origin $O, O' \in \mathbb{A}^n$ and form $\Sigma\lambda_i A_i$ in each:

$$B := O \oplus \left(\lambda_0 \overrightarrow{OA_0} + \cdots + \lambda_k \overrightarrow{OA_k}\right) \qquad B' := O' \oplus \left(\lambda_0 \overrightarrow{O'A_0} + \cdots + \lambda_k \overrightarrow{O'A_k}\right) \qquad (\Sigma\lambda = 1)$$

Claim: $B = B'$. Proof:

$$\overrightarrow{BB'} = \overrightarrow{\left(O \oplus \left(\lambda_0 \overrightarrow{OA_0} + \cdots + \lambda_k \overrightarrow{OA_k}\right)\right)\left(O' \oplus \left(\lambda_0 \overrightarrow{O'A_0} + \cdots + \lambda_k \overrightarrow{O'A_k}\right)\right)} \qquad \text{(def.)}$$

$$= \overrightarrow{OO'} + (\lambda_0 \overrightarrow{O'A_0} + \cdots + \lambda_k \overrightarrow{O'A_k}) - (\lambda_0 \overrightarrow{OA_0} + \cdots + \lambda_k \overrightarrow{OA_k}) \qquad (j)$$

$$= \overrightarrow{OO'} + \lambda_0 (\underbrace{\overrightarrow{O'A_0} - \overrightarrow{OA_0}}_{=\overrightarrow{O'A_0} + \overrightarrow{A_0O} = \overrightarrow{O'O}}) + \cdots + \lambda_k (\overrightarrow{O'A_k} - \overrightarrow{OA_k}) \qquad (\mathbb{R}^n)$$

$$= \overrightarrow{OO'} + \lambda_0 \overrightarrow{O'O} + \cdots + \lambda_k \overrightarrow{O'O} = \overrightarrow{OO'} - \lambda_0 \overrightarrow{OO'} - \cdots - \lambda_k \overrightarrow{OO'} \qquad (f), (g)$$

$$= (1 - (\lambda_0 + \cdots + \lambda_k)) \overrightarrow{OO'} = 0 \cdot \overrightarrow{OO'} = \mathbf{0} \qquad (\mathbb{R}^n)$$

$$\Longrightarrow B \overset{(a)}{=} B \oplus \mathbf{0} = B \oplus \overrightarrow{BB'} \overset{(c)}{=} B' \qquad \square$$

(a) $A \oplus \mathbf{0} = A$

(b) $(A \oplus \mathbf{x}) \oplus \mathbf{y} = A \oplus (\mathbf{x} + \mathbf{y})$

(c) $A \oplus \overrightarrow{AB} = B$

(f) $\overrightarrow{AB} = -\overrightarrow{BA}$

(g) $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

(j) $\overrightarrow{(A \oplus \mathbf{x})(B \oplus \mathbf{y})} = \overrightarrow{AB} + \mathbf{y} - \mathbf{x}$

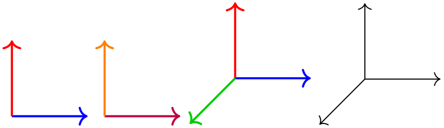
Notation

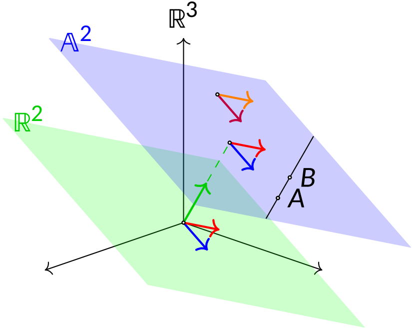
Let’s write $+$ instead of $\oplus$ for convenience from now on, now that we have gotten used to the distinction.

Definition of line in $\mathbb{A}^n$

$$\operatorname{line}(A,B) := \{(1-\lambda)A + \lambda B : \lambda \in \mathbb{R}\}$$
$$= \{A + \lambda\overrightarrow{AB} : \lambda \in \mathbb{R}\}$$

Equivalent in either coordinate system:



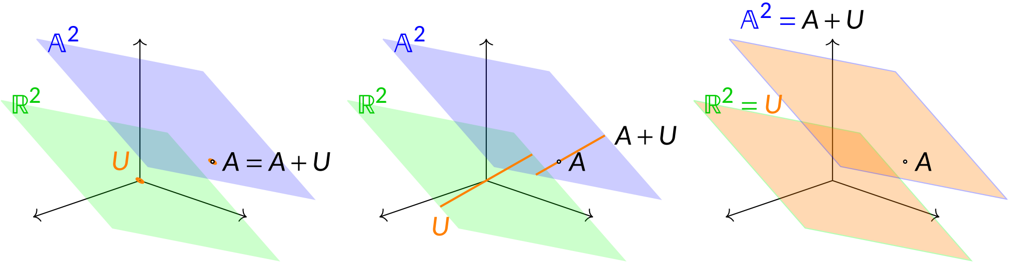


Definition of subspace in $\mathbb{A}^n$

For $A \in \mathbb{A}^n$ and U subspace $\subset \mathbb{R}^n$, we define

$$A + U := \{A + \mathbf{u} : \mathbf{u} \in U\}$$

to be an “affine subspace.”



