Euclidean geometry: 2D

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Goal

Build up all classical Euclidean geometry, using “pure” methods:

- ♡ Metrics and isometries.
- ♡ General, conceptual linear algebra.
- ⊘ Case-specific classical geometry.
- ⊘ Case-specific calculations.

For example, bring cosine theorem from ⊘ to ♡.

Euclidean space $\mathbb{E}^n$

Set:

$$\mathbb{R}^n$$

Metric:

$$d(\mathbf{x},\mathbf{y}) := |\mathbf{y}-\mathbf{x}| = \sqrt{(y_1-x_1)^2+\cdots+(y_n-x_n)^2}$$

Where:

$$\mathbf{x}\cdot\mathbf{y} := x_1y_1+\cdots+x_ny_n$$

$$|\mathbf{x}| := \sqrt{\mathbf{x}\cdot\mathbf{x}} = \sqrt{x_1^2+\cdots+x_n^2}$$

Proof that this is a metric space:

1. $d(\mathbf{x},\mathbf{y}) = 0 \iff |\mathbf{y}-\mathbf{x}| = 0 \iff \mathbf{y}-\mathbf{x} = \mathbf{0} \iff \mathbf{x}=\mathbf{y}$ 🥱
2. $d(\mathbf{x},\mathbf{y}) = |\mathbf{y}-\mathbf{x}| = |-(\mathbf{x}-\mathbf{y})| = |\mathbf{x}-\mathbf{y}| = d(\mathbf{y},\mathbf{x})$ 🥱
3. $d(\mathbf{x},\mathbf{z}) + d(\mathbf{z},\mathbf{y}) = |\mathbf{z}-\mathbf{x}| + |\mathbf{y}-\mathbf{z}| = \qquad \geq \qquad = |\mathbf{y}-\mathbf{x}| = d(\mathbf{x},\mathbf{y})$ 🤔

How to prove triangle inequality in $\mathbb{E}^n$?

- Visually? 🚫👎
- Brute algebra? 😩👎
- Conceptual linear algebra? 😎👍

👎	👍
intuitive	formal
visual	arithmetical
concrete	abstract
specific	structural
sprawling	unified

Cauchy-Schwarz inequality

Traditional form:

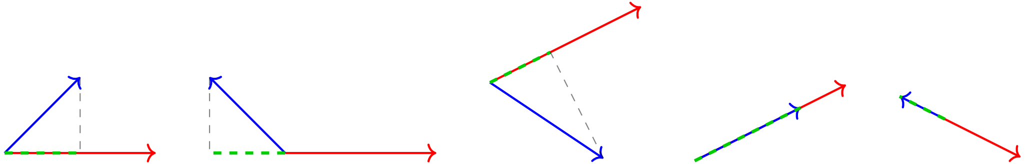
$$|\mathbf{u}\cdot\mathbf{v}| \le |\mathbf{u}|\,|\mathbf{v}| \qquad \begin{cases} = & \text{if } \mathbf{u}, \mathbf{v} \text{ are linearly dependent} \\ < & \text{otherwise} \end{cases}$$

Alternative form:

$$\mathbf{u}\cdot\mathbf{v} \le |\mathbf{u}|\,|\mathbf{v}| \qquad \begin{cases} = & \text{if } \mathbf{u}, \mathbf{v} \text{ are equidirectional} \\ < & \text{otherwise} \end{cases}$$

Informal geometric interpretation:

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta = |\mathbf{u}|\,(\text{projection of } \mathbf{v} \text{ onto } \mathbf{u}) \le |\mathbf{u}|\,|\mathbf{v}|$$



So, visually, Cauchy-Schwarz amounts to:

$$|\text{vector projected perpendicularly onto a line}| \le |\text{vector itself}|$$

Proof of Cauchy-Schwarz inequality

$$|\mathbf{u}\cdot\mathbf{v}| \le |\mathbf{u}|\,|\mathbf{v}|$$

Idea: "algebraise" the distinction between linear dependence/independence.
Proof:

$$\begin{cases} \mathbf{v},\mathbf{u} & \text{linearly dependent} \\ \mathbf{v},\mathbf{u} & \text{linearly independent} \end{cases} \iff \begin{cases} \exists \\ \nexists \end{cases} \lambda \in \mathbb{R} \text{ such that } \lambda\mathbf{u}-\mathbf{v}=\mathbf{0}$$

$$\lambda\mathbf{u}-\mathbf{v}=\mathbf{0} \iff |\lambda\mathbf{u}-\mathbf{v}|=0 \iff |\lambda\mathbf{u}-\mathbf{v}|^2=0 \iff (\lambda\mathbf{u}-\mathbf{v})\cdot(\lambda\mathbf{u}-\mathbf{v})=0$$

$$\iff \lambda^2|\mathbf{u}|^2-2\lambda\mathbf{u}\cdot\mathbf{v}+|\mathbf{v}|^2=0 \overset{*}{\iff} \lambda=\frac{2\mathbf{u}\cdot\mathbf{v}\pm\sqrt{4(\mathbf{u}\cdot\mathbf{v})^2-4|\mathbf{u}|^2|\mathbf{v}|^2}}{2|\mathbf{u}|^2}$$

$$\iff \begin{cases} 4(\mathbf{u}\cdot\mathbf{v})^2-4|\mathbf{u}|^2|\mathbf{v}|^2 \overset{**}{=} 0 \\ \phantom{4(\mathbf{u}\cdot\mathbf{v})^2-4|\mathbf{u}|^2|\mathbf{v}|^2} < 0 \end{cases} \iff \begin{cases} (\mathbf{u}\cdot\mathbf{v})^2 = |\mathbf{u}|^2|\mathbf{v}|^2 \\ \phantom{(\mathbf{u}\cdot\mathbf{v})^2} < \end{cases} \iff \begin{cases} |\mathbf{u}\cdot\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \\ \phantom{|\mathbf{u}\cdot\mathbf{v}|} < \end{cases} \quad \square$$

* Exception: doesn't work if $\mathbf{u}=\mathbf{0}$; this case of the theorem must be verified separately (which is trivial).

** >0 is not a possibility here, since this $\implies$ 2 solutions, which is seen to be impossible by the earlier forms of the equation.

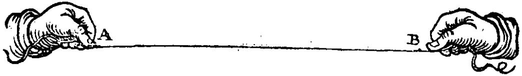
Proof of $\mathbb{E}^n$ triangle inequality from Cauchy-Schwarz

$$
\begin{aligned}
d(\mathbf{x},\mathbf{z}) + d(\mathbf{z},\mathbf{y}) &= |\underbrace{\mathbf{z}-\mathbf{x}}_{\mathbf{u}}| + |\underbrace{\mathbf{y}-\mathbf{z}}_{\mathbf{v}}| = |\mathbf{u}| + |\mathbf{v}| \\
&= \sqrt{(|\mathbf{u}| + |\mathbf{v}|)^2} \\
&= \sqrt{|\mathbf{u}|^2 + 2\,|\mathbf{u}|\,|\mathbf{v}| + |\mathbf{v}|^2} \\
&= \sqrt{\mathbf{u}\cdot\mathbf{u} + 2|\mathbf{u}|\,|\mathbf{v}| + \mathbf{v}\cdot\mathbf{v}} \qquad \text{Alt. Cauchy-Schw.: } \mathbf{u}\cdot\mathbf{v} \le |\mathbf{u}|\,|\mathbf{v}| \\
&\ge \sqrt{\mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}} \\
&= \sqrt{(\mathbf{v}+\mathbf{u})\cdot(\mathbf{v}+\mathbf{u})} \\
&= |\mathbf{v}+\mathbf{u}| = |\underbrace{\mathbf{y}-\mathbf{x}}_{\mathbf{v}+\mathbf{u}}| = d(\mathbf{x},\mathbf{y}) \qquad \square
\end{aligned}
$$

The introduction of squares is a bit more natural when the proof is read backwards.

