Mathematics for Poets

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2025

Contents

§ 0. Syllabus

Content. In this course we take a big-picture look at the nature of mathematics and its role in human thought, emphasising its interactions with society, history, philosophy, science, and culture. The course involves studying key mathematical arguments in some detail, but our goal is not to develop a repertoire of technical skills. Instead, we study accessible mathematical topics specifically selected for their rich interconnections with cultural context, and integrate our mathematical work with reflections on its broader meaning and implications. Instead of the drill and practice problems of a traditional mathematics class, we approach mathematics through seminar discussions, hands-on activities, and readings connecting it to broader issues. We thus analyse a selection of emblematic and important mathematical proofs and use them as a platform for reflecting on the nature of mathematics. In parallel, we read excerpts from seminal historical texts across the ages as well as modern scholarship from a wide range of academic disciplines that shed light on the interplay between mathematics and its societal and intellectual context. We focus especially on geometry, from the origins of mathematical reasoning in early civilizations, to Euclid's Elements that was the gold standard of exact reasoning for millennia and the model for countless philosophical systems, to the projective geometry of Renaissance art, to the more modern non-Euclidean geometry that overturned conventional wisdom about the nature of human spatial perception and the shape of space.

Aims. After completing the course students are able to:

- Discuss critically, and situate in scholarly and historical context, reflections on the nature of mathematics.
- Draw on a range of examples to discuss the role of mathematics in society, culture, and human thought.
- Read simple mathematical proofs and explain the ideas and methodologies involved.
- Reason rigorously and conceptually about fundamental notions in a simple geometric setting.
- Relate fundamental aspects of mathematical reasoning to its broader meaning and purpose.
- Discuss the nature of mathematical knowledge and axiomatic-deductive systems, especially on the basis of key foundational concepts of Euclidean and non-Euclidean geometry.

Format. We interleave mathematical topics with seminar discussions of a rich array of short readings that connect the material to a broader cultural, philosophical, and historical context. We study a selection of mathematical proofs and topics drawn from geometry with an emphasis on conceptual understanding. To this end we supplement textual mathematical sources with physical models and hands-on activities that allow us to experience and explore geometry in a concrete way.

Class preparation. The course involves three components:

- Seminar discussion. These are the sections based on lectures or other readings from various sources and associated discussion questions marked 💬. For these classes it is essential to come prepared to contribute to a discussion with your own questions, reflections, and insights. My "lectures" for this course are in the form of a podcast. In this document I link to the relevant episodes of the podcast (marked 🎙) or other web resources we will use by means of the symbol ↗. This symbol indicates a clickable link that you can open in a web browser.

- Activities. These are the sections that outline a mathematical idea and sets up some mathematical problems or investigations, often involving problems recognisable by the ☐ symbol. For these sections, it may be a good idea to look at the text before class and read the "story" part of it; the problems and activities we will then tackle together in class.
- Euclid readings. These are the sections whose title starts with "Elements." For these classes you should read the indicated parts of Euclid's *Elements* carefully before class. We will then discuss the associated problems and discussion questions in class.

Assessment.

25% Midterm exam

25% Final exam

Midterm and Final exams will consist of:

- 3 questions based on our seminar reading and discussions. These questions will be drawn from the seminar discussion questions marked 💬 in this document.
- One question consisting of one or more of the tables and figures from this document, with some parts blanked out. Your task will be to fill in the blanks with suitable text.
- One question asking you to prove a particular proposition from Euclid. This will be one of the following propositions:
  - Midterm: 2, 5, 9, 11, 35, 47, 48
  - Final: 13, 15, 16, 27, 32, 46

  You will provide a step-by-step proof with a figure and a justification for each step. Justifications can refer to previous propositions, postulates, and definitions from Euclid. At the exam, you will be allowed to use the single-page reference table of Euclidean propositions.

40% Assignments

Assignments are the questions recognisable by the ☐ symbol. Most of these questions we will investigate together during Euclid and Activity classes. As we figure them out, you will take notes and later submit your answers.

10% Participation

Participation means active engagement in discussions. Criteria for an A grade:

- For each Seminar class, you come prepared with a question or comment that you think could make for interesting discussion, in addition to at least one good bullet point in reply to each of the assigned discussion questions.
- For each Euclid class, you have studied the assigned proofs, and you have looked at the assigned discussion questions and have at least an idea or question to get you started toward collectively resolving the questions in class.
- In discussions, you make substantive connections across different readings and materials in the course, or with other relevant background knowledge you may have.
- In discussions, you make an effort to understand the points of view of others and engage with them thoughtfully.

§ 1. Cultural history of trigonometry

The following questions refer to these videos: ↗ ↗ ↗ ↗

1.1. Match the cultural, geographical, or contextual aspects with their associated applications of trigonometry.

☐ Religious ceremonies, Particular plants, Existence of mountains, Particular buildings, Expanding empire

☐ Eratosthenes, Chinese, al-Biruni motivation, al-Biruni choice of method, Snellius

1.2. Match the measurement methods with the assumptions or conditions needed for them.

☐ Horizon visibility, City-to-city visibility, Rays from sun parallel, Rays from sun not parallel, Eclipses

☐ Aristarchus, Eratosthenes, Chinese, al-Biruni, Snellius

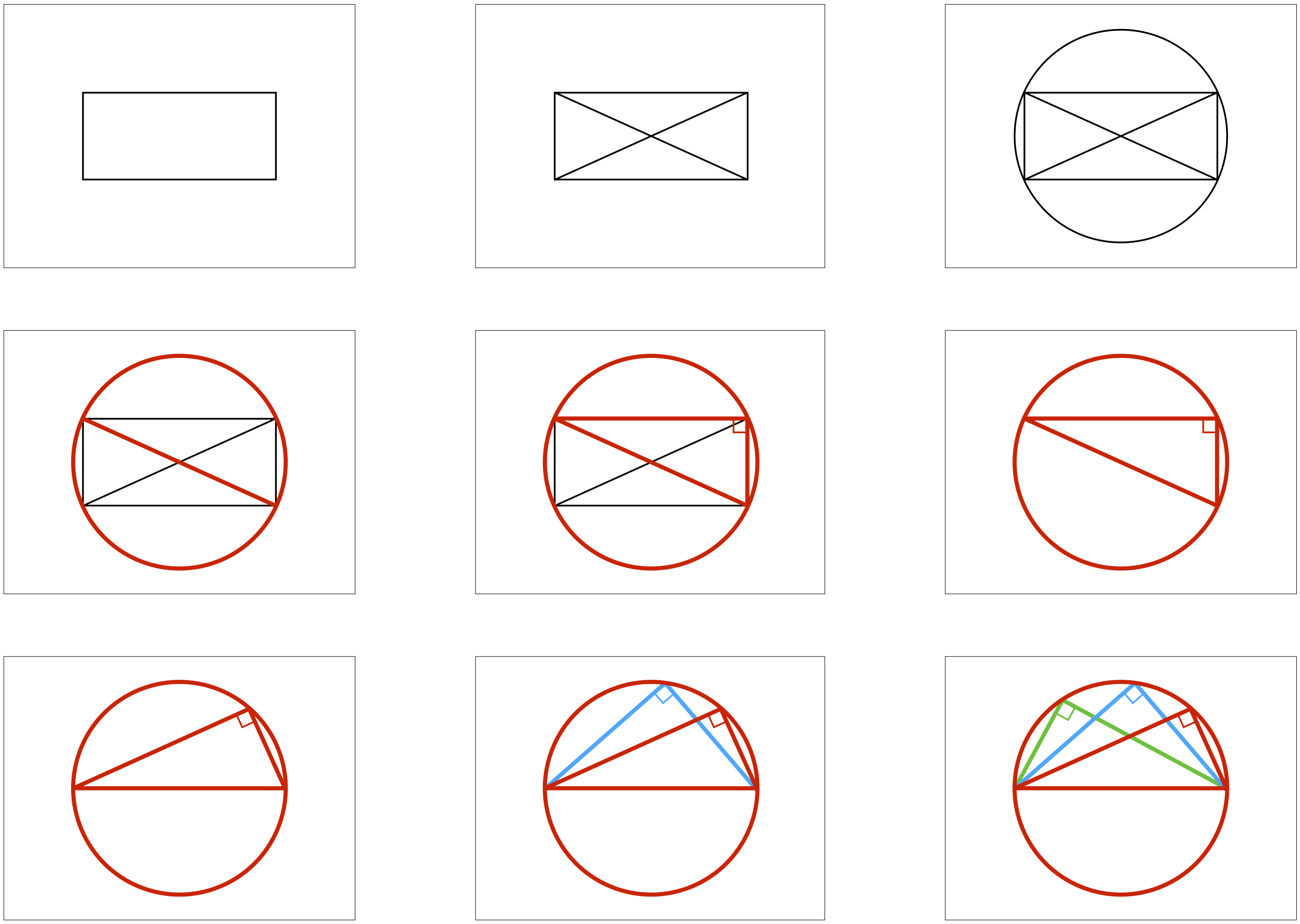
§ 2. Beginnings of mathematics

“First proofs: Thales and the beginnings of geometry”

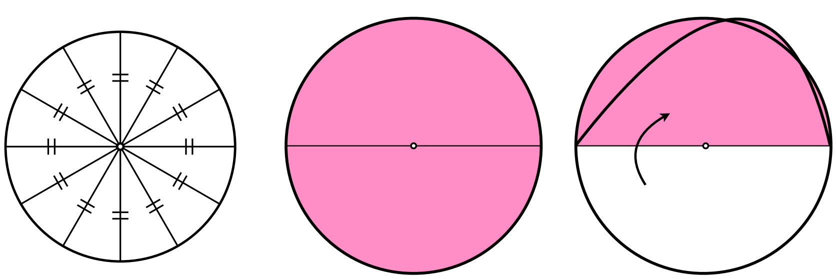
Pages 1–7 of “Lockhart’s Lament”

2.1. In what sense can the theorems attributed to Thales have been the beginning of deductive mathematics?

“Thales’ Theorem” about triangles inscribed in circles:



Proof by contradiction that diameter cuts circle in half:



The Greek tradition has it that Thales (c. −600) was the first to introduce deductive reasoning in geometry. Two of the results he supposedly proved were:

- A diameter of a circle cuts the circle into two equal halves.

- Triangles raised on the diameter of a circle all have a right angle ("Thales' Theorem").

Since this was the very beginning of geometry, Thales had to prove these things from scratch, or using arguments that his audience would accept as "obviously" true. He did not use any "formulas" or any other established principles from some textbook, because there were no textbooks. This is why it is interesting to try to recreate his steps: Forget everything you have heard about geometry and do it yourself! Enjoy the freedom of mathematical exploration! Create something out of nothing!

Here are some hints if you get stuck. The first theorem is so "obvious" that it's hard to know where to start. In such cases, a useful trick is often to use a proof by contradiction: instead of proving that the theorem is true, prove that it could not *not* be true. That is to say, assume for the sake of argument that a diameter in fact cuts a circle into two pieces that are *not* equal. Now try to prove that this is impossible. You can do this by proving that this assumption implies absurd consequences.

The second theorem probably calls for a rather different strategy. There are many ways of proving it. I will give you a hint that sets you on one fruitful path. Draw a rectangle. Get yourself a ruler and compass. How can you play or doodle with those tools, starting with the rectangle? What lines and circles feel most natural to add to the picture? Perhaps you will find yourself with a "Thales' Theorem" configuration before you know it. What does this mean?

I hope these examples gave you a taste of discovery and exploration and mathematics "in the wild," without the artificiality and shackles of a modern mathematics textbook. "The essence of mathematics is its freedom."[1] This is how mathematicians think of their subject. It is to their great regret and frustration that others perceive their subject very differently. The following reading argues this point passionately.

2.2. In "Lockhart's Lament," the author compares mathematics to art. Which of the following are true for both mathematics and art, according to the author?

- ☐ Schools destroy interest in it.
- ☐ Children have a natural interest in it.
- ☐ It cannot exist without creative expression.
- ☐ Most people don't understand what it's all about.
- ☐ Modern students are less receptive to it because of smartphones and the Internet.

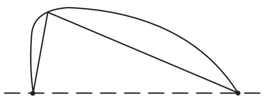
Let's consider an interesting application of Thales' Theorem. In Virgil's *Aeneid*, we encounter a woman who appears to have been mathematically educated: Dido, daughter of the king of Tyre, a major city in antiquity, whose ruins can still be seen in present-day Lebanon.

Dido was forced to flee her home and ended up on the north coast of Africa, where she struck a bargain to buy as much land as she could enclose with an oxhide.

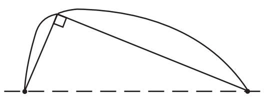
[1] Georg Cantor, *Mathematische Annalen*, Bd. 21, 1883, p. 564. "Das Wesen des Mathematik liegt gerade in ihrer Freiheit."

She cut the hide into thin strips, and then she faced the mathematical problem of enclosing the largest possible area within a given perimeter. Perhaps drawing on her royal mathematical education, it seems she decided to form a semi-circular perimeter next to the shoreline.

2.3. Show that this was a wise decision as follows. Suppose Dido had made a shape that was not a semicircle. Then by Thales' Theorem there will be some point on the boundary where lines drawn from the endpoints on the shoreline meet at an angle that is not right.



Think of there as being a void inside the triangle and think of the pieces on the sides as glued on. Slide the endpoints along the symmetry line to make the angle right.



Argue that this increases the area, and that hence a non-semicircular shape is sub-optimal.

Despite her mathematical acumen, Dido did not get to rule over her round city for very long. Aeneas, on his quest to found Rome, is shipwrecked and blown ashore at Carthage. Dido falls in love with him, but he does not return her love. He sails away and Dido kills herself.

"And so an ungrateful and unreceptive man with a rigid mind caused the loss of a potential mathematician. This was the first blow to mathematics which the Romans dealt."[2]

§ 3. Elements 47–48: Pythagorean Theorem

"Read Euclid backwards: history and purpose of Pythagorean Theorem"

[2] Morris Kline, *Mathematics for the Nonmathematician*, Dover, New York, 1985, p. 135.

3.1. 💬 A mathematical proof can serve many purposes: to explain (create an "aha" moment for the reader), the carefully verify (a result that may be doubted), to exhibit logical relations within a formal system, to reduce complex statements to more basic fundamental principles, etc. Discuss which of these goals of proofs seem to be primary in Euclid's proof of the Pythagorean Theorem (and perhaps the Elements generally).

Read Euclid's *Elements*, Propositions 47–48.

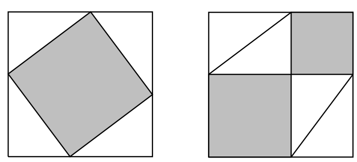
Euclid's *Elements* is one of the most important and influential works in human history. We shall read enough of it to get a sense of why. "Euclid alone has looked on beauty bare,"[3] as the poets say.

The first reading assignment is Euclid's proof of the Pythagorean Theorem. Soon we will go back and read the *Elements* from the beginning, but starting with the Pythagorean Theorem will help us appreciate the earlier parts. If we read Euclid from cover to cover, we get a strictly "bottom-up" perspective: we start with the most basic things and gradually get to higher and higher levels of sophistication. That is how mathematics is typically written down, and with good reason. But the way mathematics comes into being is a much more bidirectional process. Mathematics grows like a tree: as the branches extend, so do the roots. Starting our Euclid adventure with the Pythagorean Theorem is a way of making us think about this.

You will find when you read Euclid's proof of the Pythagorean Theorem that it is based on earlier results. We will study them in turn later, but for now it is enough to refer to the overview reference table and visual summary of the theorems of the *Elements* to know what previous results Euclid has in mind.

If we visualise the Pythagorean Theorem as the apex of a pyramid, the proof reveals which lower, more foundational stones it rests on. Those stones in turn rest on other stones, and so on. Something has to be the bedrock that is considered solid enough not to need any further support beneath it. Euclid's *Elements* can be read in two directions: as a way of *building up* a more and more elaborate structure on top of solid foundations, or as a way of *reducing* advanced results to their basic components. Indeed, by the time Euclid write the *Elements*, the theorems themselves—such as the Pythagorean Theorem—had been known for hundreds or even thousands of years. But it is one thing to have a bunch of theorems, and it is quite another to systematically derive all of them from an extremely restrictive set of basic assumptions. Euclid is very much concerned with the latter. So when we read the proof of the Pythagorean Theorem, one of the perspectives we should use is to think of it as "boiling down" this somewhat advanced result to more basic ones. This will help us appreciate the purpose and achievement of the more fundamental part of the *Elements* when we get to those.