Hyperplanes are subspaces

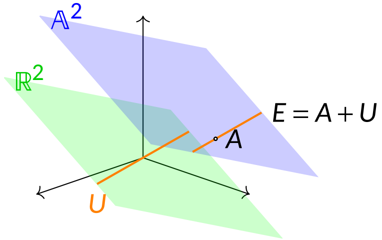
A hyperplane can be defined as the set of all solutions to a linear equation:

$$E := \{A \in \mathbb{A}^n : \underbrace{c_1 a_1 + \cdots + c_n a_n}_{\text{not all } c \text{ zero}} = d\}$$

Proof that this is a subspace:

Let $U := \{\mathbf{x} \in \mathbb{R}^n : c_1 x_1 + \cdots + c_n x_n = 0\}$, and let A be any element of E.

Want to show: $E = A + U$.



$$\begin{aligned} B \in E &\iff c_1 b_1 + \cdots + c_n b_n = d \\ &\iff c_1(b_1 - a_1) + \cdots + c_n(b_n - a_n) = d - d = 0 \\ &\iff \overrightarrow{AB} = (b_1 - a_1, \ldots, b_n - a_n) \in U \\ &\iff B = A + \overrightarrow{AB} \in A + U \end{aligned}$$

❓ Lemma: $B = A + \mathbf{v} \implies \overrightarrow{AB} = \mathbf{v}$ (where $A, B \in \mathbb{A}^n$ and $\mathbf{v} \in \mathbb{R}^n$)

❓ Prove using the $\mathbb{A}^n$ axioms. If $B = A + \mathbf{v}$, then:

$$\overrightarrow{AB} = \qquad\qquad = \mathbf{v}$$

(a) $A \oplus \mathbf{0} = A$

(b) $(A \oplus \mathbf{x}) \oplus \mathbf{y} = A \oplus (\mathbf{x} + \mathbf{y})$

(c) $A \oplus \overrightarrow{AB} = B$

(d) $\overrightarrow{A(A \oplus \mathbf{x})} = \mathbf{x}$

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(j) $\overrightarrow{(A \oplus \mathbf{x})(B \oplus \mathbf{y})} = \overrightarrow{AB} + \mathbf{y} - \mathbf{x}$

- Alternatively, the following reasoning works and secretly you can use it, but it is a problematic way of writing since it falsely suggests that 🚫 $B - A \in \mathbb{A}^n$ ⚠.

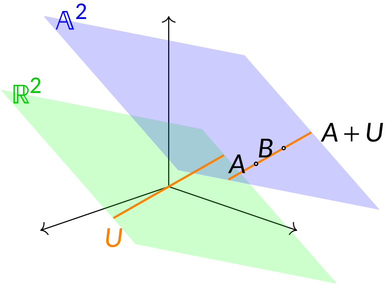
$$B = A + \mathbf{v} \implies B - A = \mathbf{v} \implies \overrightarrow{AB} = \mathbf{v}$$

Equivalent representations of affine subspaces

$$A + U = B + V \iff \begin{cases} U = V \\ \overrightarrow{AB} \in U \end{cases}$$

Proof of $\Rightarrow$:

$$A + U = B + V \implies B \in A + U \quad (\mathbf{0} \in V \implies B \in B + V)$$
$$\implies B = A + \mathbf{u} \implies B - A = \mathbf{u}$$
$$\implies \overrightarrow{AB} \in U$$



$$A + U = B + V \implies B + \mathbf{v} \in A + U \quad (\mathbf{v} \text{ arbitrary element of } V)$$
$$\implies B + \mathbf{v} = A + \mathbf{u} \implies B - A + \mathbf{v} = \mathbf{u}$$
$$\implies \overrightarrow{AB} + \mathbf{v} = \mathbf{u} \in U$$
$$\implies \mathbf{v} = \mathbf{u} - \overrightarrow{AB} \in U \quad \text{(by part (a))}$$

