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THE SIGNIFICANCE OF SLUSE'S MESOLABUM WITHIN SEVENTEENTH-CENTURY GEOMETRY AND ALGEBRA

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1. Introduction

Sluse's *Mesolabum* appeared in Liège in 1659. Its full title was (in translation) :

> "Mesolabum, or two mean proportionals between given bounds exhibited by circle and ellipse or hyperbola in an infinity of ways; also the construction of any solid problem by the same curves in the same ways and an appendix on their solution by circle and parabola." (1)

As the title indicates, Sluse's book was about geometrical construction. It dealt with finding two mean proportionals between two given lengths and in general with a class of geometrical problems called "solid"; in both cases Sluse offered an infinity of ways to perform the construction. A second edition of the *Mesolabum* was published in 1668, also in Liège. It contained the full text of the earlier edition (but with better figures) and two substantial additions, the "Analysis" and a section "Miscellanea".

The second edition was very favourably reviewed in the *Philosophical Transactions* of 1669; the review closed with the following sentence:

> "Concerning this Book, we find it to be the judgement here, (and doubtless it will have the same esteem elsewhere among the Learned) that in it there is the most excellent Advancement made in this kind of *Geometry*, since the famous Mathematician and Philosopher DesCartes." (2)

Eighteenth century mathematicians, reviewing the developments in algebra and geometry to which Sluse's *Mesolabum* had contributed, also came to favourable judgements. Wolff described Sluse's achievement in the chapter on "construction of equations" of his *Elementa Matheseos* as follows :

> "It was René François de Sluse, cannunic at Liège, who for the first time explained the true artifice to construct these equations (-) ; other writers later followed him when commenting on this matter." (3)

Reimer, writing about the history of the problem to which the *Mesolabum* was devoted, noted that Sluse found for the first time

> "the true ways of constructing equations by geometrical loci". (4)

Similar statements can be found in several other sources from the eighteenth century (5).

The title of Sluse's book and the quotations above indicate the themes which I shall have to discuss when answering the question about the significance of Sluse's *Mesolabum* within 17th century geometry and algebra. These themes are : geometrical problems, construction, equations, the new method of Descartes, and the term which provided the title of Sluse's book : *Mesolabum*.

2. The art of finding mean proportionals

Mesolabum is a term used in classical Greek geometry to denote an instrument for constructing mean proportionals. In the sixteenth century, however, the term (also spelled *mesolabium*) no longer had a strictly instrumental connotation ; it meant in general the art of constructing mean proportionals (6) ; Sluse used the term in that sense.

A mean proportional between two magnitudes a and b is a magnitude which occurs as term of a geometrical series with a and b as first and last term respectively. The simplest case occurs when the series has three terms : a, x, b, with

$$a : x = x : b;$$

x is then called the mean proportional or the geometric mean of a and b. The construction, by ruler and compass, of the geometric mean between two line-segments is given in Euclid's *Elements* (II, 13 and VI, 14).

The case of two mean proportionals occurs when the series has four terms : a, x, y, b. Hence the problem to construct these mean proportionals is :

Given : two lengths a and b;
Required : two lengths x and y such that a, x, y and b form a geometrical series, i.e. :

$$a : x = x : y = y : b.$$

The problem was formulated and solved in classical Greek geometry. It arose out of one of the classical problems, namely the duplication of the cube.

Given : a cube;
Required : a cube twice as large.

Or, to put it in terms of algebra, given a (the side of the cube) to find x (the side of the required cube) such that $x^3 = 2a^3$. Greek geometers realized that the side x of the required cube is the first of two mean proportionals between a and $2a$; thus a method to construct two mean proportionals would imply a method to double the cube.

Obviously one can generalize the problem further by requiring three, four, etc. mean proportionals. A special case is that of eleven mean proportionals

$$a : x_1 = x_1 : x_2 = ,\ldots, = x_{11} : b.$$

because that problem occurs in connection with the tuning of string instruments. To divide the octave on a monochord in twelve equal tones requires the frets to be placed on distances that are the eleven mean proportionals between the string length and its half.

3. Two mean proportionals: the tradition

Duplication of the cube, or the construction of two mean proportionals, was one of the classical geometrical problems (the others being the trisection of the angle and the quadrature of the circle). Many Greek geometers dealt with the problem ; we know about 15 different constructions from antiquity. Most of these became known to Western European mathematicians during the sixteenth century through Eutocius' commentary on Archimedes' *Sphere and cylinder*, in which no less than 12 different constructions were listed (7).