How to define "straight line"?

- Physically: "doesn't move when you pull the endpoints."



👎 Not a formal mathematical definition. Not algebraically tractable.

- "Shortest distance between two points."

👎 Not directly algebraically tractable.

- Symmetry: "same on both sides." Examples:
  - rotation axes 🏀
  - folds ✉️
  - rowing with equal force on each oar 🚣‍♀️

👎 Not directly algebraically tractable.

- $y = mx + b$

👎 $\mathbb{E}^n$ does not intrinsically have a coordinate system. (How would you define a coordinate system without already presupposing the notion of a straight line?)

- "The set of all configurations for which the triangle inequality is an equality."

👍 Based directly on definition of $\mathbb{E}^n$. Algebraically useful. Badass. 😎

Definition of lines in $\mathbb{E}^n$

$$\text{line}(\mathbf{a},\mathbf{b}) = \{\mathbf{x} \in \mathbb{E}^n : \mathbf{x} = \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a}), \lambda \in \mathbb{R}\}$$

$$= \{\mathbf{x} \in \mathbb{E}^n : \mathbf{x} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}, \lambda \in \mathbb{R}\}$$

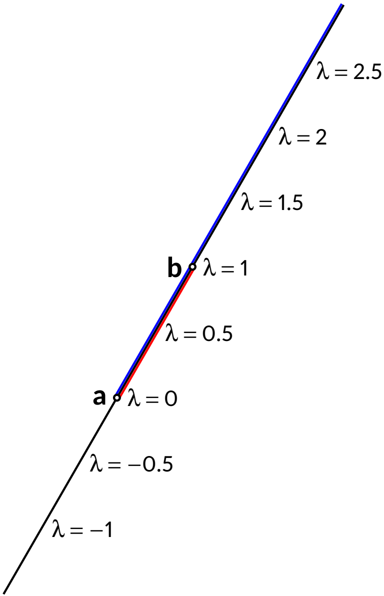
$$= \{\mathbf{a}+\mathbf{r} \in \mathbb{E}^n : \underbrace{|\mathbf{r}\cdot(\mathbf{b}-\mathbf{a})| = |\mathbf{r}|\,|\mathbf{b}-\mathbf{a}|}_{\substack{\text{equality in Cauchy-Schwarz} \\ \Longleftrightarrow \\ \mathbf{r},(\mathbf{b}-\mathbf{a})\text{ linearly dependent}}}\}$$

$$\text{ray}(\mathbf{a},\mathbf{b}) = \{\mathbf{x} \in \mathbb{E}^n : \mathbf{x} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}, \lambda \geq 0\}$$

$$= \{\mathbf{a}+\mathbf{r} \in \mathbb{E}^n : \underbrace{\mathbf{r}\cdot(\mathbf{b}-\mathbf{a}) = |\mathbf{r}|\,|\mathbf{b}-\mathbf{a}|}_{\substack{\text{equality in alt. Cauchy-Schw.} \\ \Longleftrightarrow \\ \mathbf{r},(\mathbf{b}-\mathbf{a})\text{ equidirectional}}}\}$$

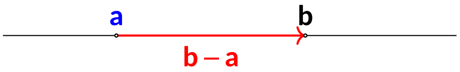
$$\text{segment}(\mathbf{a},\mathbf{b}) = \{\mathbf{x} \in \mathbb{E}^n : \mathbf{x} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}, \lambda \in [0,1]\}$$

$$= \{\mathbf{x} \in \mathbb{E}^n : \underbrace{d(\mathbf{a},\mathbf{x}) + d(\mathbf{x},\mathbf{b}) = d(\mathbf{a},\mathbf{b})}_{\text{equality in triangle ineq.}}\}$$

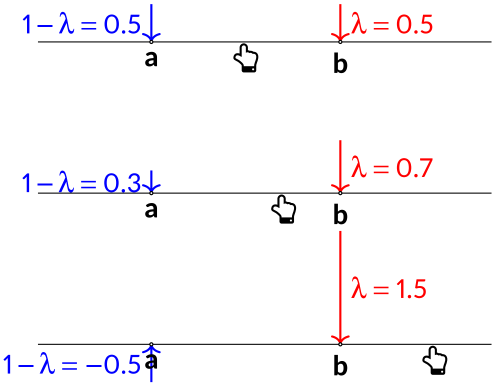


Representations of lines

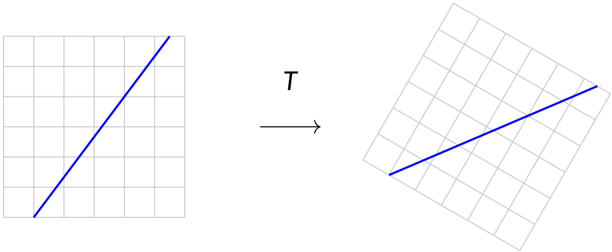
Point-direction form: $\mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$



Weighted-average form: $(1-\lambda)\mathbf{a} + \lambda\mathbf{b}$



Isometries of $\mathbb{E}^n$ preserve lines (map lines to lines)



$$\text{line segment}(\mathbf{a},\mathbf{b}) = \{\mathbf{x} \in \mathbb{E}^n : d(\mathbf{a},\mathbf{x}) + d(\mathbf{x},\mathbf{b}) = d(\mathbf{a},\mathbf{b})\}$$

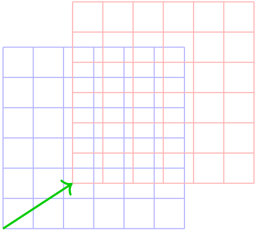
$$\mapsto \{\mathbf{x}' \in \mathbb{E}^n : d(\mathbf{a}',\mathbf{x}') + d(\mathbf{x}',\mathbf{b}') = d(\mathbf{a}',\mathbf{b}')\}$$

$$= \text{line segment}(\mathbf{a}',\mathbf{b}')$$

Translations

A translation T is a transformation that moves all points by the same vector ($\mathbf{v} \in \mathbb{E}^n$):

$$T : \mathbb{E}^n \to \mathbb{E}^n \qquad \mathbf{x} \mapsto \mathbf{x} + \mathbf{v}$$

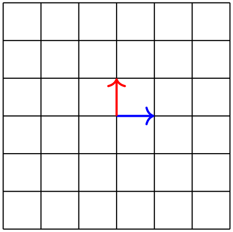


Proof that translations are isometries:

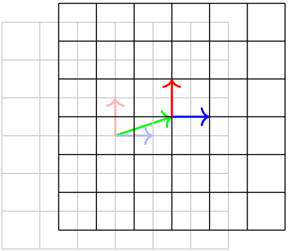
- Distance-preserving.
  $d(T(\mathbf{x}), T(\mathbf{y})) = \left|(\mathbf{y} + \mathbf{v}) - (\mathbf{x} + \mathbf{v})\right| = |\mathbf{y} - \mathbf{x}| = d(\mathbf{x}, \mathbf{y})$
- Bijective.
  - Injective: distance-preserving $\implies$ injective.
  - Surjective: $\mathbf{y} \in \mathbb{E}^n \implies \exists \mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{y}$, namely $T(\mathbf{y} - \mathbf{v}) = \mathbf{y}$ $\square$