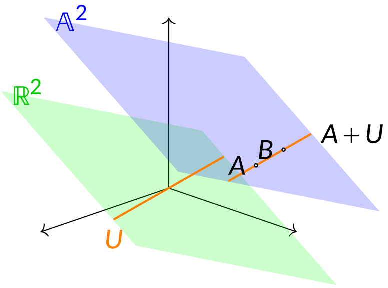
In the same way, $\mathbf{u} \in V$. □

Equivalent representations of affine subspaces

$$A + U = B + V \iff \begin{cases} U = V \\ \overrightarrow{AB} \in U \end{cases}$$

Proof of $\Leftarrow$:

$$\begin{aligned} C \in B + V &\implies C \in B + U \\ &\implies C = B + \mathbf{u} \\ &\implies \overrightarrow{BC} \in U \\ &\implies \overrightarrow{AB} + \overrightarrow{BC} \in U \\ &\implies \overrightarrow{AC} \in U \\ &\implies C - A = \mathbf{u}' \implies C = A + \mathbf{u}' \\ &\implies C \in A + U \end{aligned}$$



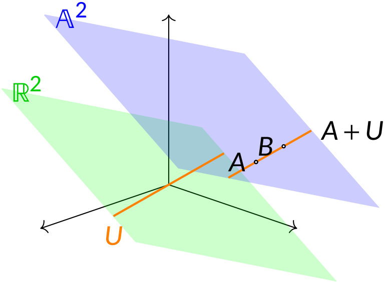
Thus $B + V \subseteq A + U$. In the same way $B + V \supseteq A + U$. $\square$

Equivalent representations of affine subspaces

$$A + U = B + V \iff \begin{cases} U = V \\ \overrightarrow{AB} \in U \end{cases}$$

Corollary (of $\Leftarrow$): Any point in an affine subspace can be used as new base point:

$$\begin{aligned} B \in A + U &\implies B = A + \mathbf{u} \implies B - A = \mathbf{u} \\ &\implies \overrightarrow{AB} = \mathbf{u} \\ &\implies \overrightarrow{AB} \in U \\ &\implies A + U = B + U \end{aligned}$$



Corollary (of $\Rightarrow$):

$$\dim(A + U) := \dim U$$

is well-defined.

subspace ∩ subspace = subspace

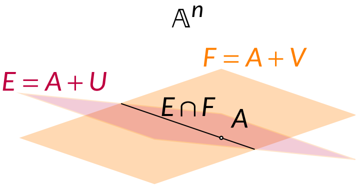
E, F affine subspaces $\subset \mathbb{A}^n$ with $E \cap F \neq \emptyset \quad \Longrightarrow \quad E \cap F$ affine subspace.

Proof:

Use $A \in E \cap F$ to represent $E = A + U$ and $F = A + V$.

$$B \in E \iff B = A + \mathbf{u} \iff B - A = \mathbf{u} \iff \overrightarrow{AB} \in U$$

$$B \in F \iff B = A + \mathbf{v} \iff B - A = \mathbf{v} \iff \overrightarrow{AB} \in V$$



so

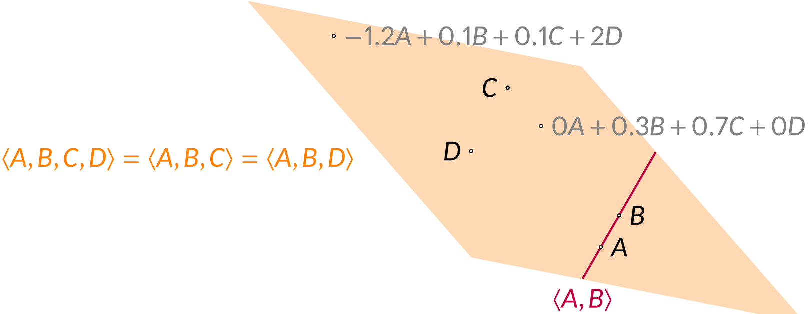
$$B \in E \cap F \iff \overrightarrow{AB} \in U \cap V$$
$$\iff B - A = \mathbf{i} \quad \text{for some } \mathbf{i} \in U \cap V$$
$$\iff B = A + \mathbf{i}$$
$$\iff B \in A + U \cap V$$

so

$$E \cap F = \underbrace{A}_{\in \mathbb{A}^n} + \underbrace{U \cap V}_{\mathbb{R}^n \text{ subspace}} = \text{affine subspace} \quad \square$$

Affine span

$$\langle A, B, C, D, \ldots \rangle := \text{smallest affine subspace containing } A, B, C, D, \ldots$$
$$\overset{*}{=} A + \operatorname{span}\left\{\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}, \ldots\right\}$$
$$\overset{*}{=} \text{all affine-linear combinations of } A, B, C, D, \ldots$$



* Proofs on next slides.