In the sixteenth and early seventeenth centuries we find the problem treated in almost all books on advanced geometry. We also find several monographs specially devoted to the problem, often under the title *Mesolabum*, for instance Salignac's *Mesolabii expositio* (1574), Scaliger's *Mesolabium* (1594) and Molther's *Problema Deliacum* (1619).

Traditionally, the origin of the problem was connected with a request of the gods that an altar of cube form be doubled. Some classical writers also related the construction of two mean proportionals to the problem how much certain ballistical instruments (for throwing stones) should be enlarged in order to the term (also spelled *mesolabium*) no longer had a strictly instrumental connotation ; it meant in general the art of constructing mean proportionals (6) ; Sluse used the term in that sense.

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In fact the significance of the problem of mean proportionals was not in its solutions but in the nature of the question itself. It was a question of construction, and indeed it concerned a crucial aspect of that geometrical procedure, namely the acceptability of other means of construction than ruler and compass.

4. Geometrical construction (10)

One characteristic feature of early modern geometry, the geometry of the sixteenth and early seventeenth centuries, was that its interest lay primarily in solving problems, not so much in proving theorems. There are very few, if any, theorems dating from that period ; it seems that geometers then felt that the stock of theorems in Euclid's *Elements* was enough for the time being and that the task of geometry now was to use these theorems in solving geometrical problems. In Euclid's *Elements* theorems and problems were treated as fundamentally different geometrical arguments ; theorems had to be proved, problems had to be constructed. From antiquity there was established a tradition in geometry which stipulated that constructions should preferably be performed by ruler and compass, or rather, by straight lines and circles. The significance of the problem of constructing two mean proportionals lay precisely in the fact that geometers had realized (though not proved) that its construction could not be performed by straight lines and circles.

The many classical constructions for finding two mean proportionals all used additional means of construction. Two categories can be discerned in these constructions beyond ruler and compass. Some used special instruments in addition to ruler and compass, for instance sliding rulers or sliding squares. If such an instrument was devised specially for the construction of mean proportionals it was called, as noted above, *mesolabum*. In the other category the construction was performed by the intersection of curves other than straight lines and circles ; for instance by intersecting two conics or a circle and a parabola (I shall give an example of such constructional use of intersections of curves below).

It should be noted that these constructions were not meant to be actually performed in any practice ; they all belonged to idealized, abstract geometry. If they involved instruments, these instruments were also considered in abstracto.

5. Two mean proportionals, the literature.

The early modern studies on constructing mean proportionals may be divided in three categories. The first is plainly nonsensical ; it consists of the studies that tried to prove that two mean proportionals could after all be found by straight lines and circles. The books by Salignac and Scaliger belong to this class. The next category may be characterized as sensible, but without much effect. These studies tried to argue that the additional means of construction necessary in solving the problem were indeed equally acceptable in geometry as straight lines and circles. Molther's book, for instance, dealt with this theme, and Descartes' *Géométrie* (1637) also contains extensive discussions on the acceptability within geometry of means of construction beyond straight lines and circles. (11)

Finally the studies that were significant and fruitful were those undertaken in the context of the application of algebra to solving geometrical problems. Viète, Fermat and Descartes are the most important mathematicians in this respect ; in their work we find the beginning of analytic geometry. Their work also forms the background of Sluse's *Mesolabum*, and therefore something more has to be said here about the use of algebra in solving geometrical problems, and especially about Descartes' approach in this matter.

6. The use of algebra in early modern geometry.

The use of algebra in early modern geometry was usually called "analysis". Mathematicians knew that the Greeks had had a method, called analysis, for finding proofs of theorems and constructions of problems, and it was generally believed that that method was more powerful than what appeared from the then extant texts about it. Some mathematicians even saw the new use of algebra as a rediscovery of the old analysis of the ancients.

The use of this new, algebraic analysis in the case of geometrical problems may be summarized as follows. First one had to translate the problem in question into an algebraic equation. Thus doubling the cube leads to the equation

$$x^3 = 2a^3,$$

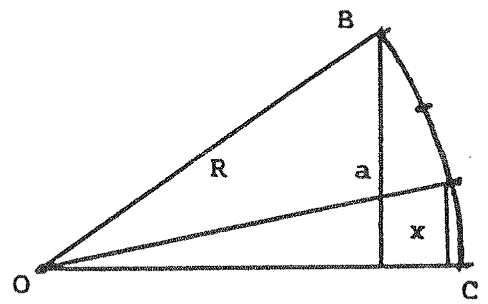
and finding two mean proportionals between a and b leads to the equations

