Euclidean geometry: 3D

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Overview

Claim:

	$\mathbf{0}$-preserving isometries = orthogonal matrices M	isometries $M\mathbf{x}+\mathbf{v}$
$\mathbb{E}^2$	R_φ, S_φ	Tr, Rot, Rfl, Gl
$\mathbb{E}^3$	$\begin{pmatrix} \pm 1 & 0 \\ 0 & R_\varphi \end{pmatrix}$	?
$\mathbb{E}^n$	?	?

Notation:

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & R_\varphi \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

Classification of matrices occurring in isometries of $\mathbb{E}^3$

$$T \text{ isometry of } \mathbb{E}^3 \implies T = M\mathbf{x} + \mathbf{v}$$

$$\begin{aligned} M\ 3\times 3 \text{ matrix} &\implies \text{eigenvalue equation } \det(M - \lambda I) = 0 \text{ has degree } 3 \\ &\implies \exists \text{ real eigenvalue } \lambda \end{aligned}$$

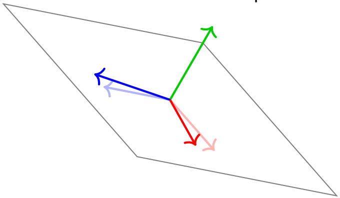
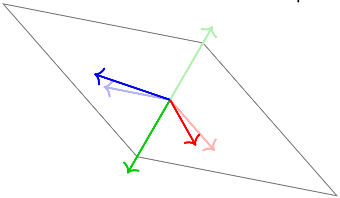
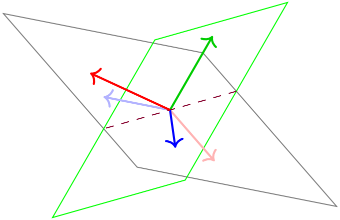
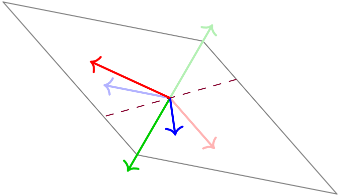
$$\begin{aligned} M \text{ length-preserving} &\implies \text{no vector magnified or shrunk} \\ &\implies \lambda = \pm 1 \end{aligned}$$

$$\Pi := \text{ plane through } \mathbf{0} \text{ perpendicular to eigenvector } \mathbf{v}_\lambda$$

$$\begin{aligned} M \text{ angle-preserving} &\implies \text{plane} \perp \mathbf{v}_\lambda \mapsto \text{plane} \perp M\mathbf{v}_\lambda = \pm \mathbf{v}_\lambda \\ &\implies \Pi \mapsto \Pi \end{aligned}$$

$$\begin{aligned} M \text{ restricted to } \Pi &= M \text{ corresponding to isometry of } \mathbb{E}^2 \\ &= R_\varphi \text{ or } S_\varphi = \mathrm{Rfl}_\ell \end{aligned}$$

Matrices occurring in isometries of $\mathbb{E}^3 = \begin{pmatrix} \pm 1 & 0 \\ 0 & R_\varphi \end{pmatrix}$

	$\lambda = 1$	$\lambda = -1$
$M\vert_\Pi = R_\varphi$	$\mathrm{Rot}_{\mathbf{v}_\lambda,\varphi} = \begin{pmatrix} 1 & 0 \\ 0 & R_\varphi \end{pmatrix}$ 	$\mathrm{Rot}_{\mathbf{v}_\lambda,\varphi} \circ \mathrm{Rfl}_\Pi = \begin{pmatrix} -1 & 0 \\ 0 & R_\varphi \end{pmatrix}$ 
$M\vert_\Pi = \mathrm{Rfl}_\ell$	$\mathrm{Rfl}_{\mathrm{span}(\mathbf{v}_\lambda,\ell)} = \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}$ 	$\mathrm{Rot}_{\ell,\pi} = \begin{pmatrix} 1 & 0 \\ 0 & R_\pi \end{pmatrix}$ 

Overview

	0-preserving isometries = orthogonal matrices M	isometries $M\mathbf{x} + \mathbf{v}$
$\mathbb{E}^2$	R_φ, S_φ	Tr, Rot, Rfl, Gl
$\mathbb{E}^3$	$\begin{pmatrix} \pm 1 & 0 \\ 0 & R_\varphi \end{pmatrix}$	**?**
$\mathbb{E}^n$	?	?