Affine span $\leftrightarrow A + \mathbb{R}^n$ span

$\langle A, B, C, D, \ldots \rangle :=$ smallest affine subspace containing $A, B, C, D, \ldots$

$\Longrightarrow \langle A, B, C, D, \ldots \rangle = A + V$ where

$V = \mathbb{R}^n$ subspace

$A, B, C, D, \ldots \in A + V$

$\Longrightarrow \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}, \ldots \in V$

$\Longrightarrow \operatorname{span}\left\{\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}, \ldots\right\} \subseteq V$

$\Longrightarrow A + \operatorname{span}\left\{\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}, \ldots\right\} \subseteq A + V = \langle A, B, C, D, \ldots \rangle$

$\Longrightarrow \langle A, B, C, D, \ldots \rangle = A + \operatorname{span}\left\{\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}, \ldots\right\}$

$A + \mathbb{R}^n$ span $\leftrightarrow$ affine-linear combinations

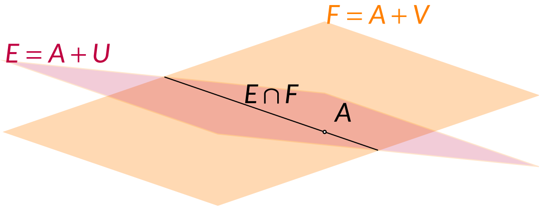
$$
\begin{aligned}
& P \in A + \operatorname{span}\left\{\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}, \dots\right\} \\
\implies & P = A + \lambda_1 \overrightarrow{AB} + \lambda_2 \overrightarrow{AC} + \cdots \\
\implies & P = A + \lambda_1 B - \lambda_1 A + \lambda_2 C - \lambda_2 A + \cdots \\
\implies & P = \underbrace{(1 - \lambda_1 - \lambda_2 - \cdots)}_{:=\lambda_0} A + \lambda_1 B + \lambda_2 C + \cdots \\
\implies & P = \lambda_0 A + \lambda_1 B + \lambda_2 C + \cdots \quad (\Sigma\lambda = 1)
\end{aligned}
$$

$$
\begin{aligned}
& P = \lambda_0 A + \lambda_1 B + \lambda_2 C + \cdots \quad (\Sigma\lambda = 1) \\
\implies & P = A - A + \lambda_0 A + \lambda_1 B + \lambda_2 C + \cdots \quad (\Sigma\lambda = 1) \\
\implies & P = A + \lambda_0 (A - A) + \lambda_1 (B - A) + \lambda_2 (C - A) + \cdots \quad (\Sigma\lambda = 1) \\
\implies & P = A + \lambda_0 \overrightarrow{AA} + \lambda_1 \overrightarrow{AB} + \lambda_2 \overrightarrow{AC} + \cdots \quad (\Sigma\lambda = 1) \\
\implies & P = A + \lambda_1 \overrightarrow{AB} + \lambda_2 \overrightarrow{AC} + \cdots \\
\implies & P \in A + \operatorname{span}\left\{\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}, \dots\right\}
\end{aligned}
$$

Dimension theorem

$$\dim(\underbrace{E\cap F}_{\neq\emptyset}) + \dim \underbrace{\langle E,F\rangle}_{:=\langle E\cup F\rangle} = \dim E + \dim F$$

$\mathbb{A}^n$



Intuitively:

$$\dim\langle E,F\rangle = \underbrace{\dim E}_{\#\text{ indep. } \mathbf{u}\text{'s in } U} + \underbrace{\dim F - \dim(E\cap F)}_{\#\text{ additional indep. } \mathbf{v}\text{'s in } V}$$

Proof of dimension theorem

The $\mathbb{A}^n$ theorem

$$\dim(\underbrace{E \cap F}_{\neq \emptyset}) + \dim \underbrace{\langle E, F\rangle}_{:=\langle E \cup F\rangle} = \dim E + \dim F$$

follows from the known $\mathbb{R}^n$ theorem

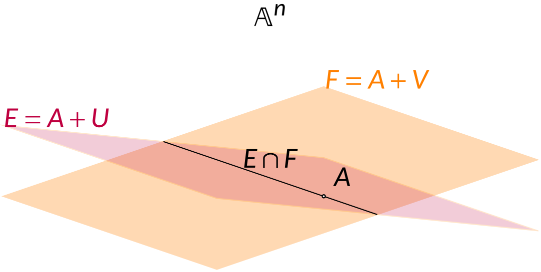
$$\dim(U \cap V) + \dim(\operatorname{span}(U \cup V)) = \dim U + \dim V$$

once everything is written in terms of a base point A in the intersection $E \cap F$.