$$x^3 = a^2b \quad \text{and} \quad y^3 = ab^2$$

(directly derivable from the condition $a:x=x:y=y:b$). Usually the equations are more complicated. For instance in the case of trisecting the angle BOC (see Figure 6.1) the equation is :

$$4x^3 - 3xR^2 + aR^2 = 0.$$

Figure 6.1



The next step is to use this equation to find the construction of the unknown x. This procedure was called the "construction of the equation". It should be stressed here that the algebraical solution of an equation is not of help in finding its construction. Algebraic solution of the equations above, for instance, involves cubic roots but leaves unanswered the question how these cubic roots should be geometrically constructed. Hence geometers were confronted with the general problem of finding geometrical constructions for the roots of any given algebraical equation. It was soon realized that roots of linear and quadratic equations can be constructed by straight lines and circles. Viète gave constructions of the roots of cubic equations without quadratic term using sliding rulers (so-called neusis constructions), and he argued that in principle the roots of all cubic and fourth degree equations can be so constructed. (12) Sluse's *Mesolabum* fits in this programme of constructing equations, its first edition dealt implicitly with the construction of roots of cubic equations ; its second edition explicitly gave the construction for third and fourth degree equations. The most important book within that programme, however, was Descartes' *Géométrie*.

7. Descartes

In his *Géométrie* (1637) Descartes studied the construction of algebraic equations in general. In doing so, he was confronted with the question what means of construction should be allowed in geometry beyond the straight line and the circle. Descartes chose to allow the use of (algebraic) curves and their intersections. He considered instruments only in as far as they traced these constructing curves. (13) He gave canonical constructions for equations ordered by their degree. These constructions were as follows :

degree of the equation	problems called	construction by intersection of
1 - 2	"plane"	straight lines and circles
3 - 4	"solid"	parabola and circle
5 - 6	"supersolid"	circle and a certain third-degree curve

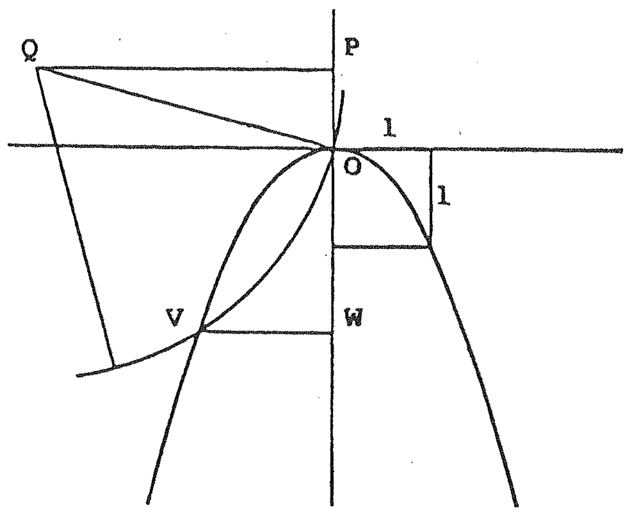
The terms "plane" and "solid" were taken over from classical geometry ; Pappus had explained in his *Collectio* (14) that problems constructible by straight lines and circles were called plane and those constructible by the intersection of conics "solid". By the beginning of the seventeenth century it had become clear that problems leading to third or fourth degree equations were "solid" in Pappus' sense. Descartes' result that they can all be constructed by parabola and circle, however, was new. Pappus had lumped all problems beyond the solid ones in one class which he called "linelike", because their construction required more complicated lines than conic sections. Descartes suggested to classify further by pairs of successive degrees of the pertaining equations. He did not, however, explicitly give these constructions beyond the sixth degree.

Descartes' construction of the third-degree equation

$$x^3 + px = q$$

may serve as an example of his procedure (his construction of general fourth degree equations is similar and not much more complicated). Descartes prescribed the following construction (see Figure 7.1) :

Figure 7.1



Draw a parabola with vertical axis and vertex O. Adjust the scale such that the point with coordinates (1,1) is on the parabola. Take $OP = (p-1)/2$ along the vertical axis, up if positive, down if negative. Take $PQ = q/2$ horizontally, to the left if positive, otherwise to the right. Draw a circle with centre Q and radius QO. The circle will intersect the parabola in a point (or in points) V. Draw VW horizontally with W on the axis. VW will be the required root, to be considered positive if V lies to the left of the axis, otherwise negative. (The construction is correct as can be checked easily.)

