Attitudes toward intuition in calculus textbooks

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Abstract

Intuition was long held in high regard by mathematicians, who considered it all but synonymous with clarity and illumination. But in the 20th century there was a strong tendency to vilify intuition and cast it as the opposite of rigorous reasoning. Calculus in particular became a battleground for these opposing views. By systematically surveying references to intuition in historical and modern calculus textbooks, I look at how its status has changed across the centuries. In particular, I argue against the veracity of the self-fashioned origin story of the modern anti-intuition movement, which relies heavily on a particular historical narrative to portray the demise of intuition as an inexorable triumph of logic and reason.

Keywords: intuition, calculus, rigour, pathological functions

The standing of intuition in mathematics has suffered fluctuating fortunes. One school of thought takes it to be the antithesis of proof and careful reasoning. Hans Hahn epitomises this view. In his famous 1933 lecture "The crisis in intuition," Hahn argued that "the failure of intuition" is a historical fact, which had the inevitable consequence that "intuition gradually fell into disrepute and at last was completely banished" from mathematics (Hahn, 1933, pp. 84, 76). Mathematicians thus came to realise that we must

> confess our faith … in careful logical inference … as opposed to bold flights of ideas, mystical intuition, and emotive comprehension (Hahn, 1930, p. 30).

This account has become canonised mathematical folklore (e.g. Bell, 1945, 278, 292, 294, 387; Boyer, 1959, 5, 13, 25, 59; Gray, 2008, 60, 62, 75, 118, 217, 275). It is nowadays a veritable party line in modern textbooks. Calculus students in particular are nowadays inculcated with a narrative that paints intuition as a corrupting temptation that must be resisted. Insofar as intuitive arguments are presented at all, textbooks make sure to undermine them at once by hastening to emphasise that they don't count as real mathematics. "These intuitive arguments do not constitute proofs" (Thomas, 2004, p. 85), we are always warned — a message that can also be efficiently conveyed by denigrating scare quotes: "Intuitive 'Proof' of the Chain Rule" (Thomas, 2004), p. 192). The dichotomy between intuition and proof is an admonition almost all modern calculus textbooks feel obligated to emphasise at every opportunity (Stewart, 2012, pp. 63, 87, 142, 199, 304, 312, 722, 982, 1008; Lang, 1986, pp. 162-163, 176, 218, 306). Likewise, intuitive conceptions of the

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fundamental notions of calculus must also be purged: "the intuitive definition of a limit" is "inadequate" and "vague" (Stewart, 2012, p. 72), and in the same way all other intuitive conceptualisations must be replaced by formal ones (Stewart, 2012, pp. 76, 142, 284, 352, 353). An emblematic formulation in one prominent calculus textbook even makes intuition the direct antonym of mathematics itself: "Intuitively, … Mathematically, …" (Strang, 1991, p. 62). In sum, although some modern calculus books occasionally pay lip service to intuition, their persistent phraseology pitting it against proof and rigour ensures that their most conspicuous message is that intuition is not real mathematics.

The anti-intuition movement has been very successful in portraying itself as the inevitable outcome of rational progress. I challenge this triumphalist narrative. The wave of anti-intuition sentiments that dominated much of the 20th century is not the end of history and the definite "right" view of mathematics; rather, it is an ideology that happened to fit the needs of a particular era. The circumstances that gave rise to it are complex and include internal mathematical developments as well as a broader philosophical context (see e.g. Volkert (1986), Jahnke (1993), and Gray (2008)). Rather than admit this, however, the movement fashioned for itself a more flattering origin story.

