Metric spaces

Viktor Blåsjö

Utrecht University

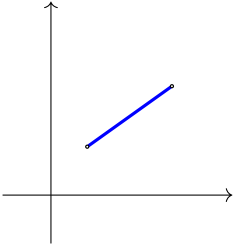
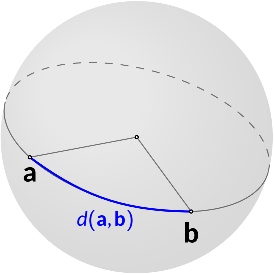
v.n.e.blasjo@uu.nl
uu.nl/staff/VNEBlasjo
@viktorblasjo

Course philosophy

“bad” mathematics	**“good” mathematics**
intuitive	formal
visual	arithmetical
concrete	abstract
specific	structural
sprawling	unified

- Non-Euclidean geometry; continuous but nowhere differentiable functions
  $\implies$ visual intuition was wrong.
- Common structures in different branches of mathematics
  $\implies$ efficient and valuable to study structures abstractly.
- Nobody can keep up with all mathematics; niches increasingly isolated
  $\implies$ unification needed.

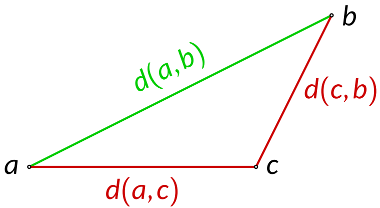
Goal: unify different concepts of distance, such as:

	metric $d(\mathbf{a},\mathbf{b}) =$
Euclidean plane	
sphere	

Definition of metric space

A metric space is a set A with a metric $d : A \times A \to \mathbb{R}_{\geq 0}$ such that (for any “points in the space” $a, b, c \in A$):

1. $d(a,b) = 0 \iff a = b$ (hence non-degenerate: $a \neq b \Rightarrow d(a,b) \neq 0$)
   Two different points cannot be “on top of each other.”
   A point cannot be “away from itself.”
2. $d(a,b) = d(b,a)$ (symmetric)
   Distances are the same “forwards” as “backwards.”
3. $d(a,b) \leq d(a,c) + d(c,b)$ (triangle inequality)
   A “detour” cannot be a “shortcut.”



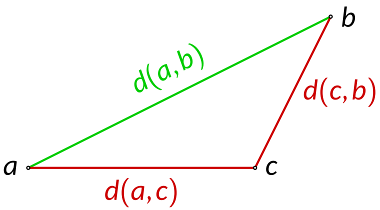
Role of visual aids and intuitive ideas

"Think geometrically, prove algebraically."

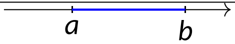
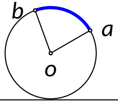
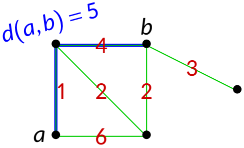
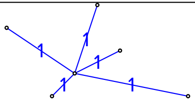
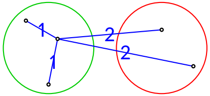
- Formally:
  - 🚫 Define objects.
  - 🚫 Justify inferences.
  - 🚫 Generalise.
- Psychologically:
  - 🥰 Help interpret and remember results.
  - 😍 Suggest proof ideas.
  - 😘 Preeeetty.

Triangle inequality:

$$d(a,b) \leq d(a,c) + d(c,b)$$



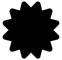
Examples of metric spaces

		set	metric $d(a,b) =$	
$\mathbb{E}^1$	Euclidean line	$\mathbb{R}$	$\lvert b-a \rvert$	
S^1	Euclidean circle	unit circle in $\mathbb{R}^2$	$\angle aob$	
	weighted graph $(\underbrace{V, E}_{\text{finite}}, w)$	set of vertices V	min. sum of weights in path from a to b	
	discrete metric	any	$\begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b \end{cases}$	
	discrete metric on disjoint set	any $A_1 \sqcup A_2$	$\begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } \neq \text{ but in same set} \\ 2 & \text{if in different sets} \end{cases}$	
	taxicab metric	$\mathbb{R}^2$	$\lvert \Delta x \rvert + \lvert \Delta y \rvert$	

❓ What does a “circle” look like in the taxicab metric space?