The example makes clear what was meant by construction by means of intersection of curves : if a certain length is to be so constructed, it is required to find two curves (in this case a parabola and a circle) such that the distance of one of its points of intersection to an axis is equal to the required length. Note that it is not explained how these constructing curves are themselves constructed, or how exactly they yield their points of intersection. In particular in the case of construction by intersection of conic sections, these curves were considered sufficiently determined if their basic parameters (top, axes, latus rectum) were given.

Descartes had shown how any equation of third or fourth degree could be constructed by intersection of a parabola and a circle, thereby proving that all solid problems could be so constructed. He mentioned in the *Géométrie* that other conics than the parabola could also be used, and even claimed that any given conic may be used (15). We will see that Sluse dealt with precisely that question in the second edition of the *Mesolabum*.

Descartes had offered no explanation of how he had determined the curves necessary for constructing an equation ; he just gave the recipe and an algebraic proof that the result was correct. Van Schooten and others later supplied the explanation : the curves can be found by the method of undetermined coefficients, and that was probably how Descartes found them (16). As we will see, Sluse provided another answer to the question of how to find constructing curves. It was this answer to which Wolff and others referred when stating that Sluse had for the first time discovered the true artifice of the construction of equations.

8. *Mesolabum* 1659

I now turn to the first edition of the *Mesolabum*. As we have seen, Sluse announced in the title that the book would provide an infinity of constructions for the problem of finding two mean proportionals and in general for all solid problems. In the preface he motivated this goal of the book. Writing about the problem of two mean proportionals he noted :

> "...even those who acknowledge that it is necessary to use either special instruments or conic sections, have given us until now so few demonstrations. For there seem to be hardly as many such demonstrations as there have been centuries since the problem was first proposed." (17)

He added that no constructions had been proposed involving an ellipse and only a few with circle and parabola or hyperbola. The argument may seem surprising : after some 20 different constructions the problem could be considered solved. However, Sluse's interest was not in solving the problem but in understanding the variety of ways it could be solved.

I shall not discuss Sluse's constructions of two mean proportionals (18) but rather the construction he gave of one type of solid problem, because that is more illustrative of his approach. I refer to his seventh proposition :

> A given line QD, cut in A, is to be cut again in C such that the ratio of the square of QA and the square of AC is equal to the ratio of AC and CD, by means of a circle and an ellipse in an infinity of ways. For this is the paradigm for the cubic equation affirmatively affected on the side. (19)

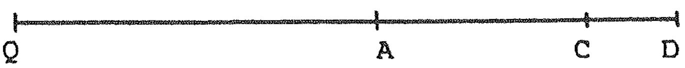


Figure 8.1

Thus (see Figure 8.1) QA and AD along a straight line are given and the point C is required such that

$$(QA)^2 : (AC)^2 = AC : CD \tag{8.1}$$

The second sentence of the proposition refers to a classification of equations introduced by Viète. (20) Call (21) $AC = x$, $AD = a$ and $QA = b$, then the required proportion is

$$b^2 : x^2 = x : (a-x), \tag{8.2}$$

which can be written out as an equation :

(a) $$x^3 + b^2x = b^2a. \tag{8.3}$$

This is one of three different types of cubic equations without quadratic terms divided according to the signs of the coefficients. The other two types are :

(b) $$x^3 - b^2x = b^2a \tag{8.4}$$

(c) $$b^2x - x^3 = b^2a \tag{8.5}$$

(with a and b positive). Sluse referred to them in terminology introduced by Viète ; "positively affected on the side" means that the coefficient of the x-term (the side) is positive (always writing the positive constant term on the right hand side and all the others left). The other two are called "negatively affected on the side" and "amphibola" respectively ; the latter term being one of the many Vietean neologisms ; the Greek amfibolos means "ambiguous". In the first edition of his *Mesolabum* Sluse referred to these equations but he did not write them out. Instead he treated "paradigm problems" that were equivalent to the equations. This practice can also be traced back to Viète (22) and the paradigm problems which Sluse used are closely related to the ones Viète gave. In connection with Sluse's analysis, to be discussed below, it is of interest to compare his "paradigm problems" with those of Viète.

Viète considered the following problems. Let two lengths a and b be given. It is required to find three lines x, y, z, such that

$$b : x = x : y = y : z \tag{8.6}$$

and :

case (a) : $x + z = a$ (8.7)

case (b) : $z - x = a$ (implying $b < x$) (8.8)

case (c) : $x - z = a$ (implying $b > x$) (8.9)

It is easily seen that the three cases lead to the corresponding three equations. But they also lead to Sluse's paradigm problems : eliminating y from the proportionality we get

$$b^2 : x^2 = x : z \tag{8.10}$$

and in case (a), where $z = a - x$, we have precisely the problem Sluse dealt with in his seventh proposition ; his other two paradigm problems also correspond in this way to the Vietean ones.