3.2. Explain how the figure below proves the Pythagorean Theorem. Why did Euclid prefer his "windmill" proof?



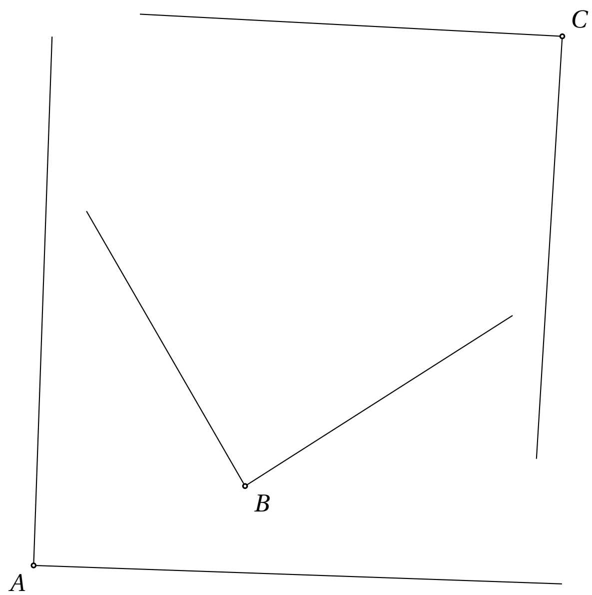
3.3. Which is a right angle? Use Proposition 48 to find out.

☐ A

☐ B

☐ C

[3] Edna St. Vincent Millay (1892–1950).



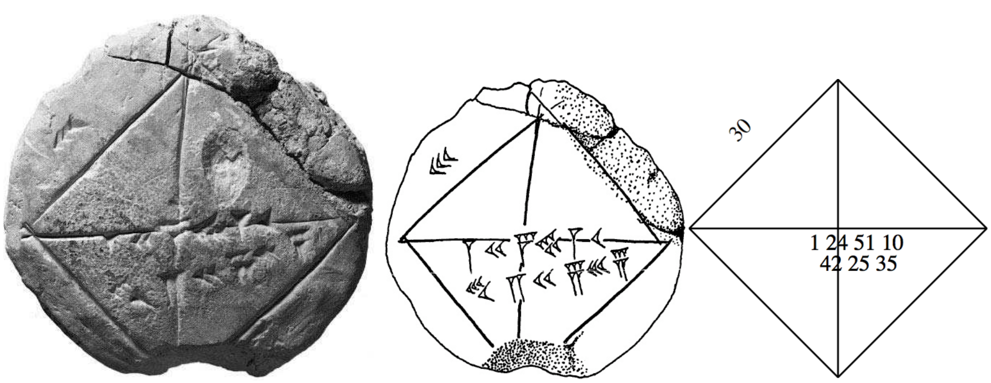
3.4. There are some indications that the ancient Egyptians knew that a triangle with sides 3, 4, 5 has a right angle. Is it realistic that they used this to construct right angles for practical purposes, such as when building the pyramids?

- ☐ Yes
- ☐ No

3.5. In , what is presented as the most likely motivation for the discovery of the Pythagorean Theorem?

- ☐ Determination of the sizes of fields.
- ☐ Determination of geographical distances.
- ☐ Practical engineering questions such as what size ladder is needed for a certain task.
- ☐ Astronomical applications having to do with eclipses.
- ☐ No practical motivation, rather general interest in mathematical challenges/play/curiosity.

The Pythagorean Theorem has little to do with Pythagoras. It was in effect known long before. It is arguably implied in this ancient Babylonian clay tablet from almost 4000 years ago:



The numbers are written in sexagesimal (base 60) form, a Babylonian invention that still lives in our systems for measuring time and angles. This means that, for example, $42,25,35 = 42 + \frac{25}{60} + \frac{35}{60^2}$.

3.6. (a) Explain the meaning of the numbers on the tablet. Hint: there are three numbers and one of them is $\approx \sqrt{2}$.

(b) Convert the tablet's value for $\sqrt{2}$ into decimal form.

(c) If you use this value to compute the diagonal of a square of side 100 meters (i.e., roughly the size of a football field), how big is the error?

- ☐ less than 1 mm
- ☐ 1 mm – 1 cm
- ☐ 1 cm – 5 cm
- ☐ 5 cm – 10 cm
- ☐ 10 cm – 20 cm
- ☐ 20 cm – 50 cm
- ☐ more than 50 cm

§ 4. Textual aspects of Greek mathematics

🎙 "Singing Euclid: the oral character of Greek geometry"

4.1. 💬 Is the written record a good representation of Greek geometrical thought?

4.2. 💬 Were Greek ways of recording mathematics constrained by technology? By tradition?

4.3. 💬 The way the Elements is written is in some ways comparable to songs, poems, tape recordings, or old movies. Discuss.

4.4. Euclid's propositional statements are set up in a form suitable for:

- ☐ mental visualisation
- ☐ phonetic memorisation
- ☐ constructive drawing

§ 5. Elements 34–41: parallelograms; area

5.1. The simplest sense in which two figures can have the same area is that of superposition: one "fits" on top of the other. This sense of equality is invoked in Euclid's Common Notion 4. But figures can also have the same area without being capable of such alignment; that is, without having the same shape. The first time Euclid talks about equality of area in this sense is in Proposition:

- ☐ 34
- ☐ 35
- ☐ 37
- ☐ 41

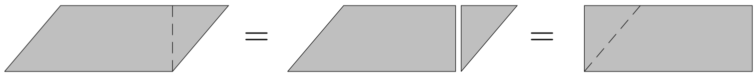
5.2. In Propositions 34–41, Euclid's strategy for proving areas equal is to:

- ☐ Transform one area into another with an area-preserving transformation.
- ☐ Cut them into pieces and use triangle congruence.
- ☐ First establish area of three-sided figures, then use this to establish area of four-sided figures, and so on upwards.

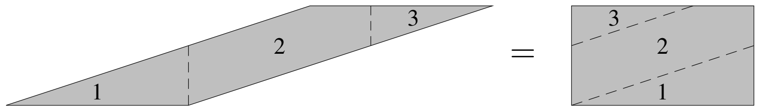
5.3. Proposition 35. If AD and EF overlap:

- ☐ Euclid's proof can be considered to still apply, with G being above the parallelograms.
- ☐ A longer proof is needed.
- ☐ A shorter proof is possible.
- ☐ The proposition no longer holds.

5.4. (Related to Proposition 35.) Parallelograms can be shown to have the same area by cutting and rearranging. In this case a single cut is sufficient:



For this more slanted parallelogram, two cuts are needed:



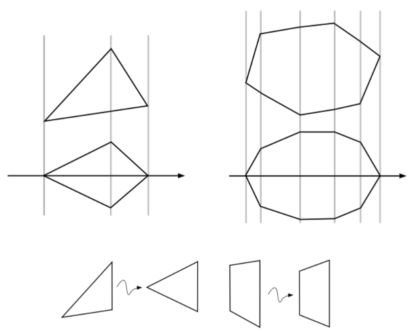
Using this method of cutting, which of the following are true?

- ☐ If part of the top side is above the bottom side, then one cut will be enough.
- ☐ If no part of the top side is above the bottom side, two cuts will always be enough.
- ☐ The more slanted the parallelogram, the greater the number of cuts needed, without bound.

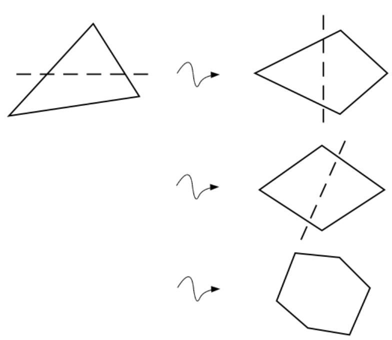
5.5. The enemy is advancing toward your capital. They have currently set up camp not far from you. In the night, you send a spy to estimate the size of their army. The spy reports back that the enemy camp has the shape of a parallelogram and is 4000 paces all the way around. It is estimated that each soldier uses about 10 square paces of area for their night camp. Approximately how many soldiers does the enemy have in the camp?

- ☐ 4000
- ☐ 40000
- ☐ 400000
- ☐ 1000
- ☐ 10000
- ☐ 100000
- ☐ Cannot be determined from the information given.

5.6. This process is called "symmetrisation":



In particular, it turns parallelograms into rectangles, which can be likened to straightening out a stack of books that has been knocked askew. Here is the symmetrisation process applied repeatedly to a specific figure:



The symmetrisation process is akin to packing a snowball; both these processes lead to a round shape for the same reason.

Among all figures with the same area, the circle has the least perimeter. I can prove this using symmetrisation if I first assume or prove that:

philosophy	both	mathematics
disagreement rival schools “subtractive” (critiques, refutations)	hyper-critical dialectic reason > experience rationalism > empiricism deduction > induction definitions of terms crucial in theory-building	consensus unity “additive” (cumulative knowledge)

Table 1: Similarities and differences between early Greek philosophy and mathematics.

- ☐ Symmetrisation preserves area but decreases perimeter for any non-circular figure.
- ☐ Symmetrisation has no effect on circles.
- ☐ Among all figures with the same area, there exists one that has the least perimeter.

§ 6. Why the Greeks?

🎙 “Why the Greeks?”

6.1. 💬 Why did the conception of mathematics as a field of knowledge characterised first and foremost by rigorous deductive reasoning arise in ancient Greece?

6.2. 💬 What does early Greek philosophy have in common with mathematics? What aspects of mathematics fits well with this view, and what aspects less well?

6.3. 💬 Which of the factors that led to mathematics in ancient Greece were or were not present in other cultures, such as for example Renaissance Europe (where mathematics was embraced) or the Roman Empire (where it was not)?

6.4. Why did the conception of mathematics as a field of knowledge characterised first and foremost by rigorous deductive reasoning arise in ancient Greece? The root cause, if anything, is arguably:

- ☐ economics
- ☐ politics
- ☐ geography
- ☐ interaction of cultures

6.5. Early Greek thought was:

- ☐ combative
- ☐ collaborative
- ☐ egalitarian
- ☐ reverential
- ☐ linked with practice

6.6. The axiomatic-deductive method arose due to:

- ☐ The nature of geometry specifically.
- ☐ The nature of mathematics generally.
- ☐ Philosophical critiques.
- ☐ The discovery of multiple possible geometries.
- ☐ Errors encountered in intuitive reasoning.

§ 7. Elements Defs.–Prop. 3: foundations; ruler and compass

7.1. Each of Euclid’s definitions may be considered a characterisation of a certain class of entities X. But in what sense? Match each of the following types of definitions with the best example from Euclid.

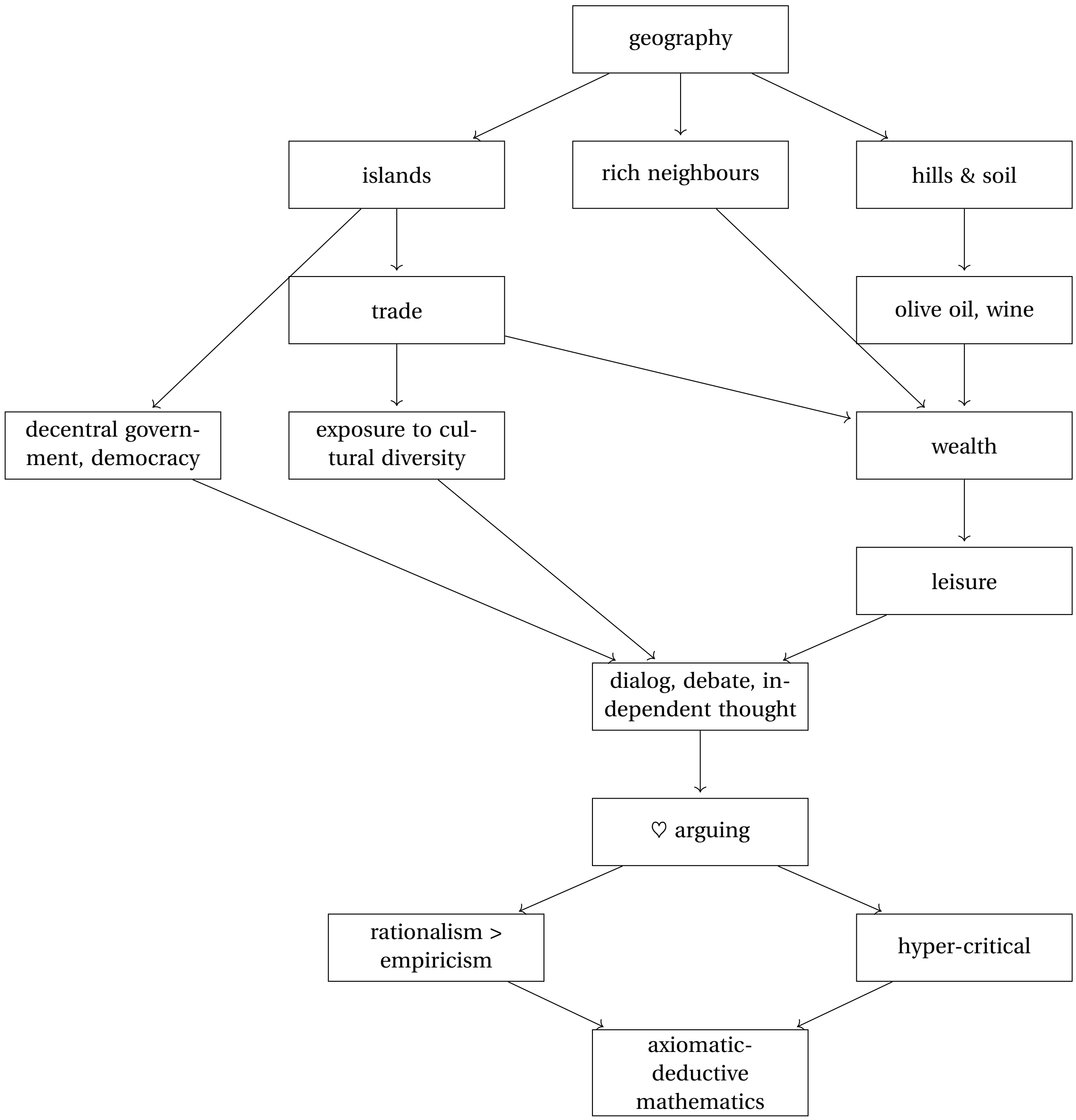


Figure 1: Factors influencing the emergence of rigorous mathematics in ancient Greece.

- ☐ “Test-condition” definition that enable us to answer, for any given object(s), the question “is this an instance of X?” in an unambiguous way that guarantees us to reach a yes or no answer by following a practically executable set of steps.
- ☐ “Exclusion” definition that specifies X negatively, in terms of what it is not.
- ☐ “Maker’s” definition that implies a way of producing an X.
- ☐ “Shorthand” definition that give a convenient label to an entire set of properties or conditions.
- ☐ “Psychological” definition that give you a hunch what it is about without being mathematically exact.
- ☐ “Fiat property” definition that stipulates the properties of X “by fiat” or “because I say so” (as a last resort when the more satisfying definition types above are not available).
- ☐ Definitions 1, 4, 10, 15, 22, 23

7.2. Match the concepts with how it would most naturally be defined.

- ☐ Test-condition, Fiat property, Maker’s, Shorthand, Psychological, Exclusion
- ☐ prime number, magnetic, cousin, stack [of pancakes, for example], intuitive, unicorn

7.3. Euclid’s definition of a straight line is vague. Can we interpret it to say: a straight line is a curve such that any piece of it fits anywhere else on the curve?

- ☐ Yes
- ☐ No

7.4. Euclid’s Proposition 1 has a logical gap or hidden assumption in it, namely that:

- ☐ C exists
- ☐ AC, BC are straight
- ☐ the two circles have the same radius
- ☐ the triangle produced is unique when in fact there are two

7.5. The gap in Euclid’s Proposition 1 could be resolved or addressed at least in part by an argument based on:

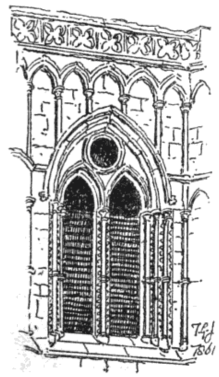
- ☐ Carrying out the constructions with ruler and compass.
- ☐ Allowing conclusions based on what is visually clear in the diagram, as long as it concerns not exact properties but only properties that would still hold even if the diagram was imperfectly drawn.
- ☐ The notions of “inside” and boundary in Definitions 13-15.