According to this standard account, the history of the calculus is a key battleground on which the anti-intuition attitude proved its superiority. An especially decisive proof of the folly of intuition, the story goes, is the existence of continuous, nowhere differentiable functions:

> A curve does not have to have a tangent at every point. It used to be thought, however, that intuition forced us to acknowledge that such a deficiency could occur only at isolated and exceptional points of a curve, never at all points. It was believed that a curve must possess an exact slope, or tangent, if not at every point, at least at an overwhelming majority of them. … Ampère … attempted to prove this conclusion. … It was therefore a great surprise when Weierstrass announced [in 1872] the existence of a curve that lacked a precise slope or tangent at any point. (Hahn, 1933, p. 82)

There are several major problems with this potted history. First of all, Hahn's diagnosis of Ampère's error is driven by his ideological commitment to discredit intuition rather than by a serious analysis of the case. For what grounds are there for taking intuition to be the culprit in his failed proof? Ampère himself doesn't say a word about intuition. Rather he claims that his proof is based on "the most rigorous possible" methods (Ampère, 1806, p. 156). Why not conclude, therefore, that rigorous mathematics is fallible, as opposed to it being intuition's fault? Of course, if one simply defines, as Hahn and others sometimes seem to do, rigorous mathematics to be exact and true mathematics, and "intuition" to be non-rigorous mathematics, then sure enough it follows that "intuition" is to blame for all errors in mathematics.

But this is a terminological sleight of hand, not a historical conclusion as Hahn would have it.

It is simplistic to claim that, according to intuition, any continuous function must be differentiable almost everywhere. The notion of function or curve involved in Weierstrass's proof is a highly formal one. A more balanced and reasonable reaction to Weierstrass's function would be to conclude that the kinds of curves we have geometric intuitions about does not correspond exactly to the particular formal definition of function assumed in this proof. And if the error lies in assuming these two classes to be equal, then this is hardly an error of intuition, but rather the error of making naive and unwarranted assumptions about formal objects.

This was in fact exactly the reaction of many people at the time. Köpcke, for instance, raised the question: To what kind of curve do our intuition apply? Curves generated by the motion of a point, or boundaries between two regions of the plane? Arguably, only the former are 'intuitable,' yet something like the latter is the notion of continuous function required for Weierstrass's proof (Köpcke, 1887, p. 136). Others made very similar points (Klein, 1894, p. 42; Perron, 1911, p. 204).

This more nuanced and less dogmatic view of the role of intuition helps explain why Hahn's ideology did not triumph until a full sixty years after Weierstrass had supposedly provided the clinching argument for it. To be sure, Hahn's view was not without precursors, even quite numerous ones. Nevertheless, given how supposedly compelling the historical evidence allegedly is, it is remarkable how many leading mathematicians of the late 19th century did not follow the script. This includes many of the major figures famous for their work in formalising mathematics and in particular the calculus.

Weierstrass himself, for one, does not seem to have drawn any anti-intuition conclusions from his own work. In his collected works, I count three mentions of the word intuition and its cognates: one in a general discussion of teaching, where he argues that the best teacher not only announces and justifies results but also makes them intuitive (Weierstrass, 1894, vol. III, p. 321), and two other occurrences where providing an intuitive interpretation of particular results is presented as a positive (Weierstrass, 1894, vol. II, p. 237, vol. IV, p. 346).

Set theory quickly became the language for formal, as opposed to intuitive, definitions and proofs in calculus and beyond. But Cantor, when he introduced the notion, had no intention of banning intuition. On the contrary, he explicitly bases the notion of set upon it: "By a set we understand any collection *M* of definite and separate objects *m* of our intuition" (Cantor, 1895, p. 481). Earlier in the century, Dirichlet (1837) had done much the same: when giving his celebrated formal definition of the integral, he explicitly takes area as an intuitively given notion.

Pro-intuition sentiments like these are also the norm in calculus textbooks from the time. Authors state with pride that their treatment is "based on geometrical intuition" (Serret, 1899, pp. iii), and invoke it repeatedly in guiding their exposition (Serret, 1899, pp. 5, 6, 7, 10, 278; Worpitzky, 1880, pp. 752, 754). Gaining "a clear intuition" of the material is explicitly stated as a goal (Bergbohm, 1892–93, pp. I.4, II.97). Giving intuitive meaning to results, beyond their formal content, is highly valued (Lipschitz, 1880, pp. 572, 633; Kiepert & Stegemann, 1897, p. 577; Kiepert & Stegemann, 1894, pp. 12, 320); one example is how the geometrical concept of curvature gives meaning to the quadratic term of a Taylor expansion (Worpitzky, 1880, p. 693). We also regularly find phrases like "as geometrical intuition allows us to recognise easily …" (Harnack, 1881, p. 64), "by means of this intuition, one easily convinces oneself that …" (Lipschitz, 1880, p. 664), and so on. In sum, Courant speaks for a long tradition when he says in his famous calculus book that "it is my aim … to give due credit to intuition as the source of mathematical truth" (Courant, 1927, p. v).