Sluse's construction of the problem posed in the seventh proposition is as follows (see Figure 8.2) :

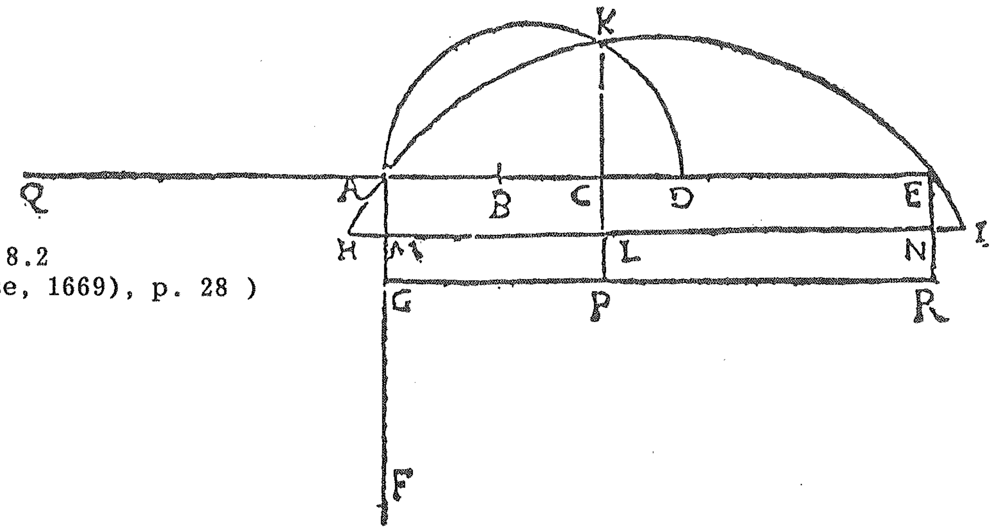


Figure 8.2
((Sluse, 1669), p. 28)

QA and AD are given. Draw a semicircle on AD ; take $AF = AQ$ perpendicular to AD, take G on AF arbitrarily. Construct the point E on AD prolonged such that $AD : AE = GF : FA$ and complete the rectangle $GAER$. Bisect AG and ER in M and N respectively. Construct points H and I on MN prolonged such that $(NH.HM) : MA^2 = AF : FG$ and $NI = HM$. (Until here all constructions are standard straight line and circle constructions.) Draw a semi-ellipse with axis HI and latus rectum (23) L such that $L : HI = FG : FA$; the ellipse passes through A and E. Let K be the point where the ellipse intersects the circle. Draw KC perpendicular to AD ; C is the required point.

Sluse added the proof that the construction is correct ; I shall omit that proof here. The origin of Sluse's construction will become clear when I discuss Sluse's analysis in Section 10.

The construction of prop. 7 well illustrates the character of Sluse's book. All its propositions concern constructions in similar style ; they are procedures to locate in a figure a circle and a parabola or an ellipse or a hyperbola such that the ordinate of the intersection of the conic and the circle provides the required length.

Sluse presented his constructions and proofs in classical Greek style ; no algebraic symbolism occurred in the book, the argument was given entirely in prose. It was Sluse's deliberate choice to present his results in that way, keeping the algebra, as it were, under the surface (24). The style contrasts strongly with Descartes' in the *Géométrie* ; although the constructions themselves have much in common (both employ the intersection of conics), Descartes used algebraic symbols and techniques explicitly throughout his work and he made no effort to adopt the classical style of geometrical reasoning.

The construction of proposition 7 also illustrates in what sense Sluse gave "infinitely many" constructions of the problem. He prescribed that the point G be taken arbitrarily on AF. Each choice gives a different ellipse so that Sluse in fact gave an infinity of ways to construct the problem by circle and ellipse. It is clear that by appropriate choice of G any form of the ellipse (i.e. any ratio of the axes) may occur. However, once G is chosen, both form and size of the ellipse are fixed. Therefore not all possible ellipses occur among those in Sluse's construction. Hence Sluse's construction did not prove Descartes' claim (cf Section 7) that any given ellipse may be used to construct the problem. We will see that Sluse later returned to this matter and proved that Descartes' claim was correct.