7.6. Propositions 2 and 3 suggest that:

- ☐ Postulate 3 is more restricted than one might think.
- ☐ Euclid’s compass is “collapsible”: when lifted from the paper, it collapses and forgets the radius it was set to.
- ☐ Euclid’s compass is “rusty”: once set, it’s stuck at a particular opening.
- ☐ Euclid’s ruler is unmarked, just a “straightedge,” that can be used to draw lines but not to measure lengths.
- ☐ Euclid’s ruler can be used to produce equal line segments (as if you can put a single mark on it corresponding to the segment you have), but not to produce segments of other sizes (as if you had a full numerical scale on it).

7.7. In the Declaration of Independence of the United States, the first part of the second sentence has a very Euclidean ring to it. It is reminiscent of one of Euclid’s postulates in particular—which one? Postulate ☐.

7.8. The Gothic style of architecture arose in the early 12th century, within a decade or two of the first Latin translation of Euclid’s *Elements*. It uses a distinctive window design:



This is reminiscent of what part of Euclid's *Elements*?

§ 8. Plato and Aristotle on geometry

"Consequentia mirabilis: the dream of reduction to logic"

"What makes a good axiom?"

8.1. What makes mathematical knowledge special? What sets it apart from other fields?

8.2. What is consequentia mirabilis? Illustrate with examples.

8.3. In the Platonic worldview, what is the relation between mathematics and physical reality? What are strengths and weaknesses of this view?

8.4. What aspects of Euclid's *Elements* or other technical mathematics fit especially well with empirical, rationalistic, or logical interpretations of mathematics respectively?

8.5. In Rafael's famous fresco "The School of Athens," Plato is pointing toward the sky and Aristotle is pointing straight ahead. Why?

8.6. Are there examples of theories in which the principles are not primitives, or the primitives are not principles, in Aristotle's sense? (Cf. §22.)

<table>
<tr><td rowspan="2"></td><td colspan="3">Axioms of mathematics are based on:</td></tr>
<tr><td>the physical world, generalised experience</td><td>the mind, intuition</td><td>pure logic</td></tr>
<tr><td rowspan="5">Examples:</td><td colspan="2">Euclid's Postulates</td><td></td></tr>
<tr><td colspan="3">Euclid's Definitions, Common Notions</td></tr>
<tr><td>Newton's law of gravity</td><td></td><td></td></tr>
<tr><td></td><td></td><td>consequentia mirabilis, e.g.:</td></tr>
<tr><td></td><td colspan="2">"I think therefore I am."
"we ought to philosophise"</td></tr>
<tr><td>View of:</td><td>Aristotle, Newton</td><td>Plato, Proclus</td><td>modern mathematics</td></tr>
<tr><td>Associated philosophy:</td><td>empiricism</td><td colspan="2">rationalism</td></tr>
<tr><td>Reading Euclid:</td><td>backwards</td><td colspan="2">forwards</td></tr>
<tr><td rowspan="3">Problems:</td><td>uniqueness of math.
(≠ botany, anthropology)</td><td></td><td></td></tr>
<tr><td></td><td colspan="2">"unreasonable effectiveness" of applied math.</td></tr>
<tr><td></td><td></td><td>consistency
(e.g. Russell's Paradox)</td></tr>
</table>

§ 9. Elements 4–11: triangle congruence; constructions

9.1. Propositions 1-3 were "problems" (showing how to do or make something) while Proposition 4 is a "theorem" (showing that a property or relation holds for certain objects). The difference is signalled by fixed stock phrases, most notably in the ______ paragraph.

- ☐ First
- ☐ Second
- ☐ Third
- ☐ Last

9.2. Proposition 7. The proof doesn't work if:

- ☐ AC=CB and AD=DB.
- ☐ C is on AD.
- ☐ C is inside triangle ADB.

9.3. Proposition 8. How can we conclude that the remaining pairs of angles are equal?

- ☐ It follows from the proof of Proposition 8 itself.
- ☐ It follows by applying Proposition 8 anew to the same triangles but with a different choice of "base".
- ☐ It can be obtained by applying Proposition 4.
- ☐ We can't; they may not be equal.

9.4. Why does Euclid bisect an angle (Proposition 9) before a length (Proposition 10)?

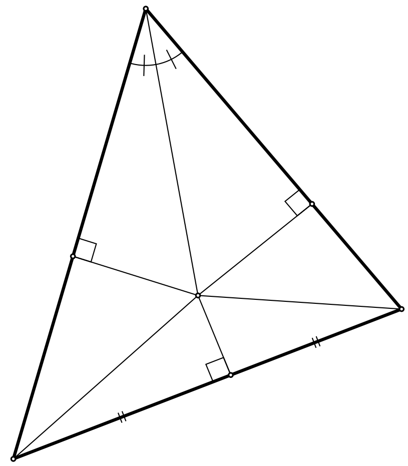
- ☐ Because the ruler-and-compass steps of the corresponding constructions proceed in this order.
- ☐ Because the only previous proposition that has equalities of sides in its conclusion is SAS.
- ☐ Because we have a postulate about equal angles (Postulate 4) but none about equal lengths.
- ☐ The order is an arbitrary choice since the two propositions are independent.

9.5. Proposition 9. Suppose I follow Euclid's construction up to the point when DE has been drawn. If I divided DE into three equal parts (assuming for the moment that I can do this somehow), and then connected the dividing points up to A, will I have cut the angle into three equal parts?

- ☐ Yes
- ☐ No

9.6. Consider the figure below. The point in the middle is defined as the intersection of the bisector of the top angle and the perpendicular bisector of the base. By construction, the top two triangles are congruent, and the two base triangles are congruent. It follows that the remaining two triangles are congruent. Which triangle congruence principles were needed to establish these three congruences?

- ☐ AAS
- ☐ ASA
- ☐ SAS
- ☐ SSS
- ☐ SSRA

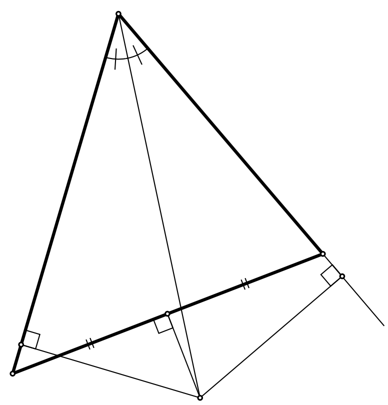


9.7. (cont.) Since we started with an arbitrary triangle, it follows that all triangles are:

- ☐ right-angled
- ☐ isosceles
- ☐ bisected in area by perpendicular bisector of each side
- ☐ decomposable into six triangles with equal area

9.8. (cont.) Actually the figure was incorrectly drawn. Below is the correct figure. On the basis of this example one could argue that:

- ☐ It can be dangerous to rely on diagrams when writing geometrical proofs.
- ☐ Exact constructions are important to safeguard rigour in geometrical reasoning.



§ 10. Ruler and compass

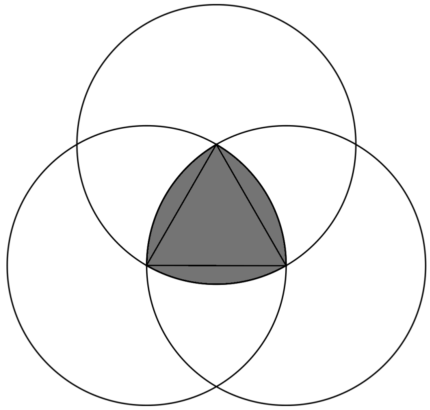
🎙 "Let it have been drawn: the role of diagrams in geometry"

10.1. 💬 (Returning question:) Is the written record a good representation of Greek geometrical thought?

10.2. 💬 (Returning question:) Were Greek ways of recording mathematics constrained by technology? By tradition?

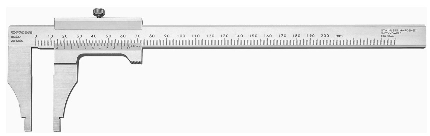
10.3. 💬 Was Euclid a Platonist who reasoned about eternal, abstract objects, or did he think of geometry as something physically produced by ruler and compass? What can we conclude in this regard from the passive formulations of construction steps in his proofs?

10.4. Consider the "fat triangle" (shaded in the picture below) obtained by adding one more circle to the construction of Euclid's Proposition 1.

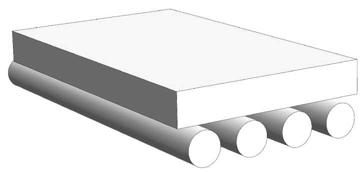


Which of the following are true? Hint: You may want to use cardboard cutouts to investigate this.

☐ If we measure the width of a "fat triangle" with a caliper, the result is the same in every direction.



☐ If the rollers in the setup below had "fat triangle" cross-sections, the platform would still roll as smoothly as before (without bobbing up and down).



☐ The "fat triangle" has a "midpoint" which is equidistant from the perimeter in every direction.

10.5. The website "Below the Surface" catalogues items found at the bottom of an excavated canal in Amsterdam. Do a keyword search for "compasses." Roughly from what year are the oldest compasses that were found in the canal? Perhaps they were thrown into the canal by frustrated geometry students.

☐ 1200

☐ 1400

☐ 1600

☐ 1800

§ 11. Constructivism

"Why construct?"

11.1. Why did constructivist philosophies of the foundations of mathematics and science gain popularity in the 20th century?

11.2. In what ways can 20th-century constructivist philosophy be considered similar to what Euclid was doing?

11.3. Suppose I add to Euclid's *Elements* the definition: "A superright triangle is a triangle each of whose angles is a right angle." So its angle sum is three right angles. But we also know from Proposition 32 that the angle sum of a triangle is two right angles. Therefore $3 = 2$. How can we avoid this paradox? How can guarantee that mathematics is free of such contradictions?

11.4. How do we know that mathematics does not contain statements that can be proven both true and false?

§ 12. Early modern reflections on mathematical method

"Maker's knowledge: early modern philosophical interpretations of geometry"

↪	"I think therefore I am"	(by "consequentia mirabilis")
⟹	my thoughts are actual	
	my thoughts include the idea of perfection	
⟹	something perfect (= God) must exist	(something perfect cannot be caused by something less perfect)
⟹	God is not a deceiver	(because a deceiver is not perfect)
⟹	intuitions (innate ideas) are reliable	(since implanted by the creator = God = not a deceiver)
⟹	Euclid's axioms are reliable	(since they are intuitive)

Table 2: Descartes, *Principles of Philosophy*, 1644, I.7–30.

12.1. 💬 How did 17th-century philosophers interpret the purpose of constructions in Euclidean geometry?

12.2. 💬 What implications did they draw from this for philosophy and other branches of knowledge?

12.3. 💬 What did Descartes see as the key aspects of geometrical reasoning that were to be generalised to other domains of thought?

12.4. 💬 What is the relation between God and geometry, according to Descartes?

12.5. 💬 What aspects of mathematics can be construed as showing that its proofs merely demonstrate propositions logically without explaining why they are true?

12.6. 💬 How can mathematics be defended against the charge that it is inferior to other sciences because its proofs are not explanatory and causal?

§ 13. Societal role of geometry in early civilisations

🎙 "Societal role of geometry in early civilisations" ↗

13.1. 💬 What aspects of mathematics were especially crucial for the societal function mathematics served in early civilisations, and why?

13.2. 💬 Mathematics textbooks often contain "pseudo-applications": problems that are supposedly about the real-world scenarios, but that are hopelessly unrealistic and artificial. Argue that history vindicates such problems and suggests that they serve some rational purposes.

§ 14. Mathematics and culture in early modern Europe

🎙 "Cultural reception of geometry in early modern Europe" ↗

📖 "When pirates studied Euclid" ↗

14.1. 💬 What is the mathematical justification for the Renaissance recipe for drawing a tiled floor (given by Alberti, *De pictura*, 1435)?

14.2. 💬 In what way has perspective painting played a role in broader philosophical and scientific developments?

14.3. 💬 Do a Google Images search for Galileo's moon drawings. Discuss.

14.4. 💬 In the early modern period, who had respect for mathematics, and why? Who were critical of mathematics, and why?

14.5. 💬 Which themes in these debates are timeless, and which were only relevant in their particular context?

14.6. Which of the following uses of mathematical methods are mentioned in the article "When pirates studied Euclid"?

- ☐ determine position by observation of the night sky
- ☐ determine time by observation of the night sky
- ☐ use geometry for carpentry-related problems on the ship
- ☐ determine the course needed to catch another ship
- ☐ determine when high and low tides will occur
- ☐ determine the area of a sail suitable for given requirements (such as the weight of the ship etc.)

☐ determine the angle of travel given the angle of the wind and the sail

§ 15. Perspective art

For a visual introduction to this topic, see

15.1. Prove the following basic facts about perspective painting.

(a) Parallel lines meet at one point in the picture. Like railroad tracks, for example.

(b) The horizon is always perpendicularly in front of the observer's eye.

(c) Parallel lines meet at the horizon.

15.2. You see a vacation photo of a person standing on a beach with the ocean behind her. The horizon is aligned with the top of her head. How tall was the person who took the photo?

☐ Taller than the subject.

☐ As tall as the subject.

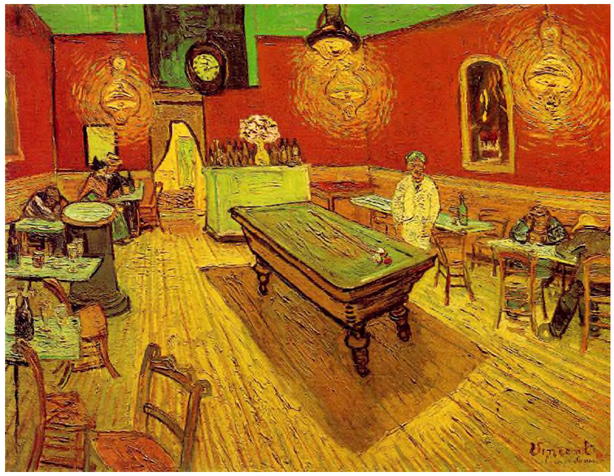
☐ Shorter than the subject.

15.3. Which of the following paintings follow correct mathematical perspective?

☐ Annunciation, Leonardo Da Vinci

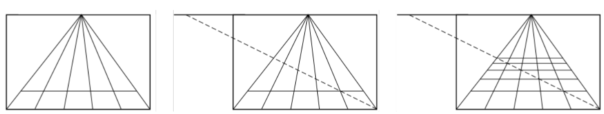
☐ The Miracle of the child falling from the balcony, Simone Martini

☐ The Night Cafe, Van Gogh



☐ Piazza San Marco, Canaletto

Alberti's *Della pittura* (1435) gives the following construction of the perspective view of a tiled floor:



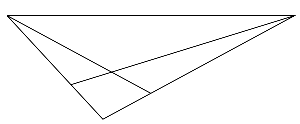
15.4. Explain why this construction gives a true perspective view of a tiled floor.

15.5. Can you use the same principles to draw an oblique view of a tiled floor, starting from a figure like the one below?

☐ Yes, and it will be the perspective image of congruent and square tiles.

☐ Yes, and it will be the perspective image of congruent though not necessarily square tiles.

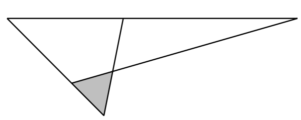
☐ No, this starting configuration does not contain sufficient information to complete the true perspective view of the floor.



15.6. Can you use the same principles to draw an oblique view of a floor tiled with congruent triangles, starting from a figure like the one below?

☐ Yes.

☐ No.