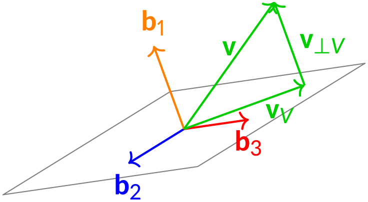
$\mathbb{E}^3$ isometry matrices composed with translations

any $\mathbb{E}^3$ isometry $T(\mathbf{x}) = \begin{pmatrix} \pm 1 & 0 \\ 0 & R_\varphi \end{pmatrix} \mathbf{x} + \mathbf{v}$

$\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 :=$ basis corresponding to this matrix representation

$V := \mathrm{span}(\mathbf{b}_2, \mathbf{b}_3) =$ plane of R_φ

$$\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ c \end{pmatrix} + \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \mathbf{v}_V + \mathbf{v}_{\perp V}$$



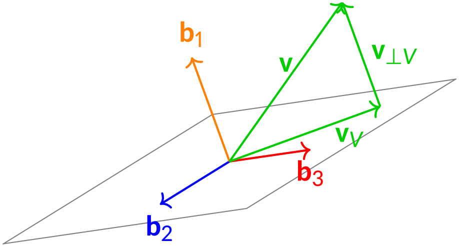
$$T = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & R_\varphi \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ b \\ c \end{pmatrix} + \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\left(\begin{pmatrix} 1 & 0 \\ 0 & R_\varphi \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ b \\ c \end{pmatrix} \right)}_{T_1} + \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}}_{T_2 \circ T_1}$$

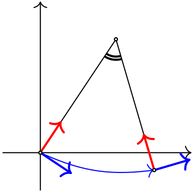
$\mathbb{E}^3$ isometry matrices composed with translations

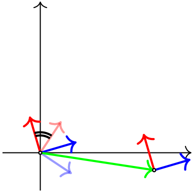
$$T = \underbrace{\begin{pmatrix}\pm1&0&0\\0&1&0\\0&0&1\end{pmatrix}\underbrace{\left(\begin{pmatrix}1&0\\0&R_\varphi\end{pmatrix}\mathbf{x}+\begin{pmatrix}0\\b\\c\end{pmatrix}\right)}_{T_1}+\begin{pmatrix}a\\0\\0\end{pmatrix}}_{T_2\circ T_1}$$

$$T_1(\mathbf{x}) = \mathrm{Tr}\circ\mathrm{Rot} = \begin{cases}\mathrm{Rot}_{\mathrm{line}\perp V,\varphi} & \\ \mathrm{Tr}_{\mathbf{v}_V} & \text{if } \varphi = 0\end{cases}$$



Recall:

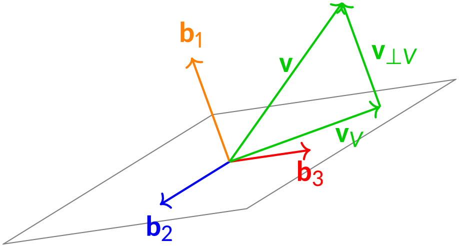
$\mathrm{Rot}_{C,\varphi}(\mathbf{x}) =$ 

$=$  $= R_\varphi\mathbf{x}+\mathbf{v}$

$\mathbb{E}^3$ isometry matrices composed with translations

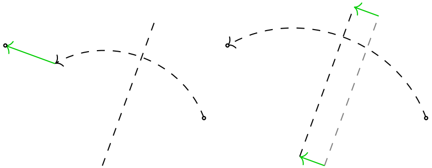
$$T = \underbrace{\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\left(\begin{pmatrix} 1 & 0 \\ 0 & R_\varphi \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ b \\ c \end{pmatrix} \right)}_{T_1} + \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}}_{T_2 \circ T_1}$$

$$T_2(\mathbf{x}) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \mathrm{Tr}_{\mathbf{v}_{\perp V}} \\ \text{or} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \mathrm{Rfl}_{V + \frac{1}{2}\mathbf{v}_{\perp V}} \end{cases}$$