$$E \cap F = A + (U \cap V)$$

(part of subspace $\cap$ subspace = subspace proof)

$$\begin{aligned}\langle E, F\rangle &= A + \operatorname{span}\{\overrightarrow{AB} \mid B \in E \cup F\} \\ &= A + \operatorname{span}(\{\overrightarrow{AB} \mid \underbrace{B \in E}_{B = A + \mathbf{u}}\} \cup \{\overrightarrow{AB} \mid B \in F\}) \\ &= A + \operatorname{span}(U \cup V)\end{aligned}$$



Affine dependence

$$A_0, A_1, A_2, \dots, A_k \in \mathbb{A}^n \text{ affinely independent}$$

$$\overset{\text{def.}}{\iff} \dim \langle A_0, A_1, A_2, \dots, A_k \rangle = k \qquad \text{("maximal dimension")}$$

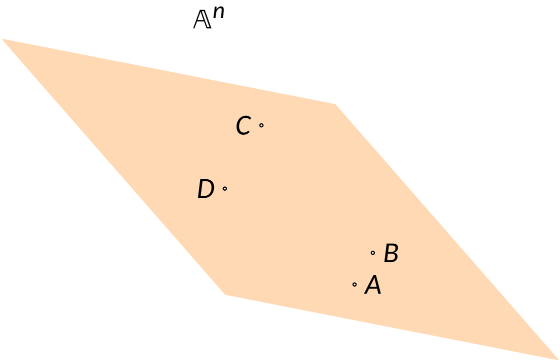
$$\iff \overrightarrow{A_0A_1}, \overrightarrow{A_0A_2}, \dots, \overrightarrow{A_0A_k} \text{ independent in } \mathbb{R}^n$$

$$\iff \text{points in } \langle A_0, A_1, A_2, \dots, A_k \rangle \overset{\text{bij.}}{\leftrightarrow} \text{affine-linear combinations of } A_0, A_1, \dots, A_k$$

Example:

$\{A, B, C, D\}$ affinely dependent.

Proper subsets of $\{A, B, C, D\}$ affinely independent.

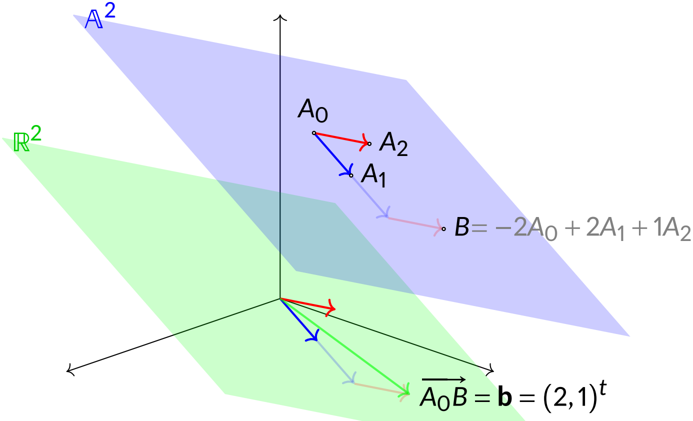


Affine bases

$n+1$ independent points define a basis in $\mathbb{A}^n$.

coordinates of B in basis $A_0, A_1, \dots, A_k$ $\quad :=\quad$ coordinates $\mathbf{b}$ of $\overrightarrow{A_0B}$ in basis $\overrightarrow{A_0A_1}, \overrightarrow{A_0A_2}, \dots \overrightarrow{A_0A_n}$

$$B = \underbrace{b_0A_0 + \cdots + b_kA_k}_{\text{affine-linear combination}} \quad \leftrightarrow \quad \mathbf{b} = (b_1, \dots, b_n)^t \qquad (\text{“because” } A_0 = \mathbf{0})$$