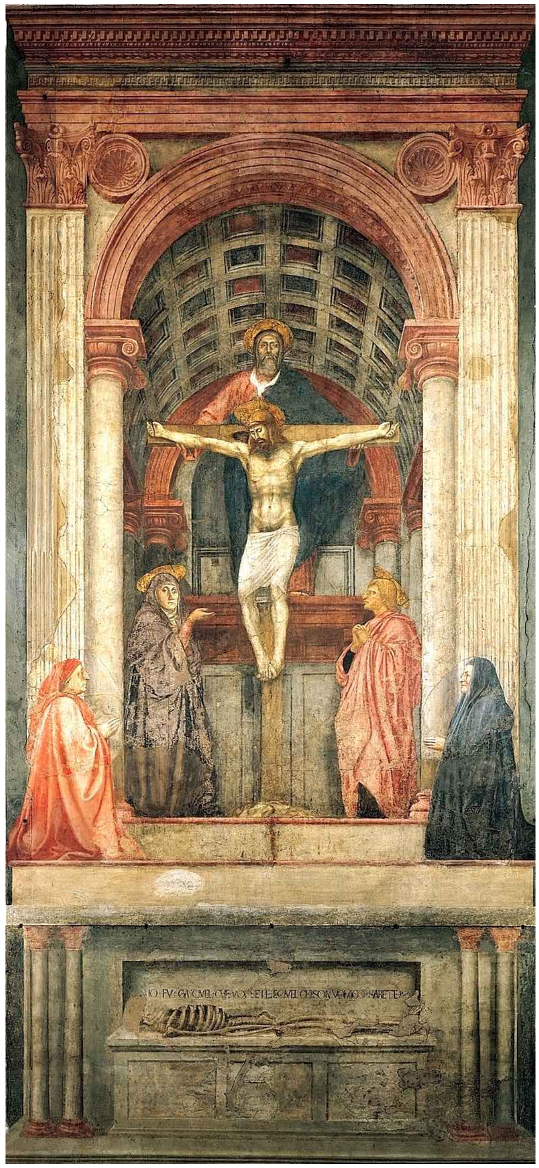
The principle of tiled floors is useful not only for pictures of floor but also as a guide to sizes generally.

15.7. In the painting “A Woman Drinking with Two Men” by Pieter de Hooch, which of the two standing women is tallest?

- ☐ The one on the left.
- ☐ The one on the right.
- ☐ They are about equally tall.

15.8. In this painting there is no tiled floor and no parallel lines. Nevertheless, estimate where the horizon is and introduce your own parallels to check whether the heights of the figures in the background are in perspective agreement with those in the foreground.

This is Masaccio’s “Holy Trinity” (1425–1428):

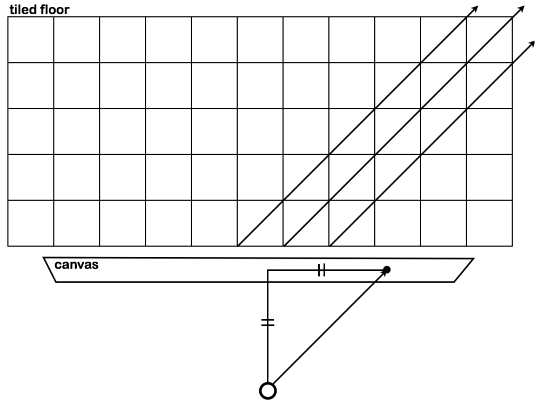


This work is painted directly on the wall of a church in Florence. It is designed to give the illusion of depth, as if the scene was really there in place of the wall. However, this only works for a specific vantage point.

15.9. What is the eye level of the observer in Masaccio's "Holy Trinity"?

- ☐ Level with the heads of the two foremost figures in the painting.
- ☐ Higher.
- ☐ Lower.

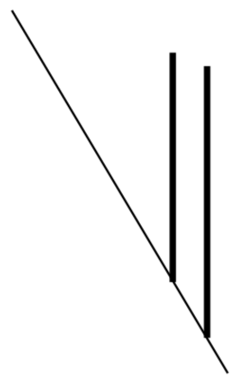
15.10. In addition to the height, the ideal viewing distance must also be taken into account. Explain how it can be determined for any painting containing a perspective view of a tiled floor using the following diagram:



15.11. Can we use this method to estimate the ideal viewing distance for Masaccio's "Holy Trinity" from the tiles in the ceiling?

- ☐ Yes. Mathematically, it doesn't matter whether we are using a curved or a flat tiled surface for the above construction.
- ☐ No. A flat tiled surface is required.
- ☐ Sort of, if we use the top tiles only and treat them as if they were flat.

15.12. The figure below shows a perspective view of two telephone poles standing on the side of a straight road. Explain how to draw the correct perspective view of the third telephone pole. Justify all your steps. (In reality, of course, the telephone poles are all of the same size and equally spaced.)



§ 16. Projective geometry

For visualisations of many of the points below, see

The principles of perspective painting also have more mathematical applications, as we shall now see. When studying the theory of projections mathematically we must idealise the painting situation somewhat to include the possibility of seeing through the back of our heads.

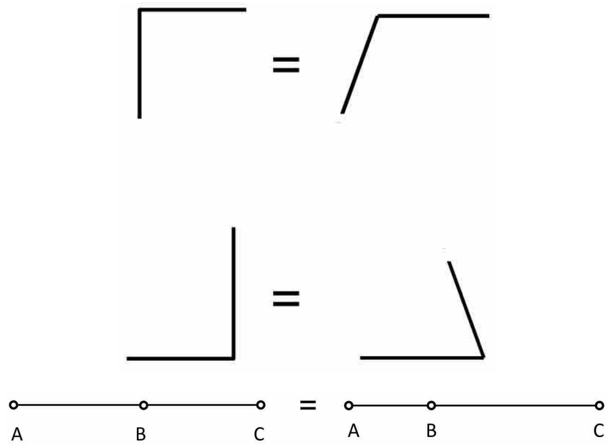
16.1. Argue that if you are painting a plane landscape everything in front of you will appear below the horizon line on the canvas. Then argue that there is a sense in which everything behind you will appear above the horizon, if it is projected onto the canvas following the same rule as the points in front of you.

16.2. What does a painting of railroad tracks look like using this kind of projection?

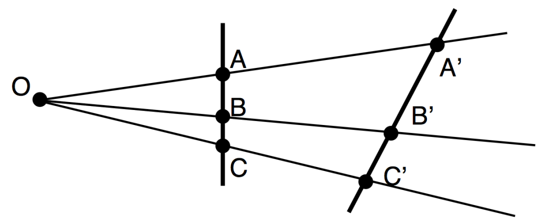
16.3. Consider two points on either side of the railroad. Is there any projection for which the images of these two points can be connected on the canvas without crossing the railroad tracks?

- ☐ Yes
- ☐ No

16.4. Convince yourself that the following figures can be projected onto one another. Hint: you can literally "see" this by having one of the two objects involved printed on a transparency sheet and aligning it with the other visually.



One-dimensional projective geometry concerns the projections of lines onto lines, such as this:



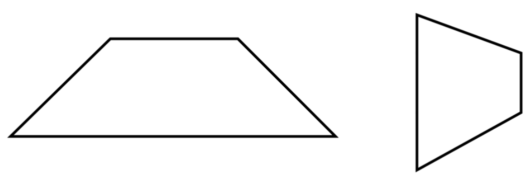
16.5. Can the first configuration below be be projected onto the second? (You are free to move the lines and choose the point O of origin of the projection.)

☐ Yes

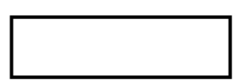
☐ No

For a projection of a plane onto a plane, four points rather than three are required to determine the projection. This can be seen from the fact that squares often appear in almost any quadrilateral shape.

16.6. In which everyday situations do you see squares looking like those below?



16.7. But can a square ever look like this:



Prove that it can. (Hint: place O at infinity.)

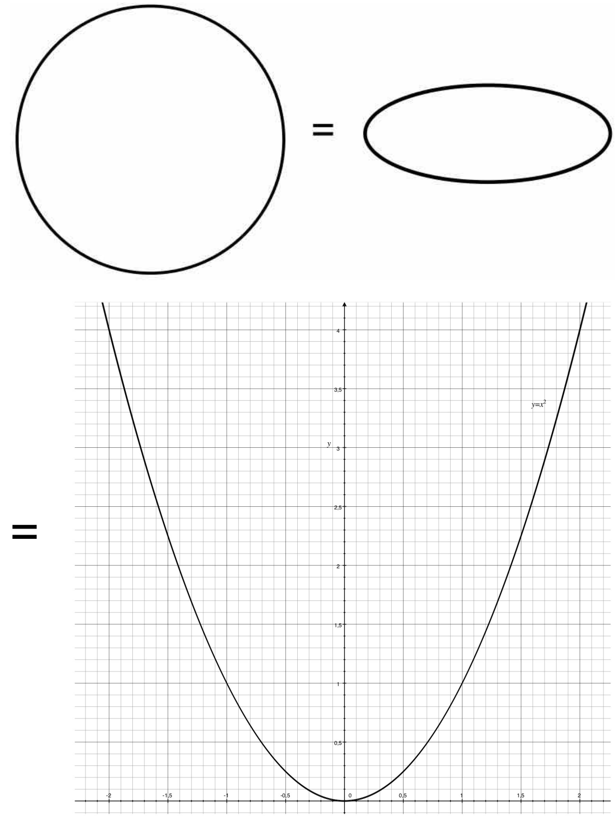
16.8. Can ↻ be projected onto ↺?

☐ Yes

☐ No

16.9. Show that, in a perspective painting, the image of a circle is always a conic section (defined in §17).

16.10. Use a circle printed on a transparency sheet to convince yourself of these projective equivalencies:



16.11. Show more generally that any conic section can be the perspective view of any other.

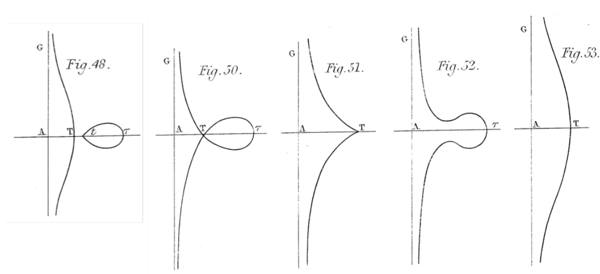
16.12. Argue that the three types of conic sections differ in how many "points at infinity" they have (i.e., number of intersections with the horizon).

The *degree* of a curve means the greatest number of intersections it can have with a line.

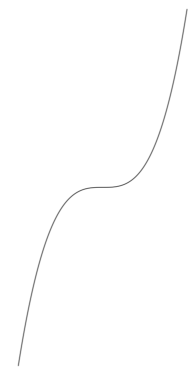
16.13. Of what degree are conic sections?

16.14. Argue that the degree of a curve is preserved under projection.

The idea thus suggests itself to study the projective classification of curves of degree three, or cubic curves. Newton did this and found that there are seven "species," as he called them:



This is indeed the zoology of mathematics, but where is the prototypical cubic curve, $y = x^3$, which we all know looks like this:



16.15. This curve is in fact projectively equivalent to Newton's

- ☐ first
- ☐ second
- ☐ third
- ☐ fourth
- ☐ fifth

16.16. Find an image of the 1,500-year-old Archway of Ctesiphon, or Taq Kasra, in Iraq. Is it parabolic? How can you tell?

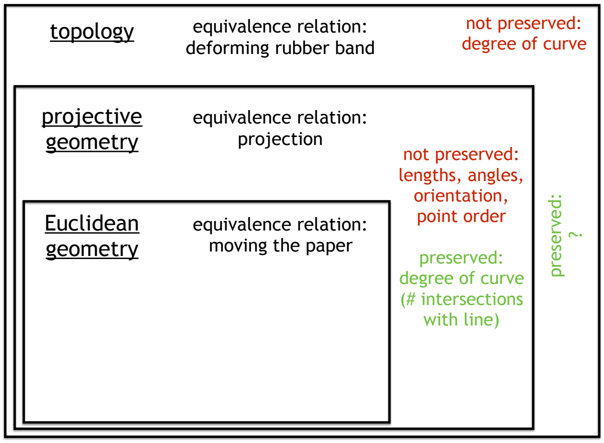
Projective geometry suggests a new way of conceptualising what geometry is. Felix Klein articulated this view of geometry in his influential Erlanger Programm of 1872. This point of view says: a geometry is characterised by its group of transformations. In other words: the theorems of a geometry is precisely the set of all properties that are invariant under certain transformations. It is difficult to see the point of this if we think about Euclidean geometry only, but when we start conceptualising projective geometry as another kind of geometry we can appreciate how Felix Klein's point of view gives us a whole new vantage point that synthesises different geometries into a satisfying general picture.

Ordinary Euclidean geometry consists of all theorems that do not change when you apply "isometries"—that is, when you move the paper around in such a way that no distances between points change. If you prove something in Euclidean geometry—let's say the Pythagorean Theorem for a certain triangle drawn on a piece of paper—and then rotate the paper, or slide it over to the other end of the table, the clearly the proof still remains valid. Such transformations are isometries. On the other hand there are transformations that change distances, such as if you tear the paper, or if the "paper" was actually a sheet of rubber that can be stretched or deformed. Of course, Euclidean theorems you proved before such a transformation do not remain valid for the deformed figure you are left with after the transformation has been applied. This is such an obvious point that we never paid attention to it before. But it turns out to be a useful perspective for generalising our concept of geometry.

Just as Euclidean geometry is about whatever is preserved by isometries, projective geometry is about whatever is preserved by projections. In Euclidean geometry we consider two figures to be "the same" is we can put one on top of the other by moving the paper. In projective geometry to figures are "the same" whenever one can be projected onto the other. Many concepts that are central in Euclidean geometry completely "collapse" in projective geometry. A circle is the same thing as an ellipse. An obtuse angle is the same thing as an acute angle. One meter is the same thing as ten meters. And so on. Yet not everything is the same as everything else. Lines are different from conic sections, and conic sections are different from curves of degree 3, and so on.

16.17. Justify these claims based on our investigations in §16.

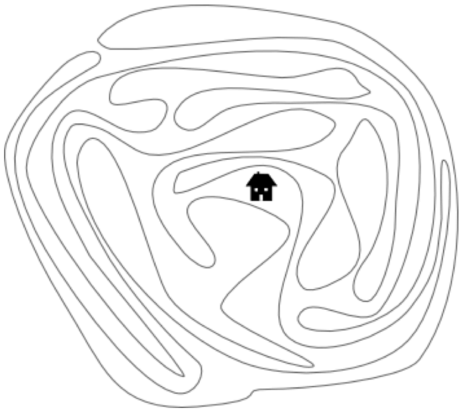
We can go further and allow a more radical group of transformations still. In topology, we study theorems that are invariant under any continuous transformation—we can think of it as rubber band transformations. Two curves are "the same" if a rubber band placed on one can be deformed into the other. Now even the distinction between the degrees of curves collapses.



You may think that we went too far with this notion of "sameness." If "everything is the same as everything else," then there is no geometry left to speak of; we can't prove any theorems at all. But in fact the collapse is not as complete as all that. Interesting properties of curves are preserved even in "rubber band geometry." One aspect of a curve that remains meaningful is that of "inside" and "outside."

16.18. Is the house inside or outside the region enclosed by the curve?

- ☐ inside
- ☐ outside

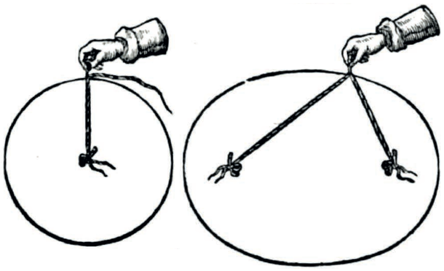


16.19. Find a systematic way—a theorem of topology—to answer questions like this for any picture of this kind.

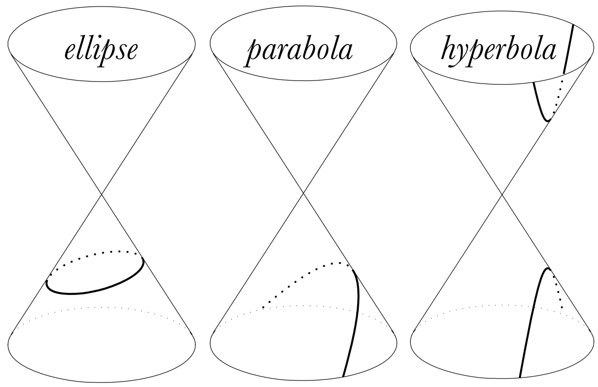
Hint: Draw a line from the house outwards. What can you say about the number of crossings?

§ 17. Conic sections

The string construction of a circle soon suggests this generalisation:



The resulting curve in an ellipse, one of the conic sections:



The terms ellipse, parabola, hyperbola mean roughly too little, just right, and too much, respectively.

17.1. Discuss how this relates to the meaning of the words ellipsis (the omission from speech or writing of words that are superfluous; also the the typographical character "…"), parable (a simple story used to illustrate a moral or spiritual lesson), hyperbole (exaggerated statements or claims not meant to be taken literally).

17.2. Match the situation with the corresponding type of conic.

- ☐ the shadow cast on a wall by the lampshade of a lamp; the water jet from the Manneken Pis statue in Brussels; the shape of a hanging chain; the orbit of Halley's Comet
- ☐ ellipse; parabola; hyperbola; none of the above

The following was another important reason for ancient man to study conic sections.