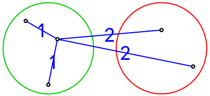
taxicab metric $\mathbb{R}^2$ $|\Delta x| + |\Delta y|$

The set of all points $\mathbf{p} \in \mathbb{R}^2$ such that $d(\mathbf{0}, \mathbf{p}) \leq 1$:

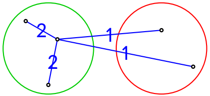
- ❓ circle 
- ❓ jagged circle 
- ❓ square 
- ❓ diamond 
- ❓ plus 

❓ Switch far and close?

Known metric space:

	set	metric $d(a,b) =$	
discrete metric on disjoint set	any $A_1 \sqcup A_2$	$\begin{cases} 0 & \text{if } a = b \\ 1 & \text{if} \neq \text{ but in same set} \\ 2 & \text{if in different sets} \end{cases}$	

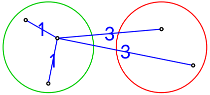
❓ Is the following also a metric space?

	set	metric $d(a,b) =$	
discrete metric on disjoint set	any $A_1 \sqcup A_2$	$\begin{cases} 0 & \text{if } a = b \\ 2 & \text{if} \neq \text{ but in same set} \\ 1 & \text{if in different sets} \end{cases}$	

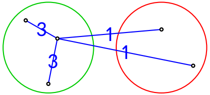
1. $d(a,b) = 0 \iff a = b$
2. $d(a,b) = d(b,a)$
3. $d(a,b) \leq d(a,c) + d(c,b)$

Hint: What would be the possible counterexamples to triangle ineq.?

❓ Does anything change if we replace 2 with 3?

	set	metric $d(a,b) =$	
discrete metric on disjoint set	any $A_1 \sqcup A_2$	$\begin{cases} 0 & \text{if } a = b \\ 1 & \text{if} \neq \text{ but in same set} \\ 3 & \text{if in different sets} \end{cases}$	

❓ Still a metric space?

	set	metric $d(a,b) =$	
discrete metric on disjoint set	any $A_1 \sqcup A_2$	$\begin{cases} 0 & \text{if } a = b \\ 3 & \text{if} \neq \text{ but in same set} \\ 1 & \text{if in different sets} \end{cases}$	

❓ Metric space?

$$d(a,b) \leq d(a,c) + d(c,b)$$

Proof that $d \geq 0$ is implied by the definition of a metric

1. $d(a,b) = 0 \iff a = b$
2. $d(a,b) = d(b,a)$
3. $d(a,b) \leq d(a,c) + d(c,b)$
4. $d(a,b) \geq 0$ (sometimes included in the definition, but follows from 1–3)

Proof of 4:

Suppose $d(x,y) < 0$. Consider the triangle inequality with x as start and end point, and y as middle point:

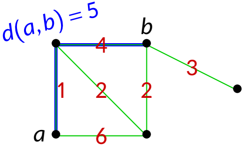
$$0 = d(x,x) \leq d(x,y) + d(y,x) = d(x,y) + d(x,y) < 0$$

Contradiction. □

intuitive	formal
visual	arithmetical
concrete	abstract

Proving triangle inequality without first knowing $d(a,b) \geq 0$.

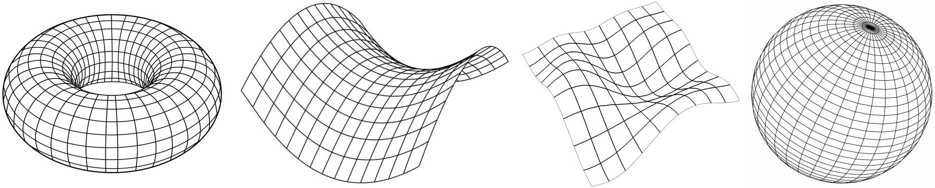
❓ What could make a definition bad?

set	metric $d(a,b) =$	
weighted graph $(\underbrace{V,E}_{\text{finite}},w)$	set of vertices V	min. sum of weights in path from a to b	

❓ Why only finite V,E? Show by an example what can go wrong without this restriction.