9. Reactions on the first edition

Directly or indirectly Sluse received comments on his book from Huygens, Oldenburg, Van Schooten and De Witt. These reactions were in general favourable, but several questions were raised, first of all concerning his method. From Sluse's text it was clear that he had used a special analysis to find his constructions. In fact he had written in the preface :

"I did not add the method, because I thought it would be more rewarding and useful if you extracted it yourself from these examples, and also because I wished to wait for your judgement about this whole business. Therefore I decided that, if your reaction would be favourable, I would soon, provided God's good help, not only submit the method to your judgement but also other matters which I have observed as well." (25)

He was indeed asked to make the method known (26) and he did so in the second edition of the *Mesolabum*.

In the first edition Sluse had dealt only with problems equivalent to cubic equations. In one of his first letters to Oldenburg (27) he mentioned that his method indeed covered all third and fourth degree equations, and he gave a construction (by parabola and circle) of the general fourth degree equation. It is not clear whether he had this construction already in 1659 (28) ; he gave a full explanation of these constructions in the second edition of his book.

Descartes' claim that solid problems can be constructed by any given conic came up in several places in the correspondence between Sluse and Huygens, both before and after 1659 (29). Huygens wrote to Sluse in October 1657 that he had a construction of two mean proportionals by means of any prescribed ellipse. It seems that Sluse at first did not understand what Huygens meant, and that Huygens, on first reading the *Mesolabum*, thought (wrongly) that Sluse's method implied a proof of the claim. Somewhat later Huygens transmitted to Sluse a remark of De Witt to the effect that it would have been more elegant to construct by an arbitrary prescribed conic than by an infinity of such curves. This remark seems to have induced Sluse to consider the matter, and in 1664 he announced that he was able to give such a construction ; he did so in the second edition of the *Mesolabum*.

Another question concerned Sluse's attitude to the analysis of Descartes. On an earlier question of Huygens he had answered in 1657 :

"To be frank, the authority of that incomparable man should of course convince me that Descartes' method is preferable over all others ; still I do not use it, because when his writings first came into my hands I had used myself already to another special method." (30)

And two years later he wrote :

"I must confess that I know no other than the analysis of Viète (I had acquainted myself with that method before I learned about Descartes' and in my opinion they don't differ much)". (31)

As we have seen, Sluse's Vietean style is indeed evident in his book ; in Section 12 I shall discuss in how far this influenced the significance of his work.

10. The second edition of the *Mesolabum*, 1668 ; Sluse's analysis

The second edition of the *Mesolabum* appeared in 1668. It consisted of the entire text of the first edition (with improved figures) and additions. Sluse changed the title :

"Mesolabum, or two mean proportionals between given bounds exhibited by a circle and by infinitely many hyperbolas, or ellipses, and by an arbitrary one, and the construction of all solid problems by the same curves. A second part on the analysis is added, and miscellanea" (Sluse, 1668)

As the title indicated, the additions were : an argument that the constructions could be performed by any prescribed conic, a section explaining Sluse's analysis, and a section containing various matters. I shall not here discuss the last mentioned section. It contains important material, (32) but mostly without direct relation to the principal theme of the *Mesolabum*.

The most important part is the Analysis. In this piece (pp. 51-95) Sluse frankly and carefully explained how he used algebra to find the constructions. I shall summarise its contents by expliciting the key ideas in Sluse's approach to the construction of third and fourth degree equations. I shall describe his analysis in the notation of modern algebra, which is, in fact, the notation that Descartes introduced. Sluse knew this notation but preferred to keep to his own (33),which was a modification of Viète's notation. Viète had used capital letters ; vowels for the unknowns, consonants for the given or constant quantities, and he used an intricate system to denote powers, based on dimensional interpretation. Sluse used lower case letters, again vowels for unknowns, consonants for constant or given quantities, he denoted powers in the Cartesian way, and he used the symbols // and / for equality and ratio respectively.

Sluse started with the problem that gave the book its name, the *Mesolabum*, the construction of two mean proportionals. We have :

$$a : x = x : y = y : b \tag{10.1}$$

Sluse remarked that the proportionalities yield three equations, each of them pertaining to a conic section :

$$xy = ab \quad \text{a hyperbola,} \tag{10.2}$$

$$ay = x^2 \quad \text{a parabola} \tag{10.3}$$

$$bx = y^2 \quad \text{a parabola} \tag{10.4}$$

Values x and y satisfying two of these equations also satisfy the third ; so the required mean proportionals x and y are the coordinates of the points of intersection of any two of the curves. Hence any two of the curves can be taken as constructing curves for the problem ; the distances x, y, of their point of intersection to the axes yield the required lengths. (The argument was not new ; it underlies, although not expressed algebraically, the construction of two mean proportionals due to Menaechmus ca. 350 bC (34).)