17.3. (a) "Gnomon" is a fancy word for a stick standing in the ground. The tip of its shadow traces a curve as the sun moves. What type of curve is it? Curves 3 and 6 in the figure below, for example, are both

- ☐ hyperbolas
- ☐ parabolas
- ☐ ellipses

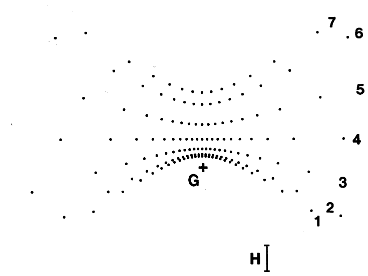


FIGURE 1.34. A sequence of shadow plots made at Seattle. On each plot, the points are half an hour apart. 1, June 22; 2, May 21 or July 23; 3, April 20 or August 24; 4, March 21 or September 23; 5, February 19 or October 24; 6, January 20 or November 23; 7, December 22. *G* labels the point at which the gnomon was set up perpendicular to the plane of the diagram. The line *H* shows the height of the gnomon that was used.

(Figure from Evans, *History and Practice of Ancient Astronomy.*)

(b) How can you find north by using the stick and the curve?

(c) Suppose the shadow cast by the gnomon in the course of a day is a straight line. What can you conclude from this?

- ☐ North cannot be determined from shadow measurements that day.
- ☐ Day and night are equally long.

☐ The sun is perpendicularly above the gnomon at mid day.

☐ Our location is not at the equator of the earth.

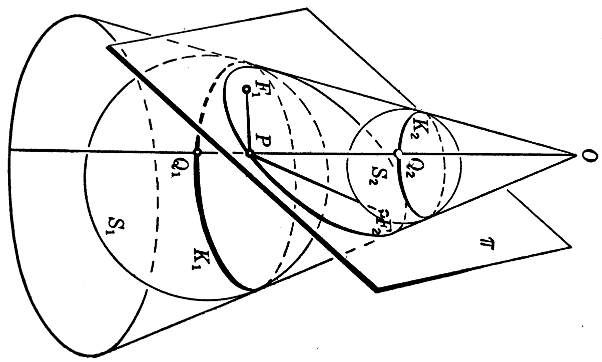
☐ The earth is no longer orbiting the sun but is travelling in a straight line toward or away form it.

17.4. An ellipse can be defined as the curve for which the sum of the distances to two fixed point are equal, as the string construction shown above suggests. In other words, the distance $F_1B + BF_2$ is the same for any point B on the ellipse, where F_1 and F_2 are the endpoints where the string is fastened. We now wish to prove that any tangent to the ellipse forms equal angles with the focal radii. Thus if light or sound is emitted from one focus it will be reflected towards the other. Reflect F_2 in the tangent, and call the image F_2'. The straight line F_1F_2' is the shortest distance between these two points. It intersects the tangent in some point B_1.

(a) Argue that B_1 must be B, the point on the ellipse.

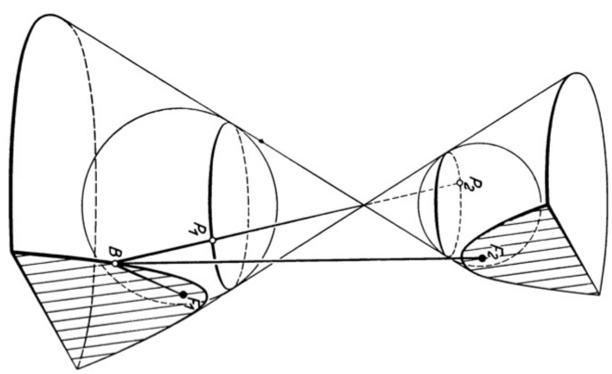
(b) Conclude the proof.

17.5. We have defined ellipses in two ways: as sections of a cone, and in terms of strings or distances from two focal points. How do we know that these two definitions amount to the same thing? Prove it based on this figure.



(Figure from Courant & Robbins, *What is Mathematics?*)

17.6. Adapt the argument for the case of a hyperbolic section of a cone:

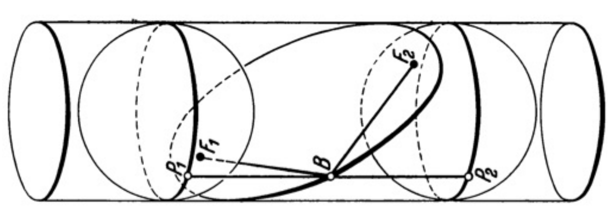


(Figure from Hilbert & Cohn-Vossen, *Geometry and the Imagination.*)

17.7. If you cut a cylinder by a plane, is the cross-section an ellipse?

☐ yes

☐ no



(Figure from Hilbert & Cohn-Vossen, *Geometry and the Imagination.*)

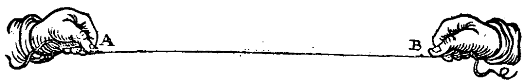
§ 18. Geometry on surfaces

🎙 "That which has no part: Euclid's definitions"

📕 Sections 1–5 of *Flatland* by Edwin Abbott

18.1. 💬 What do Euclid's definitions of point and line say about the nature of geometry, in particular with respect to Platonist versus physicalist interpretations? Use ancient sources to argue for both sides.

What is straightness? Is a straight line the same thing as the shortest distance between two points? Or to put it in more physical terms: the path of a stretched string.



To get to the bottom of the notion of straightness it is useful to consider not only the usual plane but also other surfaces. Consider:

18.2. What are the straight lines on a cylinder?

Here, "stretched string" and "shortest distance" are not the same thing. It is arguably the stretched string that gets it right. We feel that straightness is a "local" property. We can alter the distance characterisation of straightness to reflect this. Definition: a curve is a locally shortest path if, for any given point P on the curve, there is a neighborhood around that point such that the distance along the curve between any two points on the curve in that neighborhood is the shortest possible distance between those points.

18.3. Are locally shortest paths the same thing as stretched-string paths on the cylinder?

- ☐ Yes
- ☐ No

To appreciate the geometry of a surface—its "intrinsic" geometry, as we say—we should forget for a moment that it is located in three-dimensional space. We should look at it through the eyes of a little bug who crawls around on it and thinks about its geometry but who cannot leave and is unaware of any other space beyond this surface. Think of for example those little water striders that you see running across the surfaces of ponds. They know the surface of the pond ever so well. They can feel any little movement on it. But they are quite oblivious to the existence of a third dimension outside of their surface world. This makes the water strider an easy prey for a bird or a fish that strikes it without first upsetting the surface of the water.

Thinking about the intrinsic geometry of surfaces in this way forces us to realise that what we often take for granted as "obvious" objective truths in geometry are really a lot more specific to our mental constitution and unconscious assumptions than we realise. In some ways we are as ignorant of our own limitations as the water strider.

Instead of using stretched strings, straight lines can also be defined as curves possessing half-turn symmetry about every point: for any point P on the curve, there is a neighbourhood around that point such that when this neighbourhood is rotated about P by half a revolution around P the curve ends up on top of itself. More loosely, a curve is straight if it always "cuts angles in half." To test for this kind of straightness on surfaces one can use the "ribbon test": if a ribbon or band can be laid flatly along the curve without creasing on either side, then the curve is straight.

18.4. Are half-turn-symmetry lines the same thing as stretched-string lines on the cylinder?

- ☐ Yes
- ☐ No

18.5. For each of the following real-world phenomena related to straightness, try to pinpoint in mathematical terms what aspect of straightness is involved.

(a) The edge you create when you fold a piece of paper.

(b) The path of a cross-Atlantic flight.

(c) The axis of rotation when a solid body is rotated. (Such as a döner spit or a basketball spinning on your finger tip.)

(d) The heart beats through the contraction of muscular threads across its surface.

(e) Bandaging an injured limb.

(f) Light rays. But also how your finger looks “broken” when you put half of it in water.

(g) Mirrors are made flat by rubbing two of them against each other face-to-face, using fine sand or other polishing agent.

(h) Rowing a boat and applying equal or different amount of force to each oar.

18.6. According to Russo, the logically poor definition of straightness found in the Elements was inserted because:

- ☐ It is intuitively useful.
- ☐ It is constructively useful.
- ☐ It is useful for teaching purposes.
- ☐ Formal mathematics was not yet fully developed.

18.7. Argue that the “straight lines” on a sphere are parts of “great circles,” i.e., circles whose midpoint is the center of the sphere. The equator is an example of a great circle on the sphere of the earth.

18.8. Are latitude and longitude lines “straight”?

- ☐ Yes
- ☐ No

18.9. Wikipedia says: “Spherical geometry obeys two of Euclid’s postulates: the second postulate (‘to produce [extend] a finite straight line continuously in a straight line’) and the fourth postulate (‘that all right angles are equal to one another’). However, it violates the other three: contrary to the first postulate, there is not a unique shortest route between any two points (antipodal points such as the north and south poles on a spherical globe are counterexamples); contrary to the third postulate, a sphere does not contain circles of arbitrarily great radius; and contrary to the fifth (parallel) postulate, there is no point through which a line can be drawn that never intersects a given line.” Try to argue that Wikipedia is wrong. For which postulate(s) can this be done on reasonable grounds?

- ☐ Postulate 1
- ☐ Postulate 2
- ☐ Postulate 3
- ☐ Postulate 4
- ☐ Postulate 5

18.10. Which of the following are arguably true on a sphere, when Euclid’s definitions are combined with the spherical geometry concept of distance? (“Lines” here means straight lines.)

- ☐ Some circles are lines.
- ☐ Some circles are points.
- ☐ Some lines are points.
- ☐ None of the above.

18.11. Playfair’s *Elements of Geometry* (1806) defines a straight line thus: “If two lines are such that they cannot coincide in any two points without coinciding altogether, each of them is called a straight line.” Does this definition single out straight lines and only straight lines on the sphere?

- ☐ Yes
- ☐ No

Flatland gives a vivid account of what it is like to live in a surface and not know anything about a third dimension. This mirrors our concern with trying to explain geometrical concepts such as straightness “intrinsically” in terms of the surface itself: we must imagine that a “flatlander” thinks this is the only reasonable way to think of straight lines.

18.12. *Flatland* is also a societal critique. Abbott uses the imaginary country of Flatland to highlight the absurdity and folly of many practices in his own society. In particular, he uses geometry to “explain” (or rather satirise) why:

- ☐ The working class does not unite and rebel in their own interest.

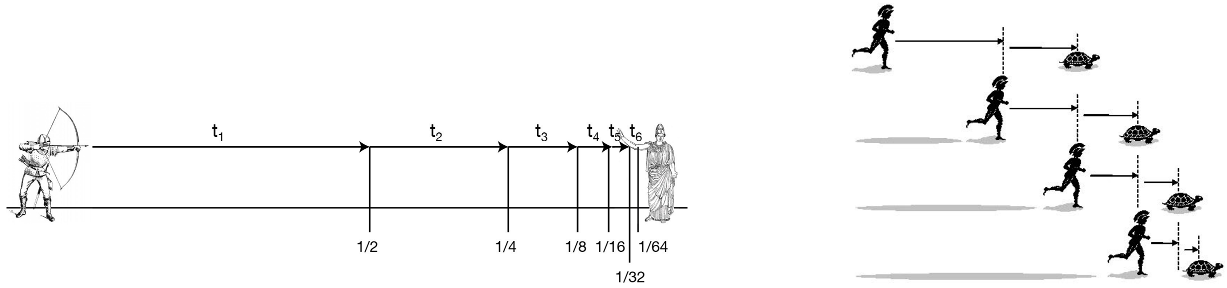


Figure 2: Zeno's paradoxes.

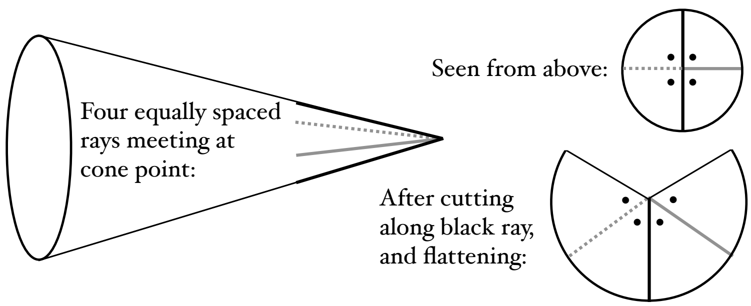


Figure 3: Right angles at a cone point. On this 240 degree cone, the right angles at the cone point are 60 degrees.

☐ A husband should treat his wife differently in public than in private.

☐ Soldiers are from the low classes.

☐ Modern architecture is of poor taste.

☐ Right hands only must be used for handshakes.

§ 19. 450 degree cone

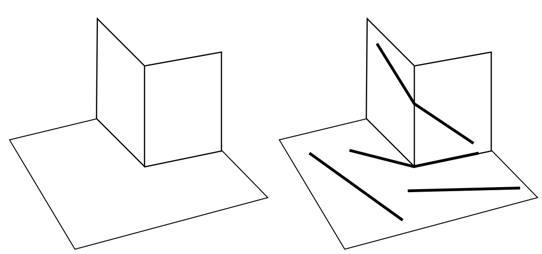
🎙 "Created equal: Euclid's Postulates 1-4"

19.1. 💬 Why did Euclid feel the need to postulate that "all right angles are equal"?

19.2. 💬 In what ways can Zeno be taken as a case in point illustrating the thesis (advanced by Lloyd and Szabó) of the dialectical origins of Greek geometry?

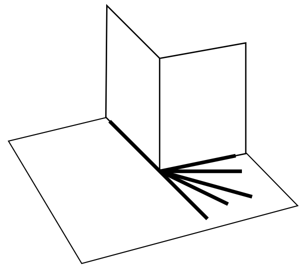
19.3. 💬 Are Zeno's dichotomy and Achilles arguments just different literary elaborations of the same idea, or is there something substantially different between the two cases?

A "450° cone" looks something like a "skyscraper corner":



You can make one yourself by cutting a slit in an ordinary piece of paper and taping in "90 extra degrees." The idea is that only the "cone point" is exceptional. All other points of the "450° cone" are indistinguishable from ordinary flat paper. Even the "edges" in the model can be flattened out, so they are not really any different from the flat part of the "walls" or "ground." The cone point, however is fundamentally different from a point on a flat piece of paper; it is impossible to flatten it out.

Let us investigate how straight lines behave on this surface. Consider for example what happens if we walk along the "wall" and reach the cone point. When we walk along the wall we are certainly going straight, but how should we continue when we reach the corner to keep going straight?



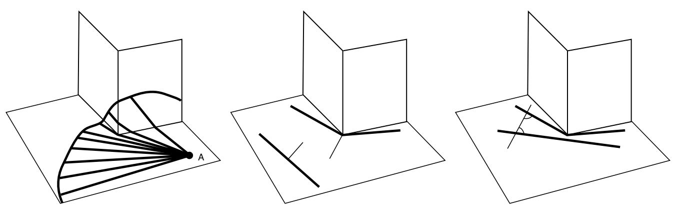
19.4. According to the stretched-string definition of straight lines, which of the lines in the figure are straight?

- ☐ all
- ☐ one
- ☐ none

19.5. According to the half-turn-symmetry definition of straight lines, which of the lines in the figure are straight?

- ☐ all
- ☐ one
- ☐ none

Let us investigate which of Euclid's postulates hold on this surface. The figures below are hints.



19.6. With the half-turn-symmetry definition of straight lines, which postulates hold on the surface of the 450 degree cone?

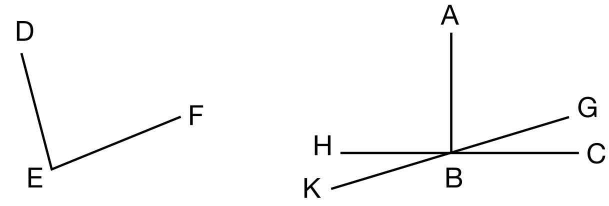
- ☐ Postulate 1
- ☐ Postulate 1' (Euclid's Postulate 1 with the added requirement that the line is unique)
- ☐ Postulate 2
- ☐ Postulate 3
- ☐ Postulate 4
- ☐ Postulate 5

19.7. With the stretched-string definition of straight lines, which postulates hold on the surface of the 450 degree cone?