The formal theory of the calculus is of course also acknowledged, but phrases like "this theorem follows already intuitively …, but can also be proved as follows …" (Lübsen, 1855, §158), "intuition teaches us the same thing directly" (Worpitzky, 1880, p. 714), or "this corresponds precisely to our intuition" (Kiepert & Stegemann, 1897, p. 578) suggest that intuition and formalism coexist and are both valid. Rather than one being real mathematics and the other only half-baked pseudo-understanding, they are both useful perspectives. For any given situation or purpose, one or both may be suitable for the task at hand. Nobody is saying that intuition can do everything, or that it must be banished from mathematics. It is notable that explicit support for intuition along such lines comes even from some of the pioneers of rigorous real analysis. Dedekind, for example, envisions such a balance:

> Resort to geometric intuition in a first presentation of the differential calculus, I regard as exceedingly useful, from the didactic standpoint, and indeed indispensable if one does not wish to lose too much time. But that this form of introduction into the differential calculus can make no claim to being scientific, no one will deny. (Dedekind, 1872, p. 1)

Lipschitz too is famous for his formal analysis work, but in his calculus textbook intuition is by no means shunned. He does temper the role of intuition with the warning that

> the geometrical interpretation serves only to make the analytically defined concepts more graspable with the help of intuition, not as foundation for the proofs. (Lipschitz, 1880, p. 502)

But he does not carry this insistence as far as modern textbooks. For instance, he is perfectly happy to consider the notion of area to be intuitively given when defining the integral (Lipschitz, 1880, p. 102), and in general he is keen to highlight that "the geometrical intuition and its analytic representation correspond to one another" (Lipschitz, 1880, p. 10).

Toeplitz was another leading analyst who saw a positive role for intuition:

> The greater number of students do not yet possess the same ability for abstract thinking in their first hour of university lectures, but have, rather, a hunger for intuitive and productive notions. The intuitive path aims to satisfy that hunger. Kiepert-Stegemann, in its earliest editions, is a perfect example of this trend carried out in a pure way; this work must contain a spark of real didactic genius from which it derives its success. (Toeplitz, 1926/2015, p. 298)

In his own semi-historical calculus textbook, Toeplitz does not repeat Hahn's story of the inevitable demise of intuition. Instead we find phrases like "this is the computational equivalent of the intuitively seen fact that …" (Toeplitz, 1949, p. 51) which treat intuition as a viable viewpoint that is respectable and indeed often equivalent to formal methods.

Altogether, Hahn's quasi-historical narrative about the cleansing of intuition is at odds with the historical record. Hahn uses a caricature of history to justify his ideological stance. Time and time again, key mathematicians who by Hahn's logic should have despised intuition instead give it a respectable place in mathematical thought.

I have used calculus textbooks in particular as indicators of the mathematical community's attitudes toward intuition. I have done this for two main reasons. Firstly, Hahn's narrative is based primarily on examples drawn from calculus. Secondly, although there is a vast philosophical literature on intuition—including authors with much affinity to mathematics such as Descartes, Kant, and Brouwer—my concern is not with philosophy but with the working mathematician's everyday attitude toward intuition. Calculus textbooks, I would argue, is where the rubber hits the road and we see how philosophical commitments play out in actual, hands-on mathematics.