Determinant condition for affine dependence in $\mathbb{A}^n$

$$\underbrace{B_0, B_1, \dots, B_n}_{\text{with coordinates } \mathbf{b}_0, \dots, \mathbf{b}_n \text{ in some } \mathbb{A}^n \text{ basis}} \in \mathbb{A}^n \text{ affinely dependent}$$

$$\iff \overrightarrow{B_0B_1}, \overrightarrow{B_0B_2}, \dots, \overrightarrow{B_0B_n} \text{ linearly dependent}$$

$$\iff \exists \text{ non-trivial solution to } \lambda_1 \underbrace{\overrightarrow{B_0B_1}}_{\mathbf{b}_1 - \mathbf{b}_0} + \lambda_2 \underbrace{\overrightarrow{B_0B_2}}_{\mathbf{b}_2 - \mathbf{b}_0} + \cdots + \lambda_n \underbrace{\overrightarrow{B_0B_n}}_{\mathbf{b}_n - \mathbf{b}_0} = \mathbf{0}$$

$$\iff \exists \text{ non-trivial solution to } \underbrace{-\left(\sum_{i=1}^{n} \lambda_i\right)}_{:=\lambda_0} \mathbf{b}_0 + \lambda_1 \mathbf{b}_1 + \lambda_2 \mathbf{b}_2 + \cdots + \lambda_n \mathbf{b}_n = \mathbf{0}$$

$$\iff \exists \text{ non-trivial solution to } \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mathbf{b}_0 & \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \mathbf{0}$$

the row of 1's reflects that we cannot choose λ_0 freely: $\lambda_0 := -\sum_{i=1}^{n} \lambda_i \iff \sum_{i=0}^{n} \lambda_i = 0$

$$\iff \det \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mathbf{b}_0 & \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{pmatrix}}_{(n+1)\times(n+1) \text{ matrix}} = 0$$

Determinant form of lines

$$\text{line}(\underbrace{A,B}_{\text{non-identical}}) \in \mathbb{A}^2 = \left\{X \in \mathbb{A}^2 : A, B, X \text{ affinely dependent}\right\}$$

$$= \text{all solutions to } \det\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{a} & \mathbf{b} & \mathbf{x} \end{pmatrix} = 0$$

in a given basis of $\mathbb{A}^2$

Example:

$$\text{line}\left((2,0),(0,1)\right) = \left\{ (x,y) \in \mathbb{A}^2 : \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & x \\ 0 & 1 & y \end{vmatrix} = 0 \right\} = \left\{(x,y) \in \mathbb{A}^2 : y = 1 - \tfrac{1}{2}x\right\}$$

(Precisely the same calculation works in $\mathbb{R}^2$, but now we have a new perspective on the meaning and significance of the row of 1's.)

Determinant form of hyperplanes

$$\text{hyperplane}(\underbrace{A_1,\ldots,A_n}_{\text{affinely independent}}) \in \mathbb{A}^n = \{X \in \mathbb{A}^n : A_1,\ldots,A_n,X \text{ affinely dependent}\}$$

$$= \text{all solutions to } \det\begin{pmatrix} 1 & \cdots & 1 & 1 \\ \mathbf{a}_1 & \cdots & \mathbf{a}_n & \mathbf{x} \end{pmatrix} = 0$$

in a given basis of $\mathbb{A}^n$

Explanation/reminder of "hyperspace" terminology:

dim(ordinary plane) : dim(ordinary space) = 2 : 3

dim(hyperplane) : dim(its ambient space) = $(n-1) : n$

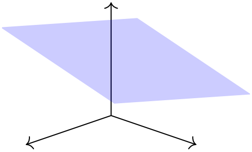
hyperplane = set of solutions to a linear equation

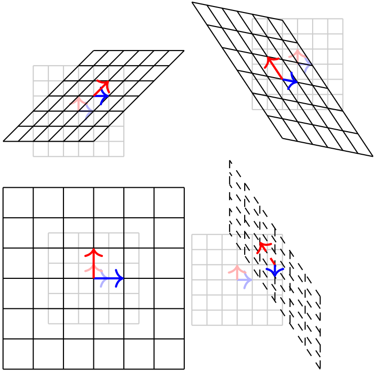
= $(n-1)$-dimensional subspace of n-dimensional space

In two-dimensional spaces such as $\mathbb{A}^2$ and $\mathbb{R}^2$, hyperplanes are lines.

Roadmap

Linear algebra point of view: A plane not through the origin is "almost" a subspace of $\mathbb{R}^n$.





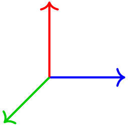
Geometry point of view: **Affine geometry** studies properties of figures invariant under *general* ($\det \neq 0$) matrix transformations.

We have now completed our task of building up as much linear algebra as we can in an "almost subspace" (affine space). We shall now see why this corresponds to taking general matrices (affine transformations) as an equivalence relation.