- ☐ Postulate 1
- ☐ Postulate 1'
- ☐ Postulate 2
- ☐ Postulate 3
- ☐ Postulate 4
- ☐ Postulate 5

Proclus tried to prove Postulate 4 as follows:

But if we must provide a graphic proof of this postulate, let us assume two right angles, ABC and DEF. I say they are equal. If they are not, one of them will be greater than the other. Let this be the angle at B. Then if DE be made to coincide with AB, the line EF will fall within the angle, say, at BG. Let BC be extended to H. Since ABC is a right angle, so also is ABH, and the two angles are equal to one another (for a right angle is by definition equal to its adjacent angle). Angle ABH, then, is greater than angle ABG. Now let BG be extended to K. Since ABG is a right angle, its adjacent angle also will be a right angle, equal to ABG. The angle ABK, then, is equal to angle ABG, so that angle ABH is less than ABG. But it is also greater, which is impossible. Hence it is false that a right angle can be greater than a right angle.



19.8. Is Proclus's proof of Postulate 4 valid?

- ☐ Yes.
- ☐ No, it makes the unwarranted assumption that the adjacent angle of a right angle is itself a right angle.
- ☐ No, it makes the unwarranted assumption that K and G are on opposite sides of HBC.

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§ 20. Elements 13–26: angles and congruence

20.1. Postulate 4 is implicitly used in:

- ☐ Proposition 11
- ☐ Proposition 13
- ☐ Proposition 14
- ☐ Proposition 15
- ☐ None of the above

20.2. (cont.) Is one or more of those propositions false on a 450 degree cone?

- ☐ Yes
- ☐ No

20.3. Proposition 13. Euclid's first equation is CBE=CBA+ABE. His second five-term equation is DBA+ABC=DBE+EBA+ABC. The last two terms in that expression are equal to CBE by the first equation. Can Euclid therefore "substitute in" CBE in place of EBA+ABC to get DBA+ABC=DBE+CBE, which is precisely what he wants (this is his second to last equation)?

- ☐ Yes, and that is what he does.
- ☐ Yes, and this would have shortened his proof.
- ☐ Yes, but this would not have shortened his proof since it needs to apply to both cases mentioned in the statement of the proposition.
- ☐ No, because the proof needs to apply to both cases mentioned in the statement of the proposition.
- ☐ No, because there is no Common Notion legitimating such a step.
- ☐ No, because we cannot assume that ABE=EBA.

20.4. Proposition 16. It is assumed that F falls within angle ABC, rather than below (the extension of) BC or above BA. This follows from:

- ☐ BE being an angle bisector.
- ☐ Two lines cannot enclose a space.

☐ None of the above.

20.5. Proposition 16 does not hold generally on a sphere. Consider the case where B is the south pole, F is the north pole, and AEC is on the equator. If we try to apply Euclid's proof to this configuration, what goes wrong?

☐ Nothing. The proof works and the theorem is true for configurations of this type (though not necessarily for all spherical triangles).

☐ The inference based on Proposition 15 fails.

☐ The inference based on Proposition 4 fails.

☐ ECD is not greater than ECF.

20.6. Proposition 22. If the condition regarding the lengths of the sides is not satisfied, it is always impossible to construct a triangle from the given sides.

☐ Yes

☐ No

20.7. Proposition 22 can be seen as a generalisation of Proposition 1. In our discussion of Proposition 1, we considered some ways of addressing a gap in the reasoning. Which of those ways are also applicable to the analogous gap in the proof of Proposition 22?

☐ Carrying out the constructions with ruler and compass.

☐ Allowing conclusions based on what is visually clear in the diagram, as long as it concerns not exact properties but only properties that would still hold even if the diagram was imperfectly drawn.

☐ The notions of "inside" and boundary in Definitions 13-15.

20.8. Proposition 23 shows how to move an angle. But didn't we already assume the ability to move angles in our proof of Proposition 4? (In fact, Proposition 23 is logically dependent on Proposition 4, so we cannot make it the foundation for that part of the proof.)

20.9. Proposition 26. Euclid has now proved SAS, SSS, and ASA triangle congruence. (Actually his Proposition 26 also includes SAA but I left this out in my edition.) Is there a triangle congruence theorem for every such letter combination? That is, if two triangles have "three things in common" then they are the same?

☐ Yes

☐ No

20.10. You are at point A and you want to know the distance to point B. However, point B is inaccessible (it is on the other side of a river, for example), so you cannot measure AB directly. Instead you proceed as follows. From A measure along a straight line at right angles to AB a length AC and bisect it at D. From C draw CE at right angles to CA on the side of it remote from B, and let E be the point on it which is in a straight line with B and D. The sought distance AB is equal to the measurable distance:

☐ DC ☐ DE ☐ CE ☐ AC

20.11. (cont.) The Euclidean propositions directly involved in setting up and justifying this procedure are:

☐ 1	☐ 7	☐ 11	☐ 16
☐ 2	☐ 8	☐ 13	☐ 22
☐ 4	☐ 9	☐ 14	☐ 23
☐ 5	☐ 10	☐ 15	☐ 26

20.12. Simple bookshelves (such as the IKEA Ivar) consist of two vertical and some horizontal planks. A problem is that they could tip askew, so that when we look at the bookshelf from the front we see, instead of a rectangle with vertical sides, a parallelogram with its longer sides inclined a few degrees with respect to the floor. Two ways of addressing this are often implemented in such bookshelves. The X method is to nail one or two long metal fortifiers diagonally across the back of the shelf. This forces the distance between diagonally opposite points to be fixed. The L method is to nail L-shaped metal fortifiers where the shelves meet the sides. This forces the angle between them to be fixed at 90 degrees. Match each method with a proposition from Euclid that can be used to explain it.

- ☐ X method
- ☐ L method
- ☐ Neither
- ☐ Proposition 4 (SAS), Proposition 8 (SSS), Proposition 26 (ASA)

§ 21. Elements 27–32, 46: parallels

21.1. Euclid's theory of parallels (Propositions 27-31) studies whether two given lines are parallel or not by cutting them with a third line (which may be called the "test line"), and then looking at the angles produced. One can look at either the "alternate angles" (the pair of angles involved in Proposition 27; also called "Z-angles" because of their configuration) or the "internal angles" (the pair of angles involved in Postulate 5).

- ☐ Euclid uses alternate angles/Proposition 16 …
- ☐ Euclid uses internal angles/Postulate 5 …
- ☐ … when he wants to prove that given lines are parallel.
- ☐ … to derive properties any parallel lines must have.

21.2. Which of the following does Euclid establish without relying on Postulate 5 in any way?

- ☐ Parallel lines exist.
- ☐ There is precisely one unique parallel to a given line through a given point.
- ☐ Alternate angles (Z-angles) are equal.
- ☐ Vertical angles (opposite pair in an X configuration) are equal.
- ☐ Angle sum of triangle is two right angles.
- ☐ Proposition 30.
- ☐ None of the above.

21.3. Proposition 31 uses nothing later than Proposition 27, so Euclid could have placed it earlier. Do the intermediate propositions add some illumination?

- ☐ Yes, they imply that the parallel constructed in Proposition 31 is unique.
- ☐ Yes, they imply that the parallel constructed in Proposition 31 is not unique.
- ☐ Yes, they justify the implicit assumption in Proposition 31 that a parallel line exists.
- ☐ None of the above.

21.4. Parallel lines can also be characterised as lines that always have the same distance between them. But the notion of not crossing is arguably more fundamental. As Postulate 2 says, lines are fundamentally things that can be extended indefinitely. Asking whether they will cross seems quite natural already from this standpoint alone, whereas talking about the distance between lines requires us to involve a lot more secondary concepts to specify what it even means.

The geometry of other surfaces also problematise the equivalence of parallelism and equidistance in interesting ways. For example, it can happen that the curve equidistant to a straight line is not a straight line. In which of the following geometries does this occur?

- ☐ sphere
- ☐ 450 degree cone with stretched-string definition of straight line
- ☐ 450 degree cone with half-turn-symmetry definition of straight line

If we wanted to base the theory of parallels on the notion of equidistance we would have to prove that these kinds of situations do not occur.

21.5. Think about how Proposition 32 is related to Proposition 16. Why didn't Euclid prove the more powerful 32 first and then derive 16 very easily from there?

- ☐ Euclid's proof of Proposition 32 is logically dependent on Proposition 16, so 16 was a necessary stepping-stone along the way; without 16 we could not have reached 32.
- ☐ Although Euclid's proof of Proposition 32 is logically independent of Proposition 16, Euclid's way of proving Proposition 16 shows that it is independent of the parallel postulate, which would not have been shown if it was derived as a corollary of Proposition 32.

21.6. Why did Euclid wait so long (until Proposition 29) to use the parallel postulate? Does this tell us something about the status of the postulate?

21.7. The parallel postulate is considerably more convoluted than the other postulates. But it transpires from the *Elements* itself that Euclid could have used a simpler, equivalent statement in place of it, such as: given any line and any point not on this line, there is no more than one parallel to the line through that point. Why did Euclid choose his formulation of the postulate instead?

21.8. Proposition 47. An odd phrase occurs: "either … or". This phrasing in effect amounts to an implicit application of Proposition ☐.

§ 22. Rationalism versus empiricism

🎙 "Rationalism versus empiricism"

22.1. 💬 How do rationalists and empiricists differ in how they interpret the Euclidean method?

22.2. 💬 Newton and Leibniz disagreed on what it means to treat gravity scientifically. In what way does their disagreement parallel their views on the nature of geometry?

22.3. 💬 Argue that rationalism and synthesis forms a natural pair, as does empiricism and analysis.

22.4. 💬 Do a Google Images search for "birth of Athena." Argue that it is a metaphor for one of the isms.

§ 23. Kant

🎙 "Rationalism 2.0: Kant's philosophy of geometry"

23.1. 💬 What is the relation between geometry and empirical data regarding spatial relations, according to Kant?

23.2. 💬 Kant reconciled elements of rationalism and empiricism that seemed irreconcilable. How?

23.3. 💬 Is geometrical knowledge in some sense specifically human or tied to human nature? Compare Kant with Plato and Descartes in this regard.

23.4. 💬 What led Descartes and Leibniz to insist on a relativistic notion of space? And Newton on an absolutist one? How does Kant reconcile the two?

23.5. 💬 Why was philosophy "dead" before Kant, and how did Kant reverse this?

§ 24. Abstract axiomatics

For thousands of years, Euclid's parallel postulate made mathematicians a bit uncomfortable. Many felt that it ought to be a theorem rather than a postulate. Various strategies for proving it were devised. One strategy was to try to prove it by contradiction, that is, to assume it to be false and try to show that this led to absurd consequences. Indeed, it was found that negating the parallel postulate had various strange consequences that felt "repugnant to the nature of the straight line" (as Saccheri put it in *Euclides Vindicatus*, 1733). For example, if the parallel postulate is false then squares do not exist. It is difficult to imagine a world in which squares are impossible. But weird is not the same as self-contradictory. Despite their best efforts, mathematicians could not find a clear-cut proof that negating the parallel postulate led to directly contradictory conclusions.

This line of inquiry pushed mathematicians toward an evermore abstract, logico-formalistic conception of geometry. We can see why. Reasoning about things contrary to our imagination requires us to use abstract logic rather than intuition and visualisation. And basing our conclusions on gut feelings about what is "repugnant to the nature of the straight line" and the like is contrary to the geometrical ideal of deducing everything from explicitly formulated assumptions. The modern trend toward formalisation

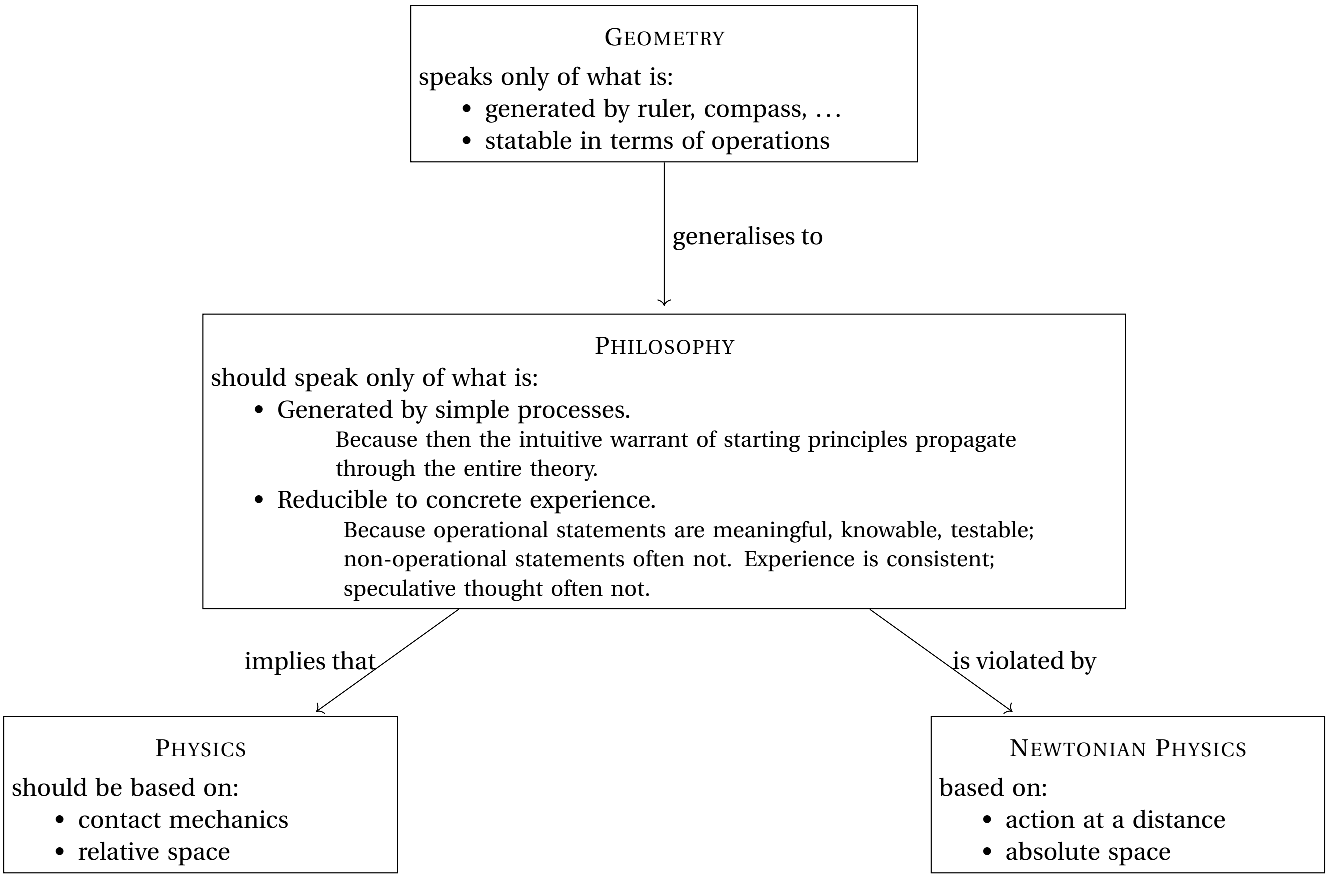


Figure 4: 17th-century operationalist view of scientific method.

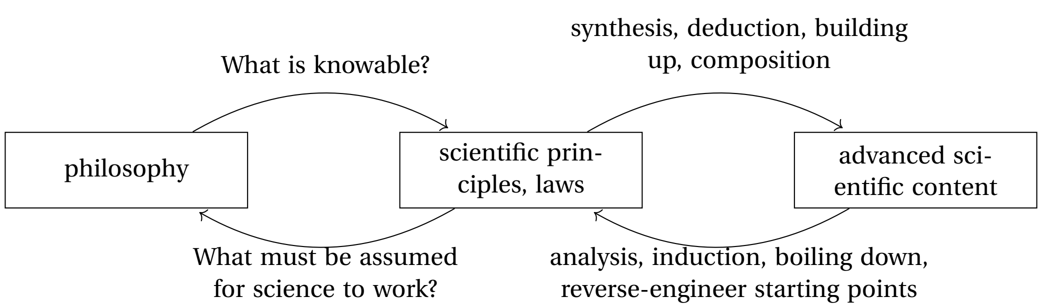


Figure 5: Opposite views on the relation between science and philosophy. Top: The view of Descartes, Leibniz, et al., according to which science must flow from philosophically justified starting principles. Bottom: The view of Newton, according to which philosophical and scientific principles are subordinated to science and retrofitted to agree with science after the fact.