Some isometries

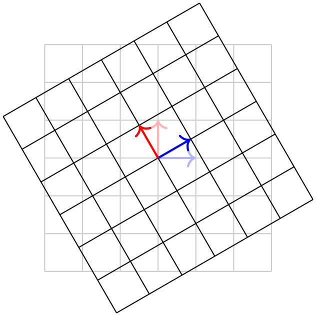
identity



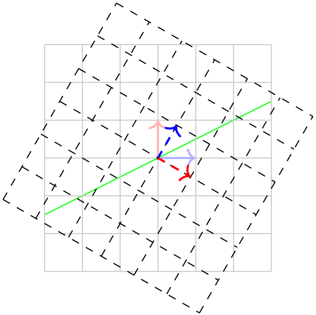
translation



rotation



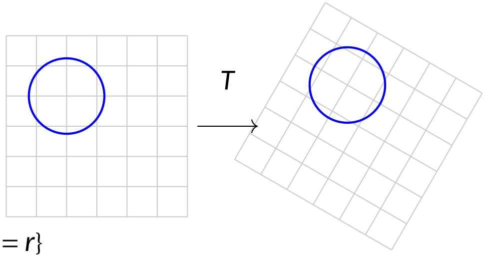
reflection



❓ circles $\mapsto$ circles under $\mathbb{E}^2$ isometries

Given:

circle before T: $C = \{\mathbf{x} \in \mathbb{E}^2 : d(\mathbf{o}, \mathbf{x}) = r\}$



transformed circle: $T(C) = \{T(\mathbf{x}) \in \mathbb{E}^2 : \mathbf{x} \in C\}$

intended image circle: $C' = \{\mathbf{x} \in \mathbb{E}^2 : d(T(\mathbf{o}), \mathbf{x}) = r\}$

Proof that $T(C) = C'$:

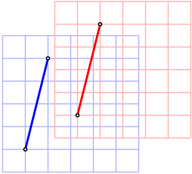
$\mathbf{p} \in C'$	$\iff$	❓	(definition of C')	$\downarrow$
	$\iff$	❓	(since T^{-1} is an isometry)	
	$\vdots$			
	$\iff$	❓	(re-expressed in terms of d)	
	$\iff$	❓	(re-expressed in terms of T^{-1})	$\uparrow$
	$\iff$	$\mathbf{p} \in T(C)$		

Intrinsic $\iff$ definable in terms of d $\iff$ invariant under isometry

Intrinsic:

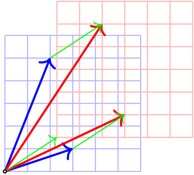
Distances

Lines



Not intrinsic:

$\lvert\mathbf{x}\rvert$	since	$\lvert T(\mathbf{x})\rvert \neq \lvert\mathbf{x}\rvert$	e.g. $\lvert\mathbf{x}+\mathbf{c}\rvert \neq \lvert\mathbf{x}\rvert$ in the case of a translation $T(\mathbf{x}) = \mathbf{x}+\mathbf{c}$
$\mathbf{x}\cdot\mathbf{y}$	since	$T(\mathbf{x})\cdot T(\mathbf{y}) \neq \mathbf{x}\cdot\mathbf{y}$	e.g. $(\mathbf{x}+\mathbf{c})\cdot(\mathbf{y}+\mathbf{c}) \neq \mathbf{x}\cdot\mathbf{y}$
$\mathbf{0}$	since	$T(\mathbf{0}) \neq \mathbf{0}$	

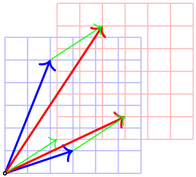


Intrinsic form of vector operations

Not intrinsic:

$|\mathbf{x}|$ since $|T(\mathbf{x})| \neq |\mathbf{x}|$

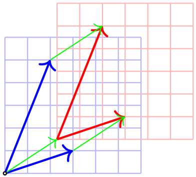
$\mathbf{x} \cdot \mathbf{y}$ since $T(\mathbf{x}) \cdot T(\mathbf{y}) \neq \mathbf{x} \cdot \mathbf{y}$



Intrinsic:

$\left|\overrightarrow{OA}\right| = d(O,A)$

$\overrightarrow{OA} \cdot \overrightarrow{OB} = f\left(d(O,A), d(O,B)\right)?$



Inner product, vector norm interdefinable

Inner product expressed in terms of lengths:

$$\begin{aligned}\mathbf{x}\cdot\mathbf{y} &= \tfrac{1}{2}\left(|\mathbf{x}+\mathbf{y}|^2 - |\mathbf{x}|^2 - |\mathbf{y}|^2\right)\\ &= -\tfrac{1}{2}\left(|\mathbf{x}-\mathbf{y}|^2 - |\mathbf{x}|^2 - |\mathbf{y}|^2\right)\end{aligned}$$

Proof:

$$\overbrace{\mathbf{x}\cdot\mathbf{y}}^{\sum x_i y_i} = \tfrac{1}{2}\Big(\underbrace{\overbrace{|\mathbf{x}+\mathbf{y}|^2}^{\sum (x_i+y_i)^2} - \overbrace{|\mathbf{x}|^2}^{\sum x_i^2} - \overbrace{|\mathbf{y}|^2}^{\sum y_i^2}}_{\sum 2x_i y_i}\Big) \qquad \square$$

Hence $\overrightarrow{OA}\cdot\overrightarrow{OB}$ is intrinsically definable in $\mathbb{E}^n$:

$$\overrightarrow{OA}\cdot\overrightarrow{OB} = -\tfrac{1}{2}\left(d(A,B)^2 - d(O,A)^2 - d(O,B)^2\right)$$

❓ Prove same result without using coordinates?

To prove:

- $\mathbf{x} \cdot \mathbf{y} = -\frac{1}{2}\left(|\mathbf{x}-\mathbf{y}|^2 - |\mathbf{x}|^2 - |\mathbf{y}|^2\right)$

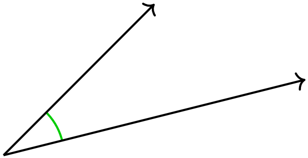
Given (definition of norm and definition of inner product):

- $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$
- $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
- $(a\mathbf{x} + b\mathbf{y}) \cdot \mathbf{z} = a\mathbf{x} \cdot \mathbf{z} + b\mathbf{y} \cdot \mathbf{z}$

Angles

Linear algebra fact:

$$\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + x_2 y_2 + x_3 y_3$$
$$= |\mathbf{x}|\,|\mathbf{y}| \cos \theta$$

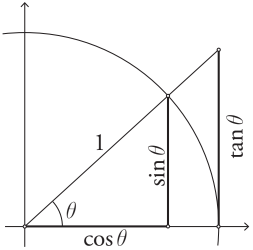


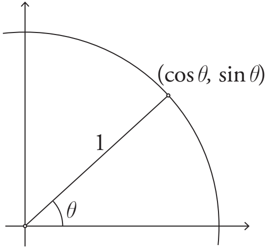
Incorporated in $\mathbb{E}^n$ by defining:

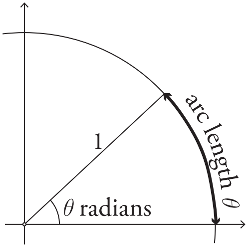
$$\angle(\mathbf{x}, \mathbf{y}) := \arccos \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}| \cdot |\mathbf{y}|} \in [0, \pi]$$