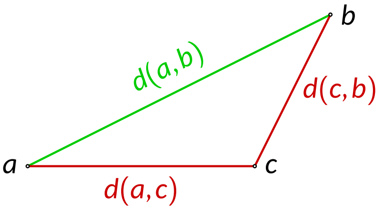
❓ Surfaces embedded in $\mathbb{R}^3$ are metric spaces

with the metric that is inherited when the metric of space is restricted to the surface:
$d(a,b)$:= length of shortest path along surface from a to b.



❓ Prove that the triangle inequality holds in all such cases.

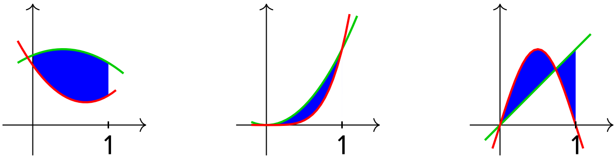
$$d(a,b) \leq d(a,c) + d(c,b)$$



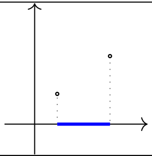
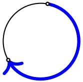
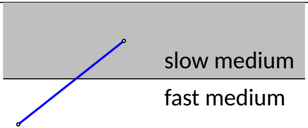
A metric space beyond this course

Set: All continuous functions on $[0,1]$.
Metric:

$$d(f,g) = \text{ area between } f,g \ = \int_0^1 |f-g|\,dx$$

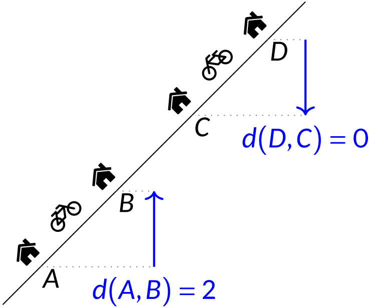


Non-examples of metric spaces

non-degen.	symmetric	triangle ineq.	set	metric	
✗	✓	✓	$\mathbb{R}^2$	horizontal distance $\lvert\Delta x\rvert$	
✓	✗	✓	circle	clockwise distance	
✓	✓	✗	bimedial	time of straight-line travel	

- Isolates what each condition contributes relative to the others.
- Shows that the conditions are independent.

❓ Bike metric

non-degen.	symmetric	triangle ineq.	set	metric	
❓	❓	❓	the line $y = x$	"bike effort": $\max(\Delta y, 0)$	

Template: Prove that … is a metric space

1. $d(x,y)=0 \iff \qquad\qquad \iff x=y$
2. $d(x,y)= \qquad\qquad =d(y,x)$
3. $d(x,z)+d(z,y)=$

   $\geq$

   $=d(x,y)$

Proof that $\mathbb{E}^1$ is a metric space

Recall definition of $\mathbb{E}^1$: Set: $\mathbb{R}$. Metric: $d(x,y) = |y - x|$.

1. $d(x,y) = 0 \iff |y-x| = 0 \iff y - x = 0 \iff x = y$
2. $d(x,y) = |y-x| = |x-y| = d(y,x)$
3. $d(x,z) + d(z,y)$
   $= |z-x| + |y-z|$
   $\geq |z - x + y - z|$ (taking $\underbrace{|a| + |b| \geq |a+b|}_{\text{“triangle inequality” for } |\ | \text{ in } \mathbb{R}}$ to be “known”)
   $= |x-y|$
   $= d(x,y)$ □

Order: what is to be proved; what this means in the concrete case; connective steps.

For the purposes of our course:

✗ “It’s intuitively/visually obvious!”

✓ “It reduces to known arithmetic facts.”
(Proving known arithmetic facts is “somebody else’s problem.”)

Proof strategy: bidirectional

Work from both ends simultaneously.