	Descartes, Leibniz Continental rationalism	Newton British empiricism
The search for knowledge starts with …	intuitively clear primitive notions	the rich diversity of phenomena
… and consists in …	deducing the diversity of phenomena from them.	reducing them to a few simple principles.
Intrinsic justification of the axiomatic principles is …	immediate by their intuitive nature	external to the matter at hand
… and is therefore …	the crucial epistemological cornerstone of the entire enterprise.	of secondary importance at best.
In the case of physics, the axiomatic principles are …	the laws of contact mechanics	Newton's three force laws and the law of gravity
… which are established by means of …	their intuitively immediate nature.	induction from the phenomena.
In the case of geometry, the study of curves starts with …	the primitive intuition of local motion	the diversity of curves conceived in any exact manner whatever
… and consists in …	constructively building up a theory of all knowable curves on this basis.	investigating their properties in a systematic fashion.
Geometrical axioms are thus …	the intuitively immediate principles that define and generate the entire subject.	the outcomes of the reductive study of curves, which it was found convenient and illuminating to take as assumptions when the time came to write a systematic account.
The certainty of geometrical reasoning …	stems directly from the axioms' intuitive warrant and the constructive manner in which the rest is built up from them.	stems not from the axioms as such, but from the general method and exactitude of geometrical reasoning.

Table 3: Overview of the two competing interpretations of the Euclidean method in the 17$^{\text{th}}$ century.

and abstraction culminated in the work of Hilbert, a world-leading mathematician and author of the seminal *Grundlagen der Geometrie* (1899). He summed up the spirit of this development like this: "one must be able to say at all times—instead of points, straight lines, and planes—tables, chairs, and beer mugs."[4] That is to say, the terms of geometry must be taken to have meaning only through formally specified axioms, without any connotations or assumptions being imported from psychological associations or intuitive conceptions of the entities in question.

In this section we shall practice thinking in the Hilbert way. This will help us when we take on non-Euclidean geometry later.

Investigating the status of the parallel postulate in the manner outlined above leads to the question: what can we prove *without* using the parallel postulate? More generally, a good way to understand the logical relations between axioms is to study what follows from a few of them at a time. Carrying this perspective to its extreme, we may ask ourselves: what are the minimum assumptions needed for something to be called geometry at all? Discarding the parallel postulate impoverishes us, but it still leaves us with lines and points and circle and triangles and so on, so it's still geometry. How much further could we go? How many more of our assumptions could we throw out? Here is one possible bare minimum set of assumptions for geometry:

Axiom 1. If L is a line, then there exists at least two points belonging to L.

Axiom 2. If L is a line, then there exists at least one point which is not on L.

Axiom 3. There exists at least one line.

Axiom 4. If A and B are different points, then there exists at least one line which contains both A and B.

From a logical point of view, "line" and "point" are undefined terms. Any formal system must involve undefined terms because everything must be defined in terms of something, and those things in terms of something else, and so on. If we did not allow undefined terms this process would never end.

[4] David Hilbert, *Gesammelte Abhandlungen*, III.403.

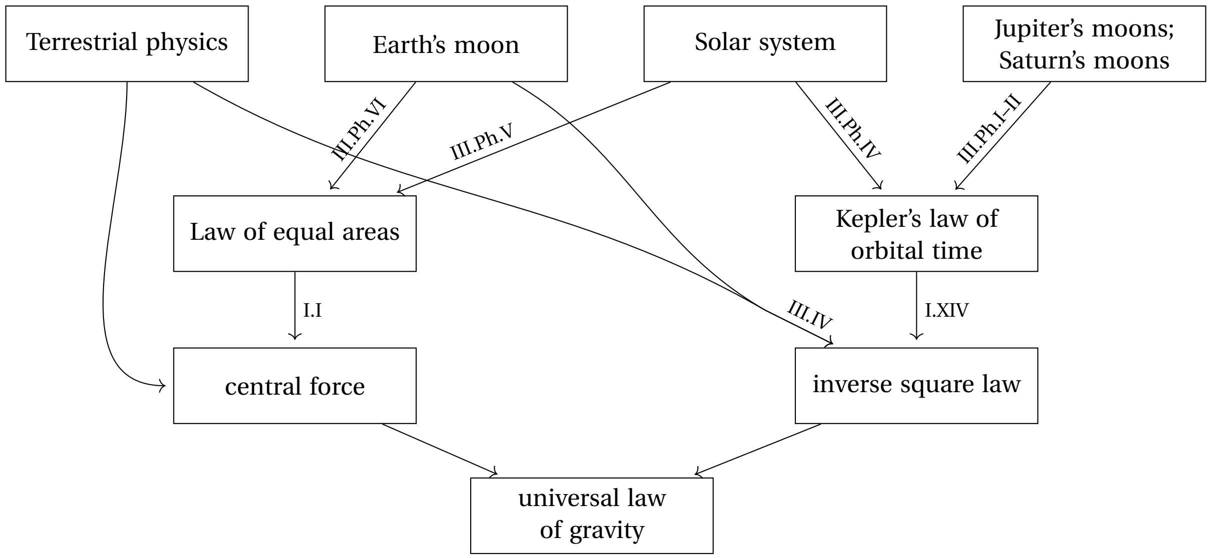


Figure 6: Flowchart illustrating Newton's method of reducing "phenomena" to universal scientific principles in the *Principia*.

rationalism		… since both made by God.
	The architecture of our mind contains seeds of truths about data from the outside world …	
Kant		… since data biased by the mind.
		… and depend on subjective presuppositions necessary for scientific thought to be possible.
	Laws/patterns in observed data are building blocks of scientific thought …	
empiricism		… and non-subjective, mind-independent.

Table 4: Kant as middle position between rationalism and empiricism: he is able to maintain bits of each by "contaminating" each with the other.

From the point of view of	rationalism	empiricism
absolute space is unknowable since it	cannot be proven by pure thought	is not susceptible to measurement
Kantian solution: yes, but it is knowable that	absolute space is a necessary precondition of our thought	sensory data will conform to this preconception of our minds
so absolute space may not be	logically necessary truth	objective fact about the world
but it is inextricably baked into	human thought	sensory data
it is delusional to hope for anything better since there is no such thing as	"pure thought" ("Critique of Pure Reason")	"objective data"
in fact, such a thing would be undesirable and amount to	a bird trying to fly in a vacuum	kaleidoscopic chaos of impressions

Table 5: Kantian outlook illustrated by the example of absolute space.

Axioms 1–4 are obviously true if we think of the usual Euclidean meaning of line and point. But it is also true for the geometry of a sphere, or a cylinder, or a cone, and so on. Each of these are a "model" of the axiom system—that is, they are ways of giving meaning to the undefined terms that make the axioms true. In fact, we can imagine much more exotic models.

24.1. Which of the following are models of the minimal geometry axioms?

- ☐ The points are: all grains of sand in Africa. The lines are: all possible sets of such points.
- ☐ The points are: all the points inside an ordinary circle. The lines are: all the cords of this circle (that is, all the line segments connecting two points on the circumference).
- ☐ The points are: all stations in the Stockholm metro system. The lines are: all the lines of the network (T10, T11, T13, T14, T17, T18, T19), each consisting of all the stations on that route.

- ☐ The points are: artichoke, Beatrice, cold. The lines are: the line consisting of artichoke and cold, and the line consisting of Beatrice and cold.
- ☐ The points are: the three vertices of a triangle. The lines are: the three sides of that triangle, each consisting of its two endpoints.

24.2. Is the statement "each point belongs to some line" always true in any model of Axioms 1-4? In any model of three of the axioms? Select all sets of axioms sufficient to make the statement true.

- ☐ 1, 2, 3
- ☐ 1, 2, 4
- ☐ 1, 3, 4
- ☐ 2, 3, 4
- ☐ 1, 2, 3, 4
- ☐ None of the above.

24.3. Select all that are true for any model of Axioms 1-4.

- ☐ Each point belongs to at least two different lines.
- ☐ Each point belongs to at least three different lines.
- ☐ For any point there exists a line that does not contain that point.
- ☐ There exists four points such that not all four are on the same line.

24.4. Mathematicians started considering "geometries" with a finite number of points at around the same time that artists took up pointillism and physicists postulated that light consists of finite quanta (photons). Coincidence?

§ 25. Abstract versus physical geometry

🎙 "Repugnant to the nature of a straight line: Non-Euclidean geometry"

Euclidean geometry is:	Plato	Kant	modern
true objectively/absolutely/unconditionally	■		
the constitution of physical space	■		
the constitution of perceived space	■	■	
innately pre-programmed into humans	■	■	
the only coherently thinkable geometry fitting Euclid's Definitions and Postulates 1–4	■	■	
axiomatic-deductive: the theorems follow logically from the axioms	■	■	■

Table 6: Evolution of the epistemology of geometry across time.

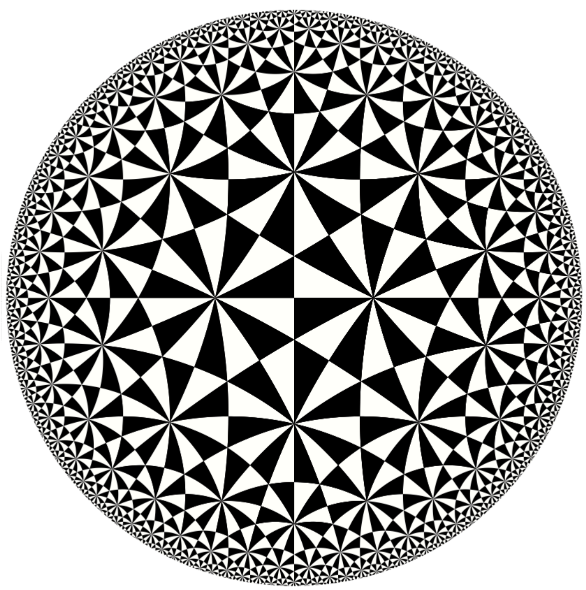
25.1. 💬 What is the standard modern view of the relation between geometry and empirical data (exemplified by mathematicians such as Hilbert and physicists such as Einstein)? What are its strengths and weaknesses compared to older conceptions?

25.2. 💬 Has the history of the epistemology of geometry been one of constant retreats? That is, has the perceived status and claim to truth of geometrical knowledge been shrinking over time?

25.3. 💬 Should all of Euclid's theorems be seen as essentially conditional statements—"*if* you accept the postulates, *then* logic compels you to also accept this theorem"—as opposed to truths in and of themselves?

§ 26. Hyperbolic geometry

This is a representation of the so-called hyperbolic plane:



Intrinsically, these tiles are all of equal size. Thus the space represented in this picture is actually infinite since the tiles become smaller and smaller as you approach the boundary. A bug living in this world could never walk to the boundary; there would always be more tiles to go. The lines in this world are arcs of circles perpendicular to the boundary (including the diameters of the disc, which are part of circles with infinite radius, so to speak). Hyperbolic angles are the same as the Euclidean angles in the picture.

A useful tool for visualisations and interactive constructions in hyperbolic geometry is: ↗

26.1. Argue that lines in this sense do indeed seem to represent the shortest (hyperbolic) distances between any two points.

26.2. Which postulates hold in hyperbolic geometry?

- ☐ Postulate 1
- ☐ Postulate 1' (Euclid's Postulate 1 with the added requirement that the line is unique)
- ☐ Postulate 2
- ☐ Postulate 3

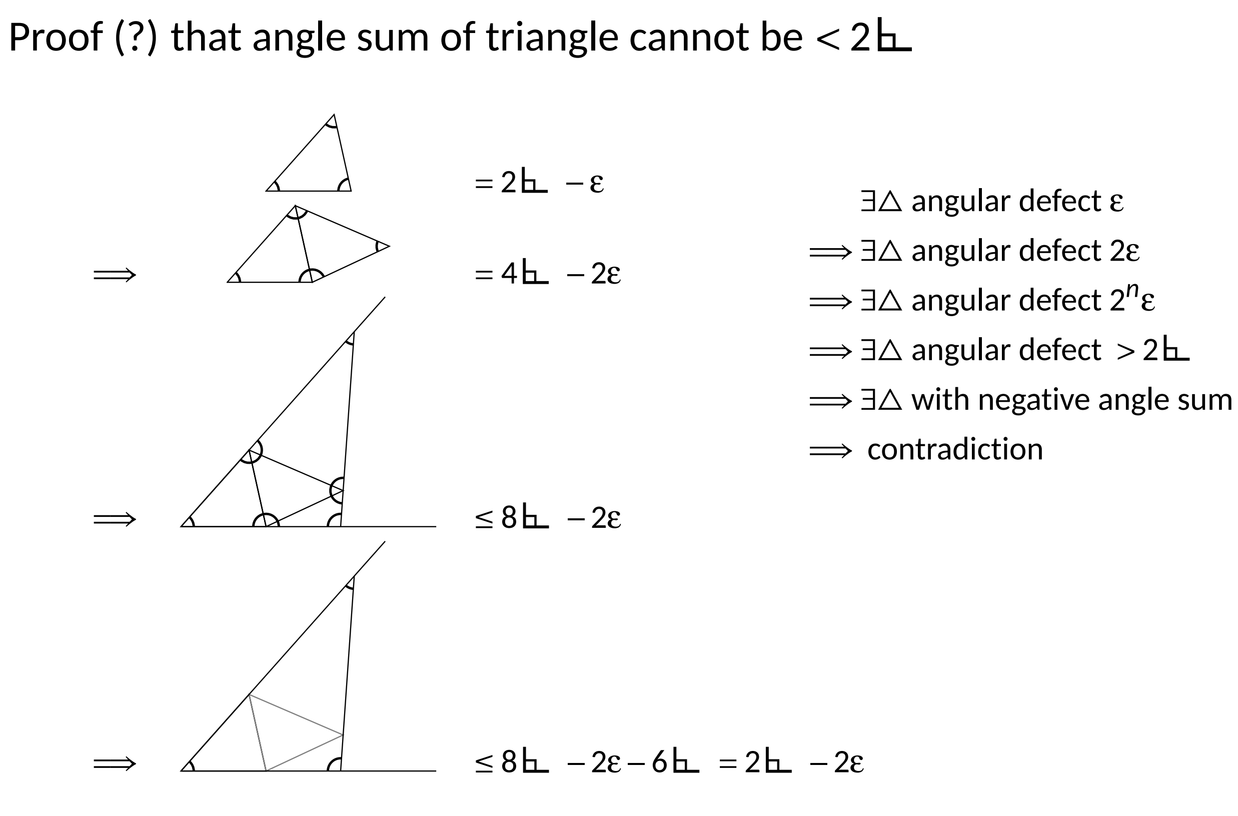


Figure 7: Legendre's attempted proof that the angle sum of a triangle cannot be less than 180°.

- ☐ Postulate 4
- ☐ Postulate 5

26.3. What does this mean for the feasibility of proving the parallel postulate from the other axioms?

26.4. Which geometries are "locally Euclidean" (i.e., practically indistinguishable from Euclidean geometry if one "zooms in far enough," i.e., restricts oneself to a small region)?

- ☐ 450 degree cone
- ☐ Spherical geometry
- ☐ Hyperbolic geometry

26.5. For a given line L and a given point P (not on L) in hyperbolic geometry, which are possible?

- ☐ There are no parallels to L through P.
- ☐ There is precisely one parallel to L through P.
- ☐ There is a finite number of parallels to L through P.
- ☐ There are infinitely many parallels to L through P.

26.6. (a) Name some other ways in which hyperbolic lines differ from plane and spherical lines.

(b) Name some ways in which hyperbolic and plane lines differ from spherical lines.

26.7. Argue that the angle sum of a hyperbolic triangle is less than 180°.

26.8. Do we live in a Euclidean or hyperbolic world? How can you tell?

Legendre tried to prove that the angle sum of a triangle cannot be less than 180° using only the first four axioms of Euclid. His proof is shown in Figure 7.

26.9. Why does Legendre's proof regarding the angle sum of a triangle not work in the hyperbolic plane?

- ☐ The placement of the second (mirrored) triangle and its congruence with the first triangle assumes properties of reflections that do not hold in hyperbolic geometry.
- ☐ When constructing the line through the outer vertex of the second triangle, it is assumed that for any pair of intersecting lines, and any point between them, one can draw a line through that point crossing both the given lines. But this assumption does not hold in the hyperbolic plane.
- ☐ It is assumed that the angular defect of the big triangle is the sum of the angular defects of its subtriangles. This is based on assuming that a "straight angle" is 180 degrees. But this assumption does not hold in the hyperbolic plane.
- ☐ Repeating the process produces triangles with greater and greater defect, but the defect does not go as low as 0; rather it approaches an upper bound, as the angle sum of the triangle approaches the original angle alpha.

Lagrange tried to prove the uniqueness of parallels from the principle that if there are two parallels to a given line through a given point, then the reflection of one of these parallels in the other should also be a parallel of the given line. This principle seems reasonable since there is no evident reason why one side of a line should be "privileged" over the other; either it's parallel or it's not; one line can't be "more parallel" than another. By reflecting again and again we can create new parallels further and further away from the ones we started with, which clearly produces a contradiction.