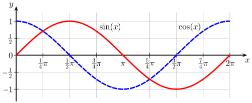
$$\angle BAC := \angle(\mathbf{b} - \mathbf{a}, \mathbf{c} - \mathbf{a}) = \arccos \frac{(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a})}{|\mathbf{b} - \mathbf{a}| \cdot |\mathbf{c} - \mathbf{a}|}$$

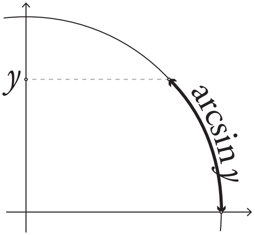
Trigonometric functions visually

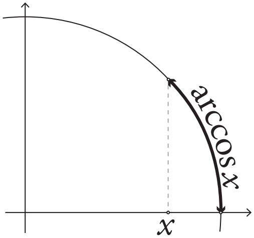


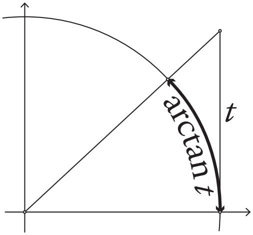


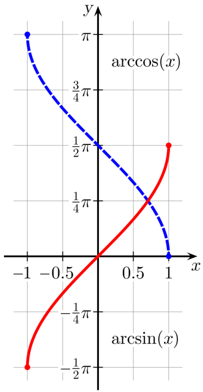












Trigonometric functions arithmetically

$$
\begin{aligned}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
\arcsin(y) &= \int_0^y \frac{1}{\sqrt{1-t^2}}\, dt \\
&= x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \ldots \\
\arccos(x) &= \int_x^1 \frac{1}{\sqrt{1-t^2}}\, dt
\end{aligned}
$$

Recall our rule:

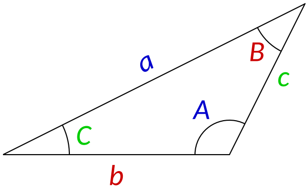
- ✗ Geometric definitions.
- ✓ Arithmetic definitions.

We reduce the foundations of geometry to arithmetic; the foundations of arithmetic, and proving purely arithmetic identities, is “somebody else’s problem.”

Trigonometric identities

Law of sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



degrees	radians	sin	cos
0°	0	0	1
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$
45°	$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$
90°	$\pi/2$	1	0

Law of cosines:

$$a^2 + b^2 - 2ab\cos C = c^2$$

Pythagorean property $\sin^2\theta + \cos^2\theta = 1$

Symmetry properties $\sin(-\theta) = -\sin(\theta)$

$\cos(-\theta) = \cos(\theta)$

Compound angle formulas $\sin(x+y) = \sin x\cos y + \cos x\sin y$

$\sin(x-y) = \sin x\cos y - \cos x\sin y$

$\cos(x+y) = \cos x\cos y - \sin x\sin y$

$\cos(x-y) = \cos x\cos y + \sin x\sin y$

Double angle formulas $\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Half angle formulas $\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}$

$\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}}$

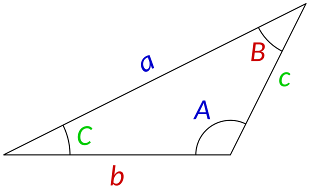
Proof of cosine rule

$$\cos\angle(\mathbf{x},\mathbf{y}) = \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}||\mathbf{y}|} \quad \text{(definition of angle)}$$

$$= -\frac{|\mathbf{x}-\mathbf{y}|^2-|\mathbf{x}|^2-|\mathbf{y}|^2}{2|\mathbf{x}||\mathbf{y}|} \quad \text{(inner product expressed in terms of norms)}$$

Hence:

$$a^2+b^2-2ab\cos C = c^2$$



Gives an easy way to see that angles are intrinsic:

$$\angle BAC = \arccos\left(\frac{d(A,B)^2+d(A,C)^2-d(B,C)^2}{2d(A,B)d(A,C)}\right)$$

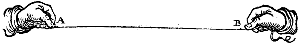
Intuitive versus formal geometry

distance

$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

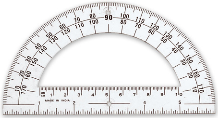
$$d(a,b) = 0 \iff a = b$$
$$d(a,b) = d(b,a)$$
$$d(a,b) \leq d(a,c) + d(c,b)$$

straight line



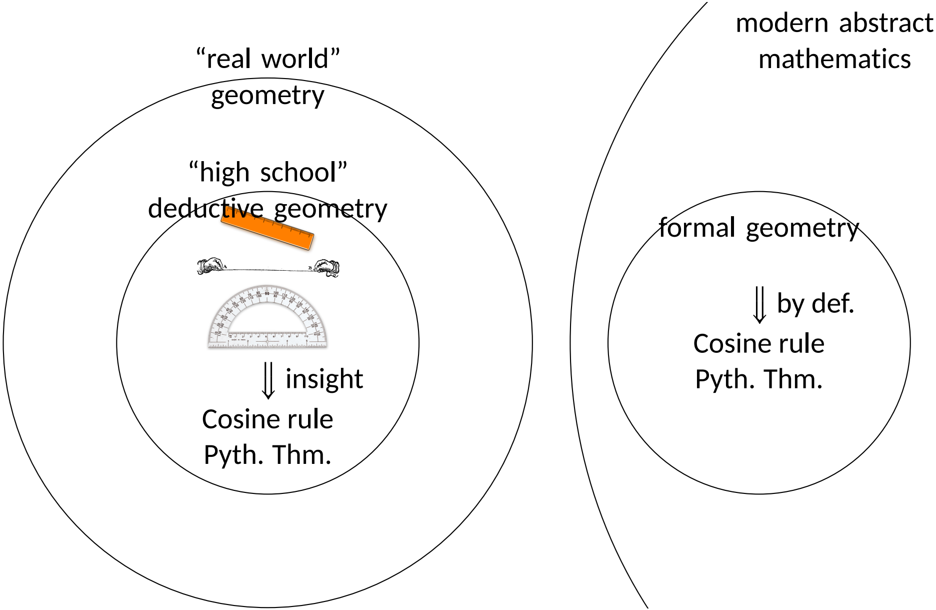
$$\text{line}(\mathbf{a},\mathbf{b}) := \{\mathbf{x} \in \mathbb{E}^n : \underbrace{d(\mathbf{a},\mathbf{x}) + d(\mathbf{x},\mathbf{b}) = d(\mathbf{a},\mathbf{b})}_{\text{equality in triangle ineq.}}\}$$

angle



$$\angle(\mathbf{x},\mathbf{y}) := \arccos \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}| \cdot |\mathbf{y}|}$$

Intuitive versus formal geometry



Triangle congruencies

	ZZZ*	ZHZ	HHZ	HZH	ZZR
	SSS*	SAS	AAS	ASA	SSR

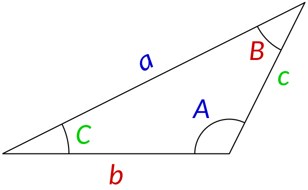
* can be taken as (intrinsic) definition of triangle congruence.