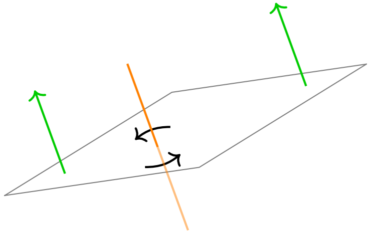
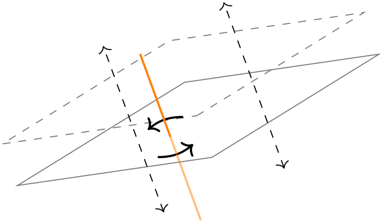
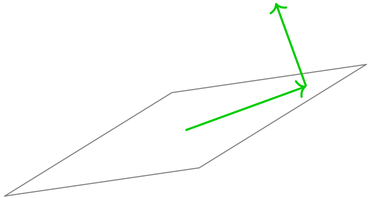
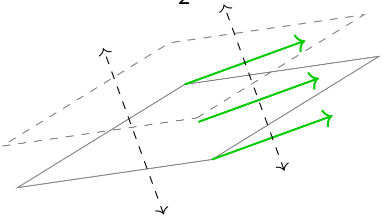


Recall:

$$\mathrm{Rfl}_\ell \mathbf{x} + \mathbf{v}_{\perp \ell} = \mathrm{Rfl}_{\ell + \frac{1}{2}\mathbf{v}_{\perp \ell}}(\mathbf{x})$$



Classification of $\mathbb{E}^3$ isometries

	$T_2 = \mathrm{Tr}_{\mathbf{v}_{\perp V}}$	$T_2 = \mathrm{Rfl}_{V+\frac{1}{2}\mathbf{v}_{\perp V}}$
$T_1 = \mathrm{Rot}_{\mathrm{line}\perp V,\varphi}$	$\mathrm{Scr}_{\mathrm{line}\perp V,\varphi,\mathbf{v}_{\perp V}}$ 	$\mathrm{RotRfl}_{V+\frac{1}{2}\mathbf{v}_{\perp V},\mathrm{line}\perp V,\varphi}$ 
$T_1 = \mathrm{Tr}_{\mathbf{v}_V}$	$\mathrm{Tr}_{\mathbf{v}}$ 	$\mathrm{Gl}_{V+\frac{1}{2}\mathbf{v}_{\perp V},\mathbf{v}_V}$ 

Overview

	0-preserving isometries = orthogonal matrices M	isometries $M\mathbf{x} + \mathbf{v}$
$\mathbb{E}^2$	R_φ, S_φ	Tr, Rot, Rfl, Gl
$\mathbb{E}^3$	$\begin{pmatrix} \pm 1 & 0 \\ 0 & R_\varphi \end{pmatrix}$	Tr, Scr, RotRfl, Gl
$\mathbb{E}^n$	**?**	?

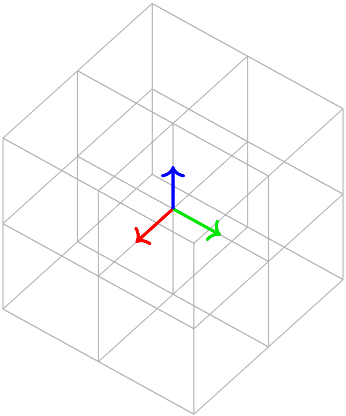
❓ Orthogonal matrices in $\mathbb{R}^n$

$$M \text{ orthogonal } n \times n \text{ matrix } \implies M = \begin{pmatrix} I_p & & & & \\ & -I_q & & 0 & \\ & & R_{\varphi_1} & & \\ & 0 & & \ddots & \\ & & & & R_{\varphi_r} \end{pmatrix}$$