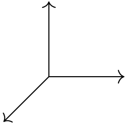
“Communicating” $\mathbb{A}^n$ results with the “outside world”

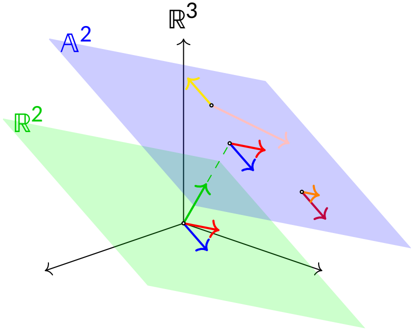
Equivalent in any “internal” coordinate system

$\implies$ equivalent in “privileged,” “canonical” (but internally inaccessible) coordinate system



$\implies$ equivalent in any “external” coordinate system





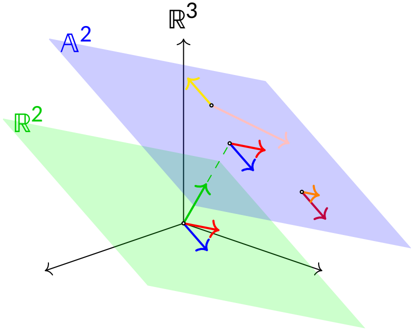
“Communicating” $\mathbb{A}^n$ results with the “outside world”

“Communicable” results must thus remain “equi-expressible” or “neutral” with respect to any changes of basis such as these:

Such changes of bases correspond to transformations of the form

$$T : \mathbb{A}^n \to \mathbb{A}^n \qquad \mathbf{a} \longmapsto M\mathbf{a} + \mathbf{v}$$

where M is any invertible $n \times n$ matrix.



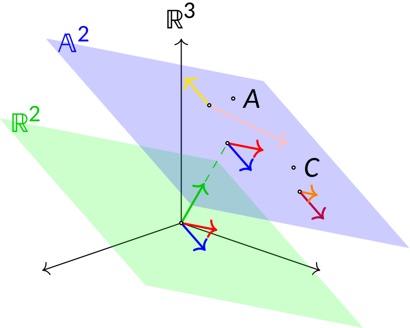
Non-equi-expressible properties

 thinks that $|\mathbf{a}| < |\mathbf{c}|$. This is not an "equi-expressible" or "coordinate-system-neutral" theorem.

 cannot re-express this theorem in his terms without retaining reference to the original coordinate system. Even if he has a "dictionary" so that he can "translate" what

 means by A etc. into his own terms,

he can still only say that "from 's point of view, $|\mathbf{a}| < |\mathbf{c}|$." The theorem doesn't "simplify" to a theorem expressible purely in terms of .

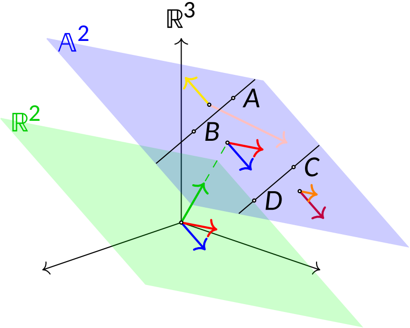


Equi-expressible properties

Suppose  hears that has proved that

$$\text{line}(A, B) \text{ is parallel to line}(C, D)$$

 can then merely translate the coordinates of the points but otherwise retain the theorem as stated. The theorem is **equi-expressible** or **coordinate-system-neutral**.



Equi-expressible properties

$$\text{line}(A,B) \text{ is parallel to line}(C,D)$$

is equi-expressible or coordinate-system-neutral because, if T is a coordinate transformation,

$$T(\text{line}(A,B)) = \text{line}(T(A),T(B))$$

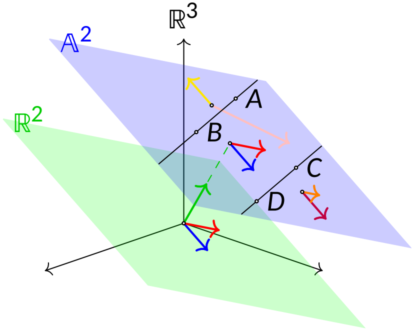
The opposite is the case with

$$|\mathbf{a}| < |\mathbf{c}|$$

because

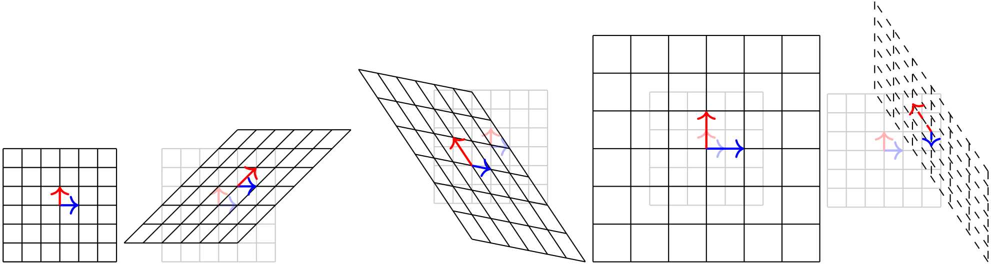
$$T(|\mathbf{a}|) \neq \left|T(\mathbf{a})\right|$$

In affine geometry we want to consider only properties that are “respected” by general coordinate transformations T. Like line(A,B) but unlike $|\mathbf{a}|$.