Law of sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Law of cosines:

$$a^2 + b^2 - 2ab\cos C = c^2$$



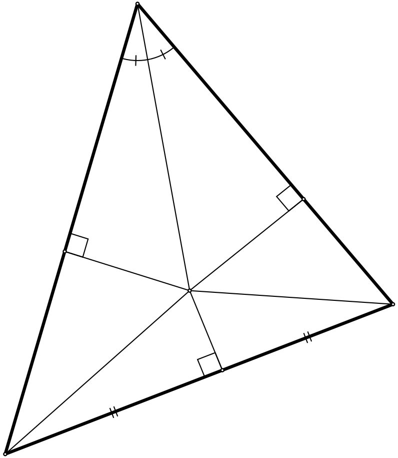
Theorems of “common”/“classical”/“high school” geometry $\subset \mathbb{E}^2$

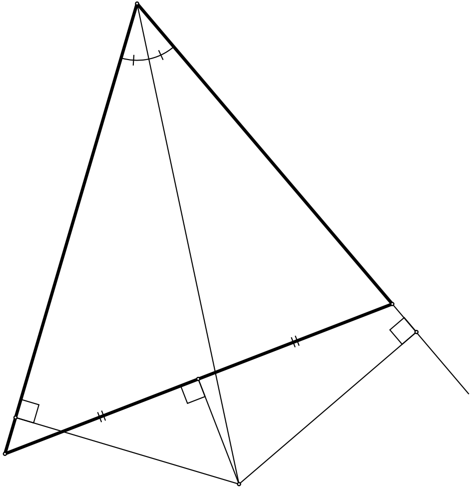
$$\begin{aligned} & \text{cosine rule, triangle congruencies, ...} \\ \Longrightarrow\ & \text{axioms of common geometry} \\ \Longrightarrow\ & \text{theorems of common geometry} \end{aligned}$$

So from now on we can assume “known” results as needed without having to prove everything from first principles in $\mathbb{E}^n$.

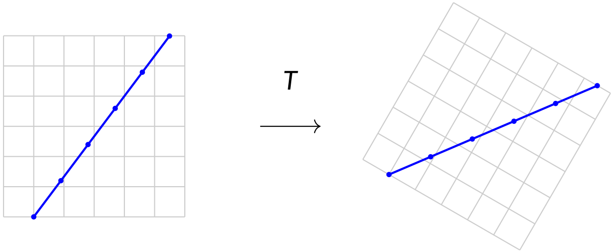
False diagrams: an argument against diagram-based geometry

Triangle congruencies in the left diagram imply that all triangle are isosceles.





Isometries of $\mathbb{E}^n$ preserve lines including their internal metric



Immediate for line segments:

$$\text{line segment}(\mathbf{a},\mathbf{b}) = \{\mathbf{x}\in\mathbb{E}^n : d(\mathbf{a},\mathbf{x}) + d(\mathbf{x},\mathbf{b}) = d(\mathbf{a},\mathbf{b})\}$$

$$\mapsto \{\mathbf{x}'\in\mathbb{E}^n : d(\mathbf{a}',\mathbf{x}') + d(\mathbf{x}',\mathbf{b}') = d(\mathbf{a}',\mathbf{b}')\}$$

$$= \text{line segment}(\mathbf{a}',\mathbf{b}')$$

Extended to lines:

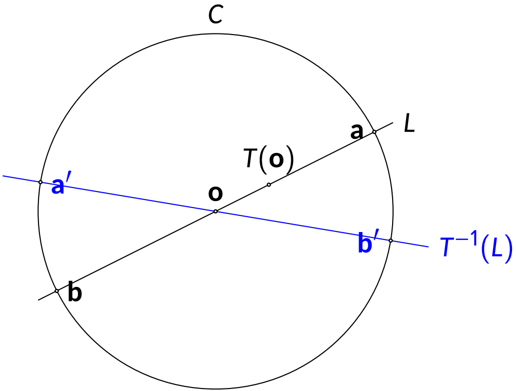
$$T((1-\lambda)\mathbf{a} + \lambda\mathbf{b}) = (1-\lambda)T(\mathbf{a}) + \lambda T(\mathbf{b})$$

(Proof in Opgave 2.6. Alternatively, argue that it follows from the result for segments.)

❓ isometry preserves circle $\implies$ preserves midpoint

Given: midpoint $\mathbf{o} \in \mathbb{E}^2$, circle $C = \{\mathbf{x} \in \mathbb{E}^2 : d(\mathbf{o}, \mathbf{x}) = r\}$, isometry $T : \mathbb{E}^2 \to \mathbb{E}^2$ such that $\mathbf{x} \in C \implies T(\mathbf{x}) \in C$.

Proof that $T(\mathbf{o}) = \mathbf{o}$:



Suppose $T(\mathbf{o}) \neq \mathbf{o}$. Justify the claims:

- ❓ $T(\mathbf{o})$ and $\mathbf{o}$ determine a line L.
- ❓ $T^{-1}(L)$ is a line ...
- ❓ ... passing through $\mathbf{o}$.
- ❓ $T(\{\mathbf{a}', \mathbf{b}'\}) = \{\mathbf{a}, \mathbf{b}\}$
- ❓ $T(\frac{1}{2}\mathbf{a}' + \frac{1}{2}\mathbf{b}') = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$
- ❓ $T(\mathbf{o}) = \mathbf{o}$

Pseudo-linearity of isometries of $\mathbb{E}^n$

Isometries “behave linearly” on certain expressions:

$$T((1-\lambda)\mathbf{a}+\lambda\mathbf{b}) = (1-\lambda)T(\mathbf{a})+\lambda T(\mathbf{b})$$

⚠ But it’s not actual linearity:

$$T(\mathbf{a}+\mathbf{b}) \neq T(\mathbf{a})+T(\mathbf{b})$$

$$T(\lambda\mathbf{a}) \neq \lambda T(\mathbf{a})$$

$$T(\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})) \neq T(\mathbf{a})+\lambda T(\mathbf{b}-\mathbf{a})$$

Counterexample: $T(\mathbf{x}) = \mathbf{x}+\mathbf{v}$.

Isometry minus translation = linear

The pseudo-linearity of an isometry

$$T((1-\lambda)\mathbf{a}+\lambda\mathbf{b}) = (1-\lambda)T(\mathbf{a})+\lambda T(\mathbf{b})$$

implies the linearity of

$$\begin{aligned} S &:= T \text{ minus its translation component} \\ &= T(\mathbf{x}) - T(\mathbf{0}) \end{aligned}$$

Proof:

$$\begin{aligned} S(\lambda\mathbf{x}) &= T(\lambda\mathbf{x}) - T(\mathbf{0}) \\ &= T((1-\lambda)\mathbf{0}+\lambda\mathbf{x}) - T(\mathbf{0}) \\ &= (1-\lambda)T(\mathbf{0}) + \lambda T(\mathbf{x}) - T(\mathbf{0}) \\ &= \lambda(T(\mathbf{x}) - T(\mathbf{0})) \\ &= \lambda S(\mathbf{x}) \end{aligned}$$

$$\begin{aligned} S(\mathbf{x}+\mathbf{y}) &= S\left(2\left(\tfrac{1}{2}\mathbf{x}+\tfrac{1}{2}\mathbf{y}\right)\right) \\ &= 2\cdot S\left(\tfrac{1}{2}\mathbf{x}+\tfrac{1}{2}\mathbf{y}\right) \\ &= 2\left(T\left(\tfrac{1}{2}\mathbf{x}+\tfrac{1}{2}\mathbf{y}\right) - T(\mathbf{0})\right) \\ &= 2\left(\tfrac{1}{2}T(\mathbf{x}) + \tfrac{1}{2}T(\mathbf{y}) - T(\mathbf{0})\right) \\ &= T(\mathbf{x}) + T(\mathbf{y}) - 2T(\mathbf{0}) \\ &= (T(\mathbf{x}) - T(\mathbf{0})) + (T(\mathbf{y}) - T(\mathbf{0})) \\ &= S(\mathbf{x}) + S(\mathbf{y}) \end{aligned}$$