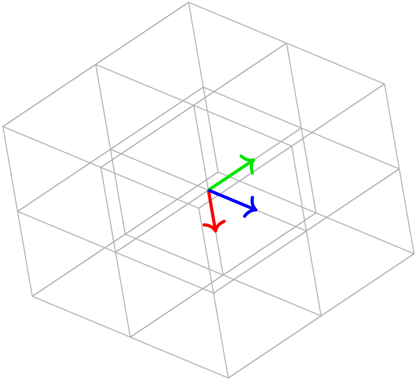
for a suitable choice of orthonormal basis; $p, q, r \geq 0$, $\varphi_1, \ldots \varphi_r \in (-\pi, \pi]$, $p + q + 2r = n$.
The M's we found for $\mathbb{E}^3$:

	p	q	r	φ_1
$\begin{pmatrix} 1 & 0 \\ 0 & R_\varphi \end{pmatrix}$				
$\begin{pmatrix} -1 & 0 \\ 0 & R_\varphi \end{pmatrix}$				
$\begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}$				
$\begin{pmatrix} 1 & 0 \\ 0 & R_\pi \end{pmatrix}$				

❓ What kind of isometry?

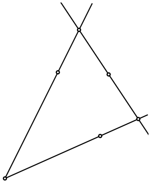
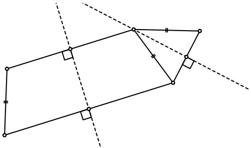
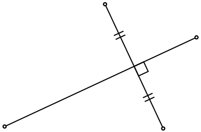


$\mapsto$



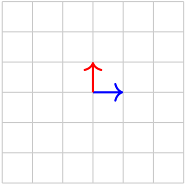
Recall: any isometry of $\mathbb{E}^3 \in \{\mathrm{Tr}, \mathrm{Scr}, \mathrm{RotRfl}, \mathrm{Gl}\}$.

Earlier results that generalise

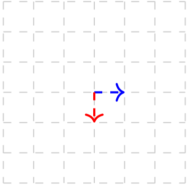
	… points* determine an isometry. (* not all in a $\mathbb{E}^{n-1}$ subspace)	Any isometry = product of … reflections.	… options for isometries left when effect on $\mathbb{E}^{n-1}$ fixed.
$\mathbb{E}^1$	2	2	2
$\mathbb{E}^2$	3	3	2
			
$\mathbb{E}^3$	4	4	2
⋮	⋮	⋮	⋮
$\mathbb{E}^n$	n+1	n+1	2

❓ Indirect isometry of $\mathbb{E}^n \sim$ direct isometry of $\mathbb{E}^{n+1}$?

$\mathbb{E}^2$ reflection in x-axis $= S_\pi$ (det $= -1$: indirect)



$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



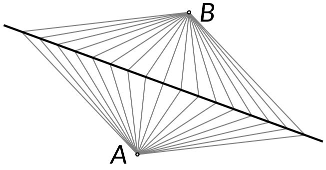
$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R_\pi \end{pmatrix} = \mathbb{E}^3 \text{ rotation about } x\text{-axis} \qquad (\det = 1 : \text{direct})$$

❓ Can we generalise?

Alternative definition of line in $\mathbb{E}^2$

Suppose we had taken as our definition of line sets of the form

$$\{P \in \mathbb{E}^2 : d(A,P) = d(B,P)\}$$



Advantages of this definition:

- Fits better with the theorem

$$\begin{aligned} &\text{perpendicular bisector of } AB \\ &= \{\text{points equidistant to } A, B\} \end{aligned}$$

  that we have used several times.
- Enables a simpler proof that lines $\mapsto$ lines under isometries T, namely:

If

$$L = \{P \in \mathbb{E}^2 : d(A,P) = d(B,P)\}$$

$$L' = \{Q \in \mathbb{E}^2 : d(T(A),Q) = d(T(B),Q)\}$$

then

$$\begin{aligned} P \in L &\implies ❓ \\ &\implies ❓ \\ &\implies T(P) \in L' \end{aligned}$$