I tried to extend my investigations also to earlier time periods, but in this endeavour I met with limited success. I went through many textbooks and did full-text searches for intuition and cognate terms in various languages, but I found that older texts contain very few explicit mentions of intuition. The vast majority of books never mention intuition at all. This includes the following: Wallis (1656), Leibniz (1678-1714), Newton (1687), Bernoulli (1692), l'Hôpital (1696), Ditton (1706), Berkeley (1734), Reyneau (1736), Reyneau (1738), Deidier (1740), Maclaurin (1742), Simpson (1750), Kästner (1770), Tempelhoff (1770), Lagrange (1797), Lacroix (1802), Neubig (1817), Cauchy (1821), Jephson (1826), Hall (1837), Raabe (1839), Snell (1846), Spencer (1847), Church (1850), Miller (1852), Woolhouse (1852), Autenheimer (1856), Price (1857), Smyth (1859), Greene (1870), Williamson (1877), Dölp (1878), Todhunter (1881), Byerly (1882), Knox (1884), Bass (1887), Bayma (1889), Kleyer (1889), Jordan (1896), Perry (1897), Czuber (1898), Murray (1898), Lorentz (1900), Thompson (1910), Landau (1934).

Nevertheless one can say something about the earlier period as well. I believe it is safe to say that intuition was held in high regard in the early history of the calculus. Like many other leading figures, Leibniz never mentioned intuition in his mathematical publications. But in his philosophical works he was very positive toward it. For example:

> The most perfect knowledge is that which is both adequate and intuitive. (Leibniz, 1969, p. 291) When my mind grasps all the primitive ingredients of a concept at once and distinctly, it possesses an intuitive knowledge. This is very rare, since for the most part human knowledge is … confused. (Leibniz, 1969, p. 319)

Leibniz surely considered mathematics no exception to these general pro-intuition convictions. Interestingly, I did find an old calculus textbook that expresses the same idea—of intuition as the opposite of confusion—specifically in the context of the calculus: "… then our Knowledge will be … more intuitive: … Now we see through a Glass darkly, or in a Riddle; but then Face to Face." (Stewart, 1745, p. 478, echoing Corinthians 13:12) I believe this passage can be taken as quite indicative of attitudes toward intuition in the early calculus generally.

The 18th century, as is well known, saw the calculus turn away from geometry and become heavily focussed on analytic expressions. It is natural that this would be accompanied by a diminished estimation of intuition. Indeed, in Euler's calculus textbooks we do find two down-putting references to intuition: "… which we will be able to prove rigorously … In the meantime it is not so difficult to see intuitively that this is true." (Euler, 1755, §170) "The truth of these formulas is intuitively clear, but a rigorous proof will be given …" (Euler, 1748, §166)

But, as Schubring (2005) has observed,

> soon after 1800, however, the pendulum moved the other way … The defamed synthetic method was restored as dominant value, and the requirement that concepts be generalizable was replaced by that of their being of easy intuitive grasp. (p. 152)

A clear expression of this revival of intuition is found in Carnot (1813), who explicitly argues that intuition is favourable to algebraic analysis:

> Far from using analysis to establish elementary truths, we must disengage them from all that prevents us from perceiving them as distinctly as possible. … Those who succeed in making us see almost intuitively the results to which

> we had only arrived before them by the aid of a complicated analysis, do they not always procure us as much pleasure as surprise …? (§161)

Thus the low tide of intuition in the era of Eulerian analysis soon gave way to "a new dominance of geometry, in the name of intuitiveness" (Schubring, 2005, p. 295).

In conclusion, intuition was largely held in high regard by leading practitioners of the calculus for nearly 200 years. It suffered a temporary dip in fortunes in the 18th century, but this was due to no fault of its own but rather to a focus on the programmatic algebraisation of mathematics. Its stock again plummeted in the first half of the 20th century. Perhaps the reasons were much the same this time, namely a zeal for a tout-court programmatic reform of mathematics along symbolic and analytic lines. The leaders of this movement, however, succeeded in portraying the demise of intuition not as their ideological doing but as an inevitable historical conclusion objectively forced upon us all by factual mathematical developments. It has been my primary goal in this essay to challenge this ingrained narrative. If history is any indication, the time may well be ripe for intuition to bounce back once again like it did two hundred years ago.

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