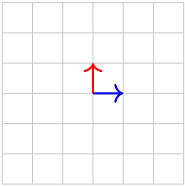
Linearity $\implies$ matrix representation

$$\begin{aligned} & T(\mathbf{x}) - T(\mathbf{0}) \text{ linear transformation } \mathbb{R}^n \to \mathbb{R}^n \\ \implies & T(\mathbf{x}) - T(\mathbf{0}) = M \quad (n \times n \text{ matrix}) \\ \implies & T(\mathbf{x}) = M\mathbf{x} + \mathbf{v} \end{aligned}$$

Furthermore:

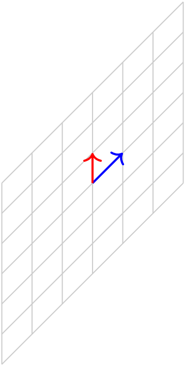
$$\begin{aligned} & T \text{ is } d\text{-preserving} \\ \implies & M \text{ is length-preserving} \\ & |\mathbf{y} - \mathbf{x}| = d(\mathbf{x}, \mathbf{y}) = d(T(\mathbf{x}), T(\mathbf{y})) = |T(\mathbf{y}) - T(\mathbf{x})| \\ & = |(M\mathbf{y} + \mathbf{v}) - (M\mathbf{x} + \mathbf{v})| = |M(\mathbf{y} - \mathbf{x})| \\ \implies & M \text{ is orthogonal (columns are orthonormal)} \end{aligned}$$

Columns of matrix = images of unit basis vectors

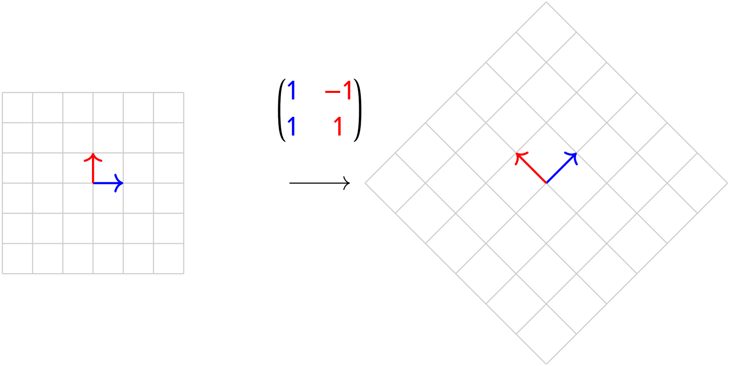


$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

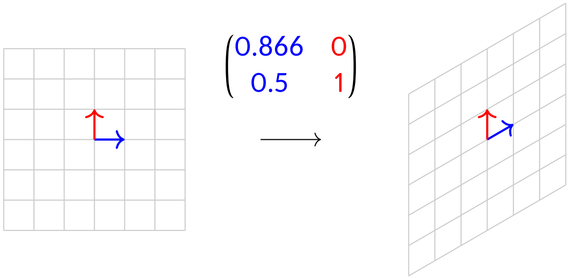
$$\longrightarrow$$



Columns not unit length $\implies$ not isometry

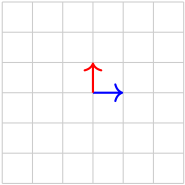


Columns not orthogonal $\implies$ not isometry

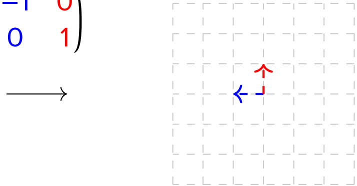


$\det M < 0 \implies$ orientation-reversing ("indirect")

The plane is flipped "upside down." I depict this with dashed lines to suggest the seams on the back of a sown fabric.



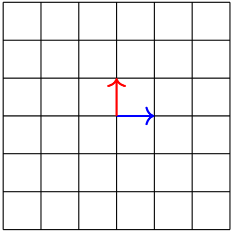
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



Representations of isometries in $\mathbb{E}^2$

identity

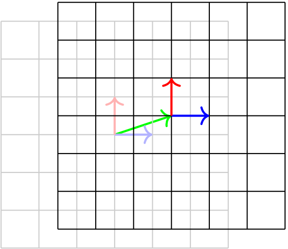
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



translation

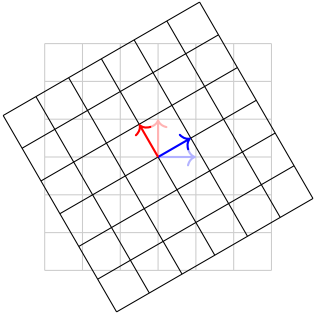
$Tr_{\mathbf{c}}$

$+\mathbf{c}$



rotation R_φ

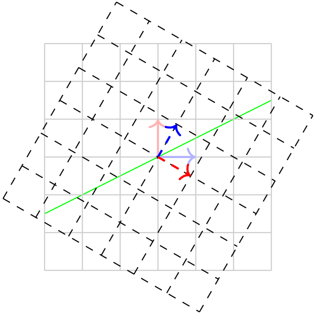
$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$



reflection S_φ

(in line w. angle $\varphi/2$)

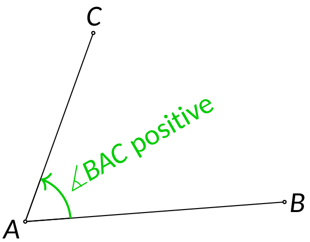
$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$$

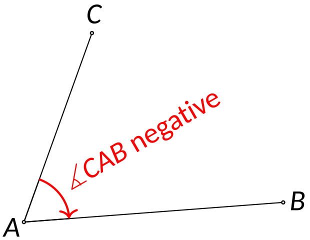


Oriented angles (not intrinsic)

$$\measuredangle(\mathbf{x},\mathbf{y}) := \begin{cases} \angle(\mathbf{x},\mathbf{y}) & \text{if } \det(\mathbf{x},\mathbf{y}) \geq 0 \\ -\angle(\mathbf{x},\mathbf{y}) & \text{if } \det(\mathbf{x},\mathbf{y}) < 0 \end{cases}$$

$$\measuredangle BAC := \measuredangle(\mathbf{b}-\mathbf{a},\mathbf{c}-\mathbf{a})$$

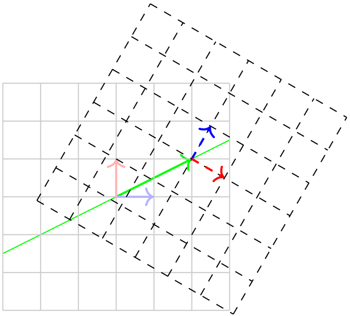




$$\sin\measuredangle(\mathbf{x},\mathbf{y}) = \frac{\det(\mathbf{x},\mathbf{y})}{|\mathbf{x}|\cdot|\mathbf{y}|}$$

Closure of {Tr, Rot, Rfl} under composition = {Tr, Rot, Rfl, Gl}

Glide reflection Gl = reflection ∘ translation along line of reflection.