26.10. Why does Lagrange's proof regarding uniqueness of parallels not work in the hyperbolic plane?

- ☐ His principle is false, because when a parallel is reflected in a parallel, the result is not necessarily another parallel.
- ☐ His principle still holds but the conclusion he draws from it does not follow, because repeated reflection does not necessarily generate new, "more spread out" parallels.
- ☐ His principle holds, as does the generation of ever more parallels through repeated reflection, but the assumption that this leads to a contradiction is false.

§ 27. Space and perception

"The universal grammar of space: what geometry is innate?"

27.1. Geometry may be shaped by sensory perception in a way similar to how one's native language is formed by one's linguistic environment. What speaks in favour of this view?

27.2. How, and under what conditions, can a mind without specific geometrical preconceptions be led to impose a geometrical structure on sensory perceptions?

27.3. What does the experience of blind people becoming sighted tell us about the relation between space and perception?

language

geometry

- When we know only one language/geometry, it seems an intuitive necessity of thought.
- But the existence of other languages/geometries shows that we "over-universalised" our subjective intuitions.
- Not all intuitive content is innate or necessary; rather, it is shaped by environmental input.
- Nevertheless, this is only made possible by pre-fixed innate concepts that:
  - Underlie and structure all possible languages/geometries. ("universal grammar"; group-transformational concepts)
  - Enable the selection of the relevant (linguistic/spatial) data from the environment.

Table 7: Similarities between language and geometry.

§ 28. Relativity theory

"Operational Einstein: constructivist principles of special relativity"

	innate ("universal grammar")	intuitive ("native" lang./geom.)
necessary for linguistic/geometrical thought	■	■
essence of language/geometry-ness	■	
synthetic a priori	■	
particular contingent instantiation		■
… but doesn't limit the scope of conceivability		■
conflated in single notion pre non-Euclidean geometry	■	■
still credibly universal post non-Euclidean geometry	■	

Table 8: Intuition versus innateness.

28.1. 🗨 What is "relative" in relativity theory?

28.2. 🗨 Relativity theory vindicates 17th-century opposition to the notion of absolute space. Discuss.

28.3. 🗨 Relativity theory is an example of a scientific advance driven by philosophical, rather than internal scientific, considerations. Discuss.

28.4. 🗨 "I had studied [the work of David Hume] avidly and with admiration shortly before discovering the theory of relativity," Einstein said. Discuss and contextualise what philosophical views Einstein and Hume had in common.

28.5. 🗨 The thought experiment of a person chained to a wall can be used to illustrate both Einsteinian (space and time in special relativity) and pre-Einsteinian (Poincaré's philosophy of space) ideas. Explain.

28.6. Which of the following principles are valid in Einstein's theory of special relativity?

- ☐ No physical experiment can determine whether a closed system (such as the inside of a train car) is at rest or is moving at uniform velocity.
- ☐ speed of a projectile = speed of projecter + firing speed
- ☐ The speed of light (in vacuum) is a universal constant, c.
- ☐ There is an absolute, universal standard of time, captured by clocks.
- ☐ There is an absolute, universal standard of space, or length, captured by rulers.

§ 29. Polyhedra

📘 The dialogue part (starting at page 8 and continuing to page 25) of Lakatos's paper "Proofs and Refutations" ↗ (You do not need to pay attention to all the mathematical details. The interesting broader question is to what extent this kind of back-and-forth dynamic between proofs and refutations should be taken as typical of mathematics altogether.)

Bodies composed of polygonal faces are called polyhedra.[5]

[5] This figure and subsequent tables are from G. Polya, *Induction and Analogy in Mathematics*, Princeton University Press, 1954.

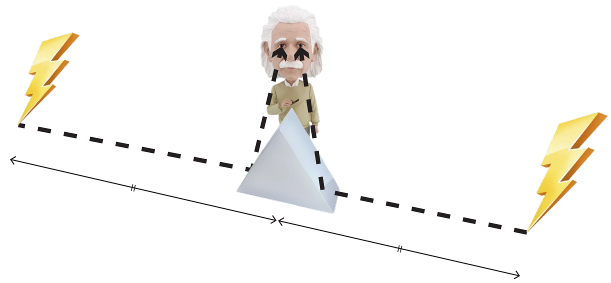


Figure 8: Einstein's definition of simultaneity. Two events occurred simultaneously if the light from them reach the eyes of an observer positioned halfway between them at the same time.

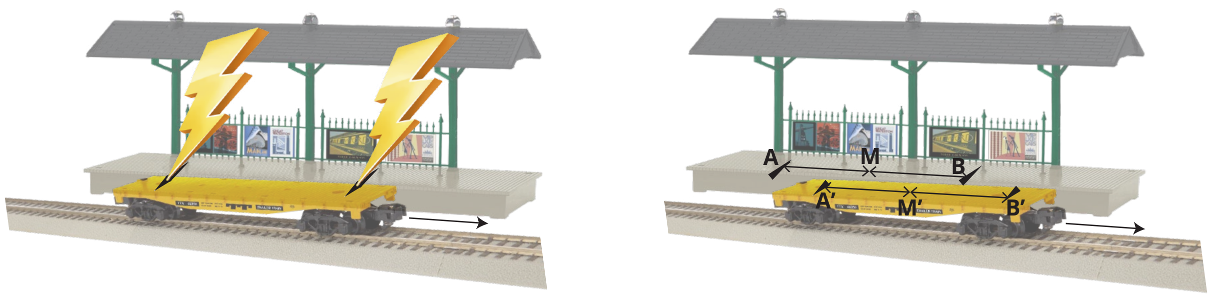


Figure 9: Relativity of simultaneity. Two lightning bolts strike a train car travelling at a certain velocity. Burn marks A, A', B, B' indicate the positions where the lighting struck. Using Einstein's definition of simultaneity, the observer on the stationary platform will say that the lighting strikes were simultaneous if the light rays from them coincide at M. However, if the light rays coincide at M, then the light from B has already passed M' while the light from A has not passed M'. Thus the observer on the train does not agree that the lighting strikes were simultaneous, and instead feels that the event at the front of the train (B, B') occurred earlier. Conversely, if the lighting strikes are simultaneous according to the observer on the train (light rays coincide at M'), then the observer on the platform (at M) will feel that the event at the back of the train (A, A') occurred first (since the light from this event reaches M before it reaches M').

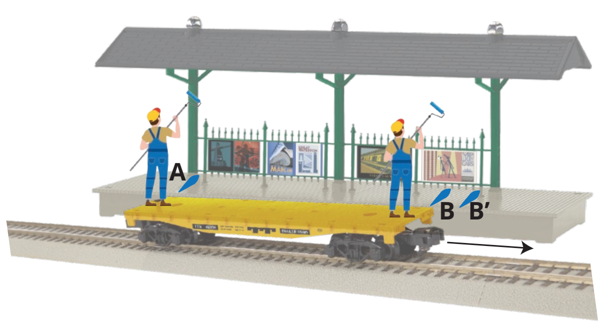
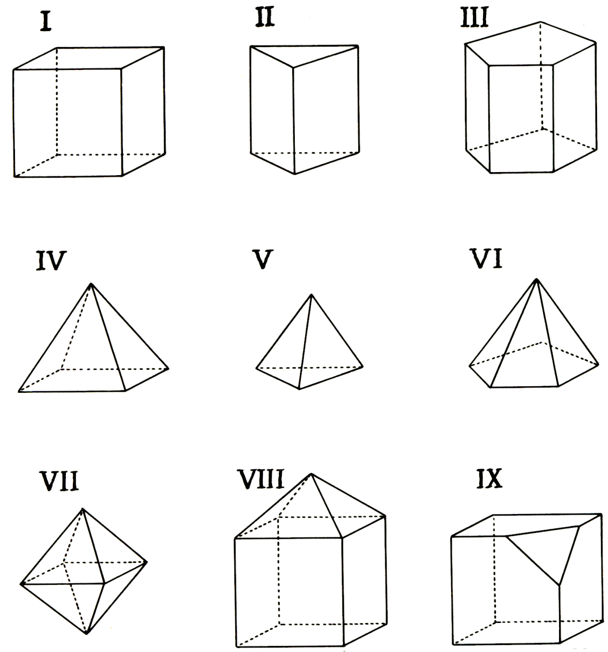
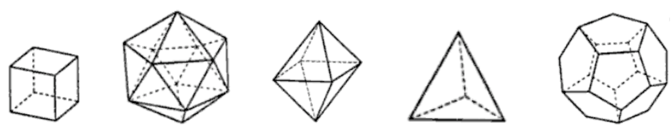


Figure 10: Relativity of length. The length of the train is transferred onto the platform by two paint marks drawn simultaneously. Because of the relativity of simultaneity, this gives different results for different observers. Marks A, B were made simultaneously according to an observer on the platform. Marks A, B' were made simultaneously according to an observer on the train.



There are precisely five *regular* polyhedra; that is, polyhedra where the faces are regular polygons and the corners as all alike.



Cube Icosahedron Octahedron Tetrahedron Dodecahedron

Is there a relationship between the number of faces F, vertices V, and edges E of a polyhedron? If we count these numbers for the above examples of polyhedra we find:

	Polyhedron	F	V	E
I.	cube	6	8	12
II.	triangular prism	5	6	9
III.	pentagonal prism	7	10	15
IV.	square pyramid	5	5	8
V.	triangular pyramid	4	4	6
VI.	pentagonal pyramid	6	6	10
VII.	octahedron	8	6	12
VIII.	"tower"	9	9	16
IX.	"truncated cube"	7	10	15

Looks quite complicated. But what of we sort the same data according to the last column instead?

Polyhedron	F	V	E
triangular pyramid	4	4	6
square pyramid	5	5	8
triangular prism	5	6	9
pentagonal pyramid	6	6	10
cube	6	8	12
octahedron	8	6	12
pentagonal prism	7	10	15
"truncated cube"	7	10	15
"tower"	9	9	16

And for the regular polyhedra:

Polyhedron	*F*	*V*	*E*
triangular pyramid . .	4	4	6
square pyramid . .	5	5	8
triangular prism . .	5	6	9
pentagonal pyramid .	6	6	10
cube	6	8	12
octahedron . . .	8	6	12
pentagonal prism . .	7	10	15
"truncated cube" . .	7	10	15
"tower" . . .	9	9	16

29.1. Find a formula that expresses the relation between F, V, and E.

We could find a formula for these examples, but does the formula work for *any* polyhedron? One way of investigating this conjecture is to check whether the relationship expressed by the formula is preserved by various polyhedral manipulations.

29.2. Consider the following ways of manipulating a polyhedron. "Roofing," as in going from I to VIII. "Truncating," as in going from I to IX. "Pinching," as in collapsing the top face of I to get IV. How are the quantities F, V, E changed under such an operation? (In the answer options below, n indicates the number of sides of the face to which the manipulation is applied, or, in the case of truncation, the number of edges meeting in the vertex to which the manipulation is applied.)

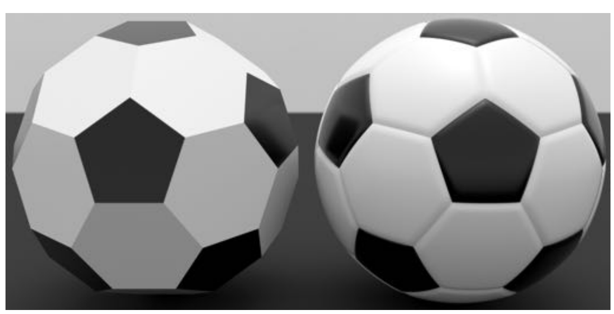
- ☐ Roofing, Truncating, Pinching
- ☐ -1 $\quad -n+1$ $\quad -n$
$+1$ $\quad +n-1$ $\quad +n$
$+n-1$ $\quad +1$ $\quad +n$

29.3. Suppose we started with a polyhedron for which the formula of problem 29.1 holds. In which cases do the formula also hold after the transformation has been applied?

- ☐ roofing
- ☐ truncating
- ☐ pinching
- ☐ none of these operations

29.4. A "soccer ball" polyhedron can be obtained from one of the polyhedra above by

- ☐ roofing
- ☐ truncating
- ☐ pinching
- ☐ none of these operations



29.5. One of the historical footnotes quite early on in Lakatos's "Proofs and Refutations" mentions crystals. What is the point being made?

- ☐ Crystals provided motivation for classifying polyhedra.
- ☐ Crystals provided motivation for focussing on V, F, E.
- ☐ Crystals suggested a counterexample to the formula.

☐ The formula can be used to understand how crystals form.

☐ The formula can be used to understand how crystals break.

29.6. Which of the characters in the dialogue think that trying to poke holes in arguments and looking for counterexamples for its own sake is a good way of doing mathematics?

☐ Teacher

☐ Alfa

☐ Delta

☐ Gamma

29.7. Lakatos leaves us with the impression that mathematical theorems are:

☐ Conjectures that have not yet been refuted.

☐ Infallible by virtue of the precision of mathematical language.

☐ Socially constructed beliefs.

§ 30. General discussion questions

30.1. 🗩 Argue that geometry, as conceived in Euclid's *Elements*, is distinct from the physical world; that it is purely logical, abstract, Platonic.

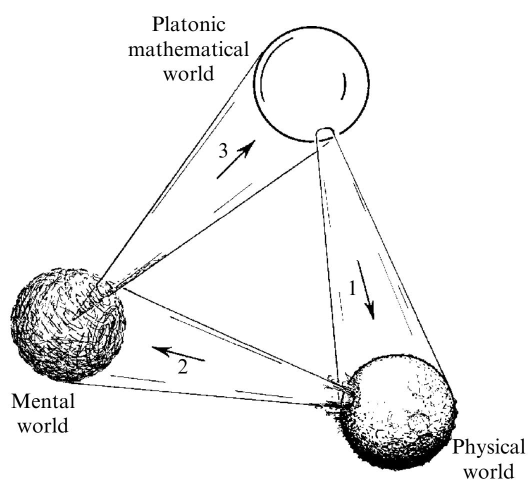
30.2. 🗩 Argue that geometry, as conceived in Euclid's *Elements*, concerns physical space and is anchored in concrete and hands-on operations with real-world tools like ruler and compass.

30.3. 🗩 What are some examples of "assumption minimalism" in Euclid? That is, situations where it would have been natural to assume a bit more but Euclid would rather start from only the least possible assumption even if this means more work.

30.4. 🗩 If Greek civilisation had been eradicated in some war before it got going, would there have been "another Euclid" somewhere else?

30.5. 🗩 If there is intelligent life on another planet, do they know Euclidean geometry? Consider how thinkers such as Plato, Descartes, Kant, and Poincaré might answer this question.

30.6. 🗩 Consider the "three worlds" picture of Penrose, quoted below. Consider two thinkers mentioned in the course (such as Plato and Kant) and discuss which aspects of this picture they would agree with and which they would disagree with.



"I am allowing that only a small part of the world of mathematics need have relevance to the workings of the physical world. ... Likewise, ... I am not insisting that the majority of physical structures need induce mentality. ... Finally, ... I regard it as self-evident that only a small fraction of our mental activity need be concerned with absolute mathematical truth! These three facts are represented in the smallness of the base of the connection of each world with the next.

However, it is in the encompassing of each entire world within the scope of its connection with the world preceding it that I am revealing my prejudices. Thus, ... the entire physical world is depicted as being governed according to mathematical laws.

Likewise, many might object ... that I am taking too hard-boiled a scientific attitude by drawing my diagram in a way that implies that all of mentality has its roots in physicality. A further prejudice of mine is ... that ... I have represented the entire Platonic world to be within the compass of mentality. This is intended to indicate that—at least in principle—there are no mathematical truths that are beyond the scope of reason.

[There are] three deep mysteries [associated with the three arrows in the diagram:]

[1] Why mathematical laws should apply to the world with such phenomenal precision.

[2] How it can come to pass that appropriately organized physical material—... living human (or animal) brains—can somehow conjure up the mental quality of conscious awareness.

[3] How it is that we perceive mathematical truth [given that] mathematics ... seems to have a robustness that goes far beyond what any individual mathematician is capable of perceiving. Those who work in this subject, ... usually feel that they are merely explorers in a world that lies far beyond themselves—a world which possesses an objectivity that transcends mere opinion, be that opinion their own or the surmise of others, no matter how expert those others might be." (Roger Penrose, *The Road to Reality*, 2004, §§1.3–1.4.)