Proof strategy: linear

Incrementally build from starting point, getting closer to the goal at each step.

Proof strategy: sketch first

Start with outline, fill in details.

Proof strategy: pillars first

Start by finding the most reliable principles to build on.

Proof fail

Counterexample discovered in the course of writing a proof; work wasted.





Isometries

An isometry is a mapping T from one metric space (A, d) to another (A', d') that is

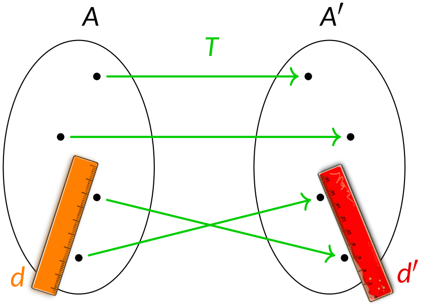
- Distance-preserving.

  $d'(T(a), T(b)) = \qquad = d(a, b)$

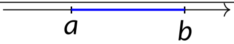
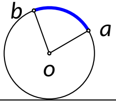
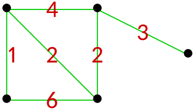
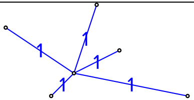
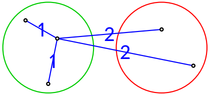
- Bijective.
  - injective: $T(a) = T(b) \implies \qquad \implies a = b$

    Actually distance-preserving $\implies$ injective, so this does not need to be included in the definition or checked separately:

    $T(a) = T(b) \implies d'(T(a), T(b)) = 0 \implies d(a, b) = 0 \implies a = b$
  - surjective: $b \in A' \implies \exists a \in A$ such that $T(a) = b$, namely $T(\qquad) = b$

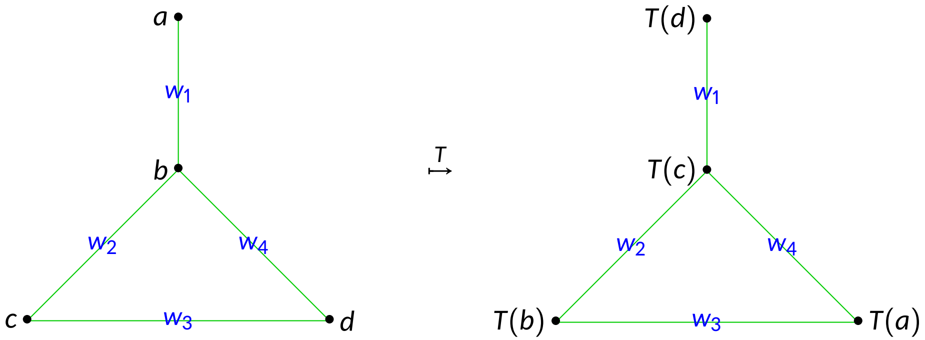


❓ Examples of distance-preserving bijections from a space to itself

metric $d(a,b) =$	❓ examples of isometries?
$\lvert b-a \rvert$	
$\angle aob$	
min. sum of weights in path from a to b	
$\begin{cases} 0 & \text{if } a=b \\ 1 & \text{if } a \neq b \end{cases}$	
$\begin{cases} 0 & \text{if } a=b \\ 1 & \text{if} \neq \text{but in same set} \\ 2 & \text{if in different sets} \end{cases}$	
$\lvert \Delta x \rvert + \lvert \Delta y \rvert$	

❓ Can an isometry of a weighted graph be “topologically nontrivial”?