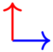


$$\mathrm{Gl}_{\ell,\mathbf{v}} = \mathrm{Tr}_{\mathbf{v}} \circ \mathrm{Rfl}_{\ell} = \mathrm{Rfl}_{\ell} \circ \mathrm{Tr}_{\mathbf{v}}$$

(where **v** is parallel to ℓ)

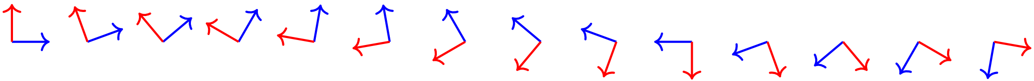
Classification of matrices occurring in $T(\mathbf{x}) = M\mathbf{x} + \mathbf{v}$ in $\mathbb{E}^2$

M orthogonal $\implies M = R_\varphi$ or $M = S_\varphi$ because these are all the ways one can transform

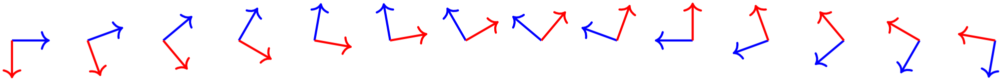


while preserving orthonormality.

R_φ's:



S_φ's:



Classification of isometries of $\mathbb{E}^2$

Any isometry of $\mathbb{E}^2$ = $M\mathbf{x}+\mathbf{v}$ = $R_\varphi\mathbf{x}+\mathbf{v}$ or $S_\varphi\mathbf{x}+\mathbf{v}$.

Each of these two options can be reduced to single isometries:

- $R_\varphi\mathbf{x}+\mathbf{v} = \mathrm{Rot}_{C,\varphi}(\mathbf{x})$ or $\mathrm{Tr}_\mathbf{v}$ if $\varphi = 0$

$$\mathrm{Rot}_{C,\varphi} = \mathrm{Tr}_\mathbf{c} \circ \mathrm{Rot}_{O,\varphi} \circ \mathrm{Tr}_{-\mathbf{c}} = R_\varphi(\mathbf{x}-\mathbf{c})+\mathbf{c} = R_\varphi\mathbf{x}+(\mathbf{c}-R_\varphi\mathbf{c})$$

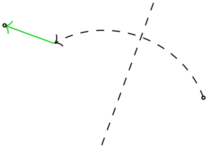
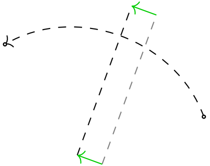
- $S_\varphi\mathbf{x}+\mathbf{v} = \mathrm{Gl}$ or Rfl if $\mathbf{v}=\mathbf{0}$

$$\mathrm{Rfl}_{(\text{general line } m)}(\mathbf{x}) = \mathrm{Tr}_\mathbf{c} \circ \mathrm{Rfl}_{(\ell \text{ through } O)} \circ \mathrm{Tr}_{-\mathbf{c}} = S_\varphi(\mathbf{x}-\mathbf{c})+\mathbf{c} = S_\varphi\mathbf{x}+(\mathbf{c}-S_\varphi\mathbf{c})$$

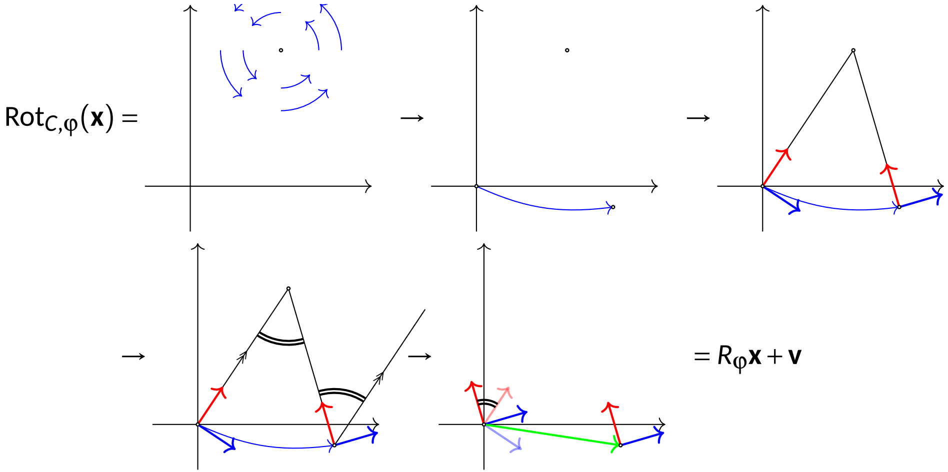
Hence any isometry $M\mathbf{x}+\mathbf{v} \in \{\mathrm{Tr}, \mathrm{Rot}, \mathrm{Rfl}, \mathrm{Gl}\}$.

$S_\varphi \mathbf{x} + \mathbf{v} = \mathrm{Gl}$

$$S_\varphi \mathbf{x} + \mathbf{v} = \mathrm{Rfl}_\ell \mathbf{x} + \underbrace{\mathbf{v}_{\perp\ell}}_{\text{perp. to } \ell} + \underbrace{\mathbf{v}_\ell}_{\text{paral. to } \ell} = \underbrace{\left(\mathrm{Rfl}_\ell \mathbf{x} + \mathbf{v}_{\perp\ell}\right)} + \mathbf{v}_\ell = \mathrm{Tr}_{\mathbf{v}_\ell}\big(\underbrace{\mathrm{Rfl}_{\ell+\frac{1}{2}\mathbf{v}_{\perp\ell}}(\mathbf{x})}\big) = \mathrm{Gl}$$

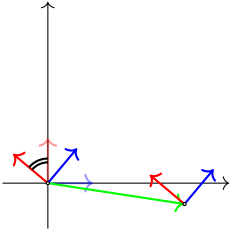
Each $\mathrm{Rot}_{C,\varphi}(\mathbf{x})$ is a $R_\varphi \mathbf{x} + \mathbf{v}$



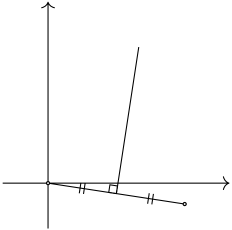
- Agree on 3 points $\implies$ agree on all points.
- $\mathrm{Rot}_{C,\varphi}(\mathbf{x}) \subseteq R_\varphi \mathbf{x} + \mathbf{v}$. Opposite inclusion?

Reconstruction of $\mathrm{Rot}_{C,\varphi}(\mathbf{x})$ from given $R_\varphi\mathbf{x}+\mathbf{v}$

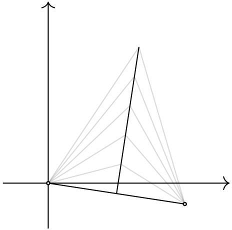
$R_\varphi\mathbf{x}+\mathbf{v} =$



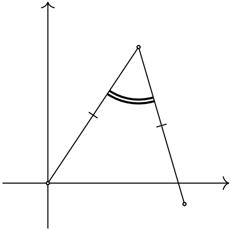
$\rightarrow$



$\rightarrow$



$\rightarrow$



$= \mathrm{Rot}_{C,\varphi}(\mathbf{x})$

Three non-collinear points determine an isometry in $\mathbb{E}^2$

Recall: $T(A)$, $T(B)$ determined $\implies$ "no choice left" for T on line AB. (Since the isometry must preserve the internal metric of the line.)