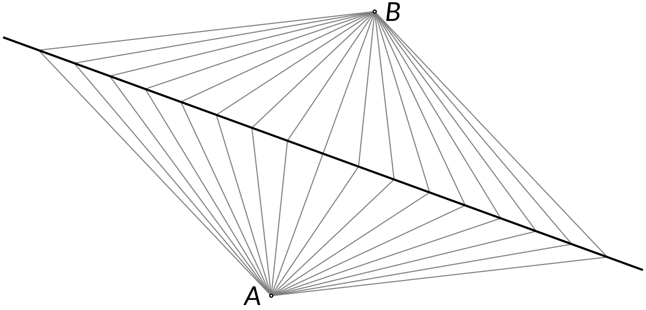
and

$$\begin{aligned} Q \in L' &\implies ❓ \\ &\implies ❓ \\ &\implies T^{-1}(Q) \in L \end{aligned}$$

Hence $T(L) = L' =$ a line.

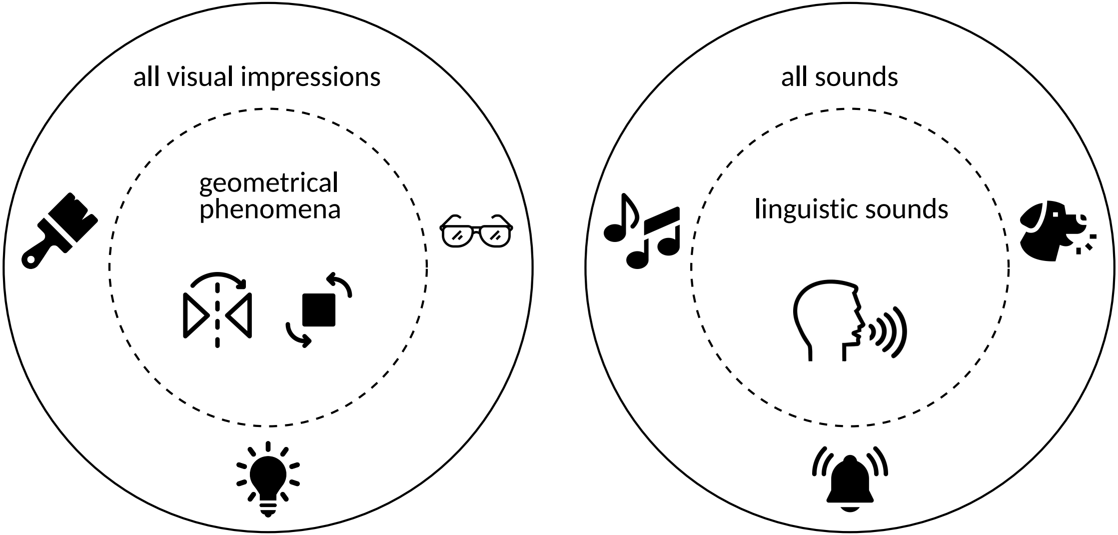
❓ What disadvantages arise when trying to generalise this definition of line to $\mathbb{E}^n$?

$$\{P \in \mathbb{E}^2 : d(A,P) = d(B,P)\}$$



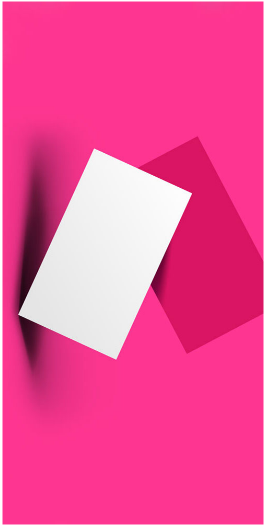
What is geometry?

How to “demarcate” or “pick out from the environment”?



Demarcating geometry according to Poincaré

flipping a card
= cancellable through isometry/motion
= geometrical transformation



changing colour of a liquid
= not cancellable through motion/isometry
= not a geometrical transformation