Affine transformations

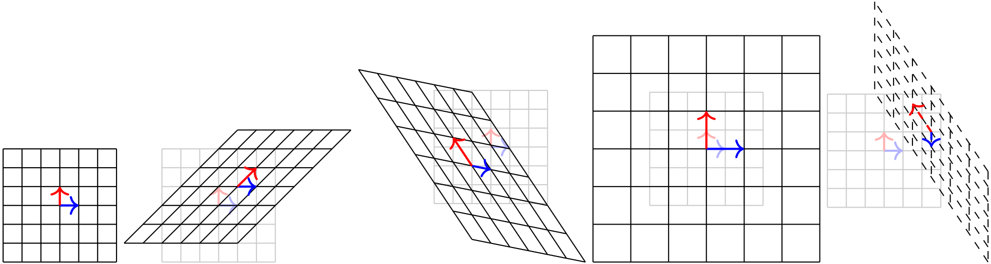
$$\mathbb{R}^n \to \mathbb{R}^n \qquad \mathbf{a} \longmapsto M\mathbf{a} + \mathbf{v} \qquad (M = \text{any invertible } n \times n \text{ matrix})$$



preserved by …	straightness	parallelism	angles	orientation	congruence	similarity	d	d-ratios	d-ratios ⊂ line	squares	triangles	circles	ellipses	conics	degrees
orthogonal M	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$k\times$ orthogonal M	✓	✓	✓	✗	✗	✓	✗	✓	✓	✓	✓	✓	✓	✓	✓
invertible M	✓	✓	✗	✗	✗	✗	✗	✗	✓	✗	✓	✗	✓	✓	✓

“Affinity”

$\mathbb{R}^n \to \mathbb{R}^n$ $\quad$ $\mathbf{a} \mapsto M\mathbf{a} + \mathbf{v}$ $\quad$ (M = any invertible $n \times n$ matrix)



Affinely equivalent figures are not equal, congruent, similar, isomorphic, etc.—they merely have a weak “affinity” with one another.

Calculation rules for affine transformations $\implies$ preserve affine properties

$$T : \mathbb{A}^n \to \mathbb{A}^n \qquad \mathbf{a} \mapsto M\mathbf{a} + \mathbf{v} \qquad M = \text{ invertible } n \times n \text{ matrix}$$

$$\Downarrow$$

$$\overrightarrow{T(A)T(B)} = (M\mathbf{b} + \mathbf{v}) - (M\mathbf{a} + \mathbf{v}) = M\mathbf{b} - M\mathbf{a} = M(\mathbf{b} - \mathbf{a}) = M\overrightarrow{AB}$$

$$T(A + \mathbf{x}) = M(\mathbf{a} + \mathbf{x}) + \mathbf{v} = (M\mathbf{a} + \mathbf{v}) + M\mathbf{x} = T(A) + M\mathbf{x}$$

$$\Downarrow$$

Affine transformations preserve affine spaces:

$$T(\text{affine subspace}) = T(A + U) = \underbrace{T(A)}_{\in \mathbb{A}^n} + \underbrace{MU}_{\mathbb{R}^n \text{ subspace}} = \text{ affine subspace}$$

Affine transformations preserve affine-linear combinations:

$$T(\lambda_0 A_0 + \cdots + \lambda_k A_k) = T(A_0 + \lambda_1 \overrightarrow{A_0 A_1} + \cdots + \lambda_k \overrightarrow{A_0 A_k}) = T(A_0) + M(\lambda_1 \overrightarrow{A_0 A_1} + \cdots + \lambda_k \overrightarrow{A_0 A_k})$$

$$= T(A_0) + \lambda_1 M\overrightarrow{A_0 A_1} + \cdots \lambda_k M\overrightarrow{A_0 A_k} = T(A_0) + \lambda_1 \overrightarrow{T(A_0)T(A_1)} + \cdots + \lambda_k \overrightarrow{T(A_0)T(A_k)}$$

$$= \lambda_0 T(A_0) + \cdots + \lambda_k T(A_k)$$