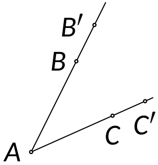
Therefore:

isometries T, T' agree on A, B, C

$T(ABC) = T'(ABC)$

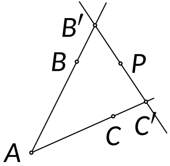
$\implies$ T, T' agree on each $B' \in AB$ and $C' \in AC$

$T(B') = T'(B')$, $T(C') = T'(C')$



$\implies$ T, T' agree on any $P \in B'C'$

$T(P) = T'(P)$

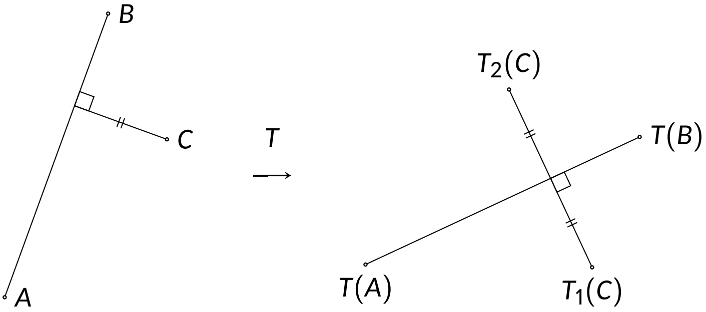


= any point $P \in \mathbb{E}^2$ if A, B, C non-collinear

Degrees of freedom of isometries

$\mathbb{E}^1$	$T(A)$, $T(B)$ determined $A \neq B$	$\Longrightarrow$	no choice left
$\mathbb{E}^2$	$T(A)$, $T(B)$, $T(C)$ determined A, B, C non-collinear	$\Longrightarrow$	no choice left
$\mathbb{E}^2$	$T(A)$, $T(B)$ determined $A \neq B$	$\Longrightarrow$	only one choice left: reflect or not in $T(A)T(B)$

$T(A)$, $T(B)$ determined $\Longrightarrow$ $T_1(C)$, $T_2(C)$ the only possible choices for $T(C)$, since T must preserve distances and angles.

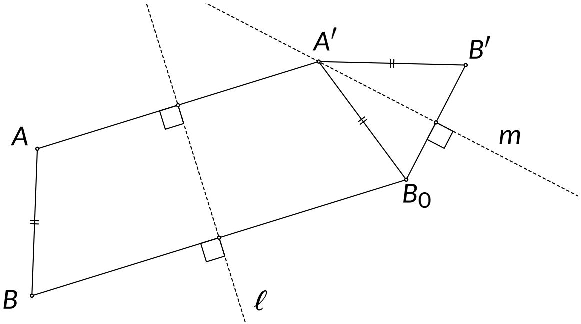


Given $A, B, A', B' \in \mathbb{E}^2$, $d(A,B) = d(A',B')$ construct isometry that sends $A \mapsto A'$ and $B \mapsto B'$

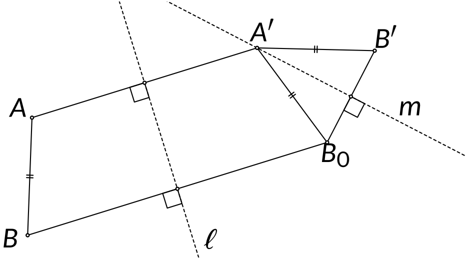
Let

$$\begin{aligned} \ell &:= \text{perpendicular bisector of } AA' \\ B_0 &:= \mathrm{Rfl}_\ell(B) \\ m &:= \text{perpendicular bisector of } B_0B' = \{\text{points equidistant to } B_0,\, B'\} \ni A' \end{aligned}$$

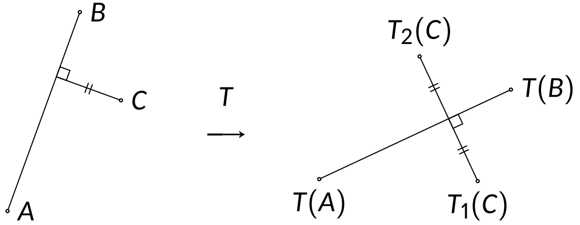
Possible choices for T are $\mathrm{Rfl}_m \circ \mathrm{Rfl}_\ell$ and $\mathrm{Rfl}_{A'B'} \circ \mathrm{Rfl}_m \circ \mathrm{Rfl}_\ell$.



Summary of previous two slides



$$\left|\left\{\mathrm{Rfl}_m \circ \mathrm{Rfl}_\ell,\ \mathrm{Rfl}_{A'B'} \circ \mathrm{Rfl}_m \circ \mathrm{Rfl}_\ell\right\}\right| = 2 \leq \left|\left\{\begin{array}{c}\text{isometries}\\ \text{that send}\\ A,B \mapsto A',B'\end{array}\right\}\right| \leq 2 = |\{T_1,\ T_2\}|$$



“Three reflections theorem” in $\mathbb{E}^2$

$$
\begin{aligned}
\text{any isometry} &= \text{isometry that sends } A, B \mapsto A', B' \\
&= \mathrm{Rfl}_m \circ \mathrm{Rfl}_\ell \quad \text{or} \quad \mathrm{Rfl}_{A'B'} \circ \mathrm{Rfl}_m \circ \mathrm{Rfl}_\ell \\
&= \text{product of two or three reflections}
\end{aligned}
$$

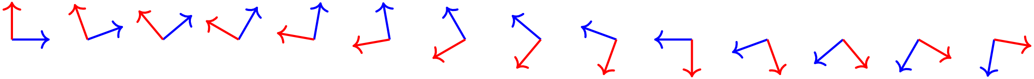
“Construct all isometries such that $A, B \mapsto A', B'$.”

“All isometries are products of two or three reflections.”

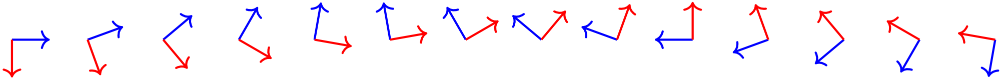
❓ Give a simple argument that $\mathrm{Rfl}_{\ell'} \circ \mathrm{Rfl}_{\ell} = \mathrm{Rot}_{\ell' \cap \ell}$ (assuming $\ell' \cap \ell \neq \emptyset$)

Hint: Recall:

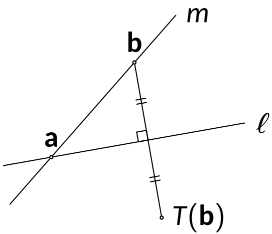
$$R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \qquad \underbrace{\det R_\varphi = 1}_{\text{direct}}$$



$$S_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} \qquad \underbrace{\det S_\varphi = -1}_{\text{indirect}}$$



❓ isometry T of $\mathbb{E}^2$ has precisely 1 fixed point ($T(\mathbf{a}) = \mathbf{a}$) $\implies$ $T = \mathrm{Rot}$



- ❓ $\mathbf{a} \in \ell := \{\mathbf{x} : d(\mathbf{x}, \mathbf{b}) = d(\mathbf{x}, T(\mathbf{b}))\}$ ❓
- ❓ **a** and **b** are fixed points of $\mathrm{Rfl}_\ell \circ T$ ❓
- ❓ There are precisely two isometries with this property ($\mathbf{a} \mapsto \mathbf{a}$, $\mathbf{b} \mapsto \mathbf{b}$), namely:
  - $\mathrm{Rfl}_\ell \circ T =$ ❓

    $\implies T =$ ❓
  - $\mathrm{Rfl}_\ell \circ T =$ ❓

    $\implies T =$ ❓

❓ isometry T of $\mathbb{E}^2$ has precisely 1 fixed point ($T(\mathbf{a}) = \mathbf{a}$) $\implies T = \mathrm{Rot}$

Give a second proof, using the classification theorem for $\mathbb{E}^2$ isometries:

Any isometry of $\mathbb{E}^2$ = $M\mathbf{x} + \mathbf{v} \in \{\mathrm{Tr}, \mathrm{Rot}, \mathrm{Rfl}, \mathrm{Gl}\}$.