Sluse's first new idea was that from the three equations one can find further constructing curves by combination. For instance adding (10.3) and (10.4) yields

$$ay + bx = x^2 + y^2, \tag{10.5}$$

a circle, which may be combined with any of the three curves (10.2-4), whereby we find two constructions with circle and parabola and one with circle and hyperbola. Sluse also remarked that (10.3) is equivalent to

$$\lambda ay = \lambda x^2, \tag{10.6}$$

where λ is any (positive) ratio. Combining (10.6) with (10.4) yields

$$\lambda ay + bx = \lambda x^2 + y^2 \quad \text{(ellipses),} \tag{10.7}$$

$$\lambda ay - bx = \lambda x^2 - y^2 \quad \text{(hyperbolas).} \tag{10.8}$$

Any of these ellipses or hyperbolas together with the circle (10.5) yields a construction of two mean proportionals ; this is the way Sluse found his infinitely many constructions for the *Mesolabum*.

The next step was to relate the procedure above to the Vietean paradigm problems corresponding to cubic equations without quadratic term. As was explained in Section 8, these equations correspond to the problem of constructing, for given lengths a and b, a series of four proportionals

$$b : x = x : y = y : z, \tag{10.9}$$

such that $z = \pm a \pm x$. Obviously we can now form and combine equations in the same way as above, and, because z is linear in x, we again get conic sections. In the case of $z = a - x$ (corresponding to the "paradigm problem" of Prop. 7, see Section 8) we have

$$xy = b(a-x) \qquad \text{hyperbola} \tag{10.10}$$

$$x^2 = by \qquad \text{parabola} \tag{10.11}$$

$$x(a-y) = y^2 \qquad \text{circle.} \tag{10.12}$$

Sluse added (10.11) and (10.12) to get the parabola

$$ax = y^2 + by, \tag{10.13}$$

which he combined with arbitrary multiples of (10.11) to get infinitely many ellipses

$$y^2 + by - \lambda by = ax - \lambda x^2 \tag{10.14}$$

His construction, discussed in Section 8, is by intersecting the ellipses (10.14) with the circle (10.12). By taking G arbitrarily on AF (cf Figure 8.2), Sluse fixed $\lambda = GF/AF$ - it is easily checked that his construction of the points G and E through which the ellipse passes conforms to the equation (10.14) with that value for λ .

Thus far this is certainly the analysis which Sluse used in deriving the constructions presented in the 1659 edition of the *Mesolabum*. He went on to present the analysis for third degree equations with a quadratic term and for fourth degree equations. That analysis is somewhat cumbersome. Perhaps a study of his manuscripts could reveal the date of that part of the analysis, but from the printed texts it seems most likely that Sluse worked it out after the first edition, perhaps as a reaction upon the questions concerning fourth degree equations raised in his correspondence with Oldenburg.

Sluse's further analysis may be summarized as follows. In the case of the cubic equations without quadratic terms, discussed above, the equation is first rewritten as a proportionality

$$b^2 : x^2 = x : z, \quad z = \pm a \pm x. \tag{10.15}$$

The geometric mean y of x and z is then introduced to reduce this proportionality to a continued proportion between terms linear in x and y, namely

$$b : x = x : y = y : z, \tag{10.16}$$

and the separate proportions of (10.16) yield the constructing curves. Sluse found that this approach is applicable, with modifications, in the case of general third and fourth degree equations, but the y can no longer be interpreted as a mean proportional ; it has to be chosen such that the analogue of (10.15) is reduced to a proportionality between terms linear in x and y. Sluse formulated a *Regula universalis* (pp. 90-95) and illustrated it with examples. I paraphrase his example of a fourth degree equation :

$$x^4 - 2bx^3 + bax^2 + b^2cx = b^2d^2. \tag{10.17}$$

The equivalent proportionality is

$$b^2 : x^2 = (x^2-2bx+ba) : (d^2-cx). \tag{10.18}$$

Putting

$$x^2 - bx = qy \tag{10.19}$$

we can make all terms in (10.18) linear in x and y :

$$b^2 : (qy+bx) = (qy-bx+ba) : (d^2-cx) \tag{10.20}$$

and rewrite it as an equation :

$$q^2y^2 - b^2x^2 + abqy + ab^2x + b^2cx = a^2d^2. \tag{10.21}$$

This equation represents infinitely many ellipses (because q is still undetermined). Sluse then showed how, by combining (10.19) and (10.21) one can get a circle, and the intersection of this circle with the ellipses (10.21) gives the construction of the fourth degree equation.