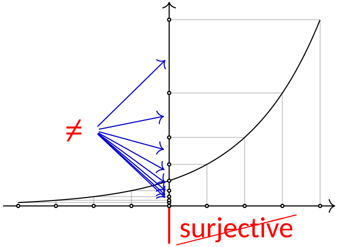
Such as the following (for some choice of weights w_i):



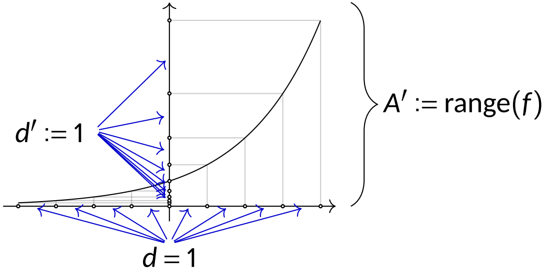
General isometries of limited use since easily “fakeable”

If (A,d) is a metric space, any injective map f from A to any set is an isometry in a boring way: simply define $A' = \text{range}(f)$ and $d'(x,y) = d\left(f^{-1}(x), f^{-1}(y)\right)$.

Example:
$f(x) = e^x$ is neither bijective nor distance-preserving as a map from $\mathbb{E}^1$ to $\mathbb{E}^1$.



Yet one can trivially redefine (A', d') so that f is an isometry nevertheless.
$A' := \text{range}(f) = \mathbb{R}_{>0}$
$d'(x,y) := d\left(f^{-1}(x), f^{-1}(y)\right) = \left|\log(y/x)\right|$



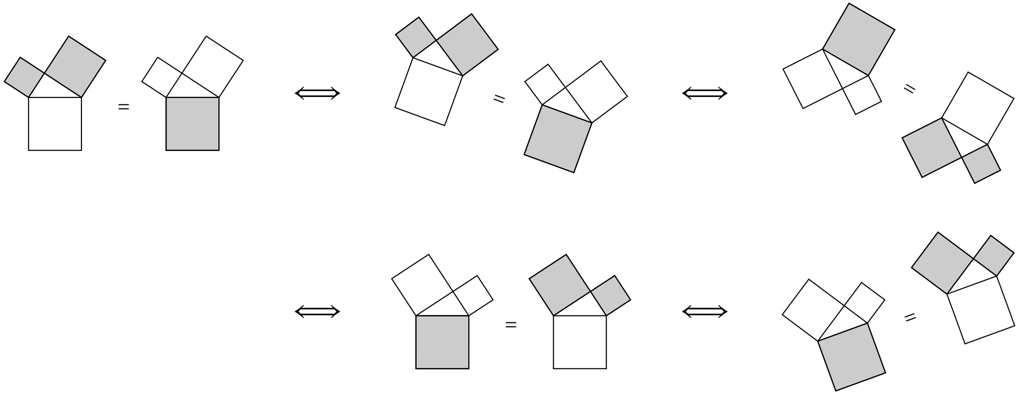
“Self”-isometries are profound

The set (in fact, group) of all isometries from a space to itself captures the essence of the geometry of that space. This is the idea of Felix Klein’s Erlanger Programm (1872), and a theme of our course.

the set of all theorems of Euclidean geometry

=

the set of all properties of figures invariant under Euclidean isometries



Vergleichende Betrachtungen

über

neuere geometrische Forschungen

von

Dr. Felix Klein,
o. ö. Professor der Mathematik an der Universität Erlangen.

Programm

zum Eintritt in die philosophische Facultät und den Senat der k. Friedrich-Alexanders-Universität

zu Erlangen.

Erlangen.

Verlag von Andreas Deichert.

1872.

Als Verallgemeinerung der Geometrie entsteht so das folgende umfassende Problem:

Es ist eine Mannigfaltigkeit und in derselben eine Transformationsgruppe gegeben; man soll die der Mannigfaltigkeit angehörigen Gebilde hinsichtlich solcher Eigenschaften untersuchen, die durch die Transformationen der Gruppe nicht geändert werden.

In Anlehnung an die moderne Ausdrucksweise, die man freilich nur auf eine bestimmte Gruppe, die Gruppe aller linearen Umformungen, zu beziehen pflegt, mag man auch so sagen:

Es ist eine Mannigfaltigkeit und in derselben eine Transformationsgruppe gegeben. Man entwickele die auf die Gruppe bezügliche Invariantentheorie.

Dies ist das allgemeine Problem, welches die gewöhnliche Geometrie nicht nur, sondern namentlich auch die hier zu nennenden neueren geometrischen Methoden und die verschiedenen Behandlungsweisen beliebig ausgedehnter Mannigfaltigkeiten unter sich begreift.

SUR LA GÉOMÉTRIE ENVISAGÉE COMME UN SYSTÈME PUREMENT LOGIQUE

Par Mario Pieri,

Professeur à l'Université de Catane.

Le mouvement intellectuel touchant les principes directeurs de la Mathématique en général et spécialement de la Géométrie, quant à l'analyse logique des prémisses (définitions, axiomes, etc.) et quant à la critique des méthodes qui en forment les doctrines fondamentales, a atteint aujourd'hui un développement si remarquable, que l'étudiant, aussi bien que le savant, n'est pas toujours en état d'en suivre les phases principales, ni d'en recueillir et d'en coordonner les détails. Au contraire, ils sont en comparaison assez rares, les travaux de reconstruction où l'on fait usage de quelque idée nouvelle pour réformer ou modifier une partie de l'édifice de la science en conformité avec les exigences spéculatives croissantes.

Il ne paraîtra donc pas hors de propos que je me permette de soumettre au jugement des gens studieux, même non mathématiciens (à l'occasion de ce grand bilan intel-

moins un point qui soit transformé en un autre différent.

7° *Pour tout couple de points distincts, il existe au moins un mouvement propre qui les laisse fixes tous les deux.* On peut apercevoir cette proposition dans le fait sensible qu'un corps est toujours capable de *se mouvoir* quand on fixe deux quelconques de ses points. Et le *mouvement des corps rigides* servira d'image concrète, intuitive et pleinement conforme à l'idée abstraite de mouvement, pourvu qu'on fasse abstraction du *temps*, et qu'on porte l'attention exclusivement sur *deux* états du mobile (positions *initiale* et *finale*) en observant la distinction entre les mouvements propre et impropre. Du principe 7° on peut conclure tout de suite l'existence d'un mouvement effectif, vu les précédents 2° et 3°.

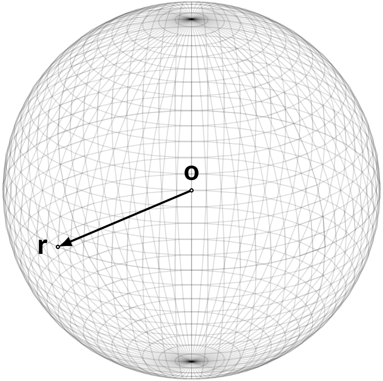
8° *Étant donné que* a, b, c *sont des points distincts, s'il existe un mouvement propre qui les représente chacun par lui-même, tout mouvement qui laisse fixes individuellement* a *et* b *laissera fixe aussi* c. C'est là un principe d'une grande capacité déductive, et qui par suite énonce une condition assez restrictive pour les classes « point » et « mouvement ». Il nous permet de produire la notion de « droite » et de reconnaître quelques-unes de ses propriétés les plus remarquables. En effet, nous pourrons appeler *collinéaires* trois points *a, b, c*, s'il existe un mouvement *effectif* qui les représente chacun par lui-même; et « joignante *a* avec *b* » ou bien « *ab* » la classe de tous les points *x* tels que les trois points *a, b, x* soient collinéaires. Alors, des prémisses déjà établies il suit que, « si *a* et *b* sont des

❓ Sphere defined in terms of isometries

$$\text{sphere } S = \left\{ \mathbf{x} \in \mathbb{E}^3 : d(\mathbf{o},\mathbf{r}) = d(\mathbf{o},\mathbf{x}) \right\}$$

$$= \underbrace{\{ \quad ❓ \quad : \quad ❓ \quad \}}$$

expressed purely in terms of **o**, **r**, and *T* where *T* denotes an isometry of $\mathbb{E}^3$



Hint: For suitable *T*,

$$d(\mathbf{o},\mathbf{r}) = d(T(\mathbf{o}),T(\mathbf{r})) = d(\mathbf{o},T(\mathbf{r}))$$

$$\implies T(\mathbf{r}) \in S$$