The other addition to the second edition was Sluse's method to construct solid problems by means of any given conic. The addition consists of four pages (pp. 43-46) and it only contains one example, from which the general approach should become clear. The example is the construction of two mean proportionals by means of a given ellipse. I shall not give the details but only sketch the idea behind the construction. Let A and B be the given lengths between which two mean proportionals X and Y have to be constructed ; let Γ be the given conic. In his first proposition Sluse had given a construction of two mean proportionals between lengths a and b by means of an infinity of ellipses. Sluse now found that it is possible to construct (by straight lines and circles) two segments a and b such that

$$a : b = A : B \tag{10.22}$$

and such that Γ occurs among the infinitely many ellipses by which the mean proportionals x and y between a and b can be constructed. So x and y are found by means of Γ and we have

$$a : x = x : y = y : b \tag{10.23}$$

and

$$a : A = b : B. \tag{10.24}$$

The required mean proportionals X and Y can now be constructed from x and y because

$$x : X = y : Y = a : A \tag{10.25}$$

and the ratio $a : A$ is known.

11. Sluse's geometrical style

Mesolabum was an apt title for Sluse's book ; his approach to the construction of third and fourth degree equations can be understood, as I noted in Section 10, as a generalization of the method to construct two mean proportionals. The fact that the name of a geometrical problem served as title of the whole book points to the importance, for Sluse, of the geometrical side in the interplay between algebra and geometry. There is more evidence for this : Sluse's analysis keeps to dimensional homogeneity of equations (thus assuring their direct geometrical interpretability independently of the choice of a unit length) ; algebraic equations are related to corresponding geometrical paradigm problems ; and there is the curious role of the proportionalities (10.15) and (10.18). Sluse stated explicitly that the reduction of the equations to these proportionalities was the key to his method (35). However, for the algebraical side of his argument the proportionalities are superfluous, one can pass immediately from equation (10.17) to equation (10.21) by substituting (10.19). The role of the proportionalities is to provide a geometrical interpretation of the algebraic manipulations and especially of the new variable *y*.

But this geometrical interpretability also had its disadvantages ; it blocked a further generalization of Sluse's approach to equations of higher degree. For such equations an analysis by means of proportionalities analogous to (10.17) - (10.21) is not practicable. Here Sluse's method contrasts with that of Descartes, which, because of its more openly algebraic character, could easily be extended beyond equations of the fourth degree.

12. The significance of the *Mesolabum* within 17th century geometry and algebra

In 17th and 18th century geometry a process occurred which can be described as the de-geometrization of analysis. Analysis comprised algebra and the infinitesimal methods as far as they involved the use of algebraic symbols and formulas. It arose as a very useful tool in solving geometrical problems. But soon its methods also acquired a life of their own ; the formulas became interesting in themselves regardless of their relation to diagrams and geometrical context. The interest of mathematicians shifted from the geometrical figure to the analytical formula, from the curve to the function.

With respect to this process of de-geometrization, Sluse's *Mesolabum,* with its elegant balance of geometrical and algebraical interpretability, was conservative in style. Strongly connected to the Vietean tradition, the book was in a sense more geometrical than Descartes' *Géométrie* and for that reason stood outside of the main development of geometry and algebra.

Sluse's contribution in the *Mesolabum* was, as has been shown, that he explained a geometrical and algebraical rationale behind the methods of finding constructing curves and that he showed how solid problems could be constructed in an infinity of ways by means of conics. From the contemporary judgements it may be inferred that these were significant contributions to be field.

But with the advantage of hindsight we can see that Sluse's work also incorporated aspects which were later to cause a stagnation in the tradition to which the *Mesolabum* belongs : the construction of equations. Sluse presented infinitely many ways to construct the problems he discussed. Adapting his motivational statement about the number of centuries elapsed since the problem of two mean proportionals was first proposed, one might say that Sluse now gave more constructions than there had been minutes or even seconds since that time. And it was difficult to indicate among the infinitely many constructions that he gave certain single ones that were particularly better or more preferable than the others. So Sluse's results implied that problems of geometrical construction had too many solutions ; such an implication erodes the concept of construction and weakens the motivation of the theory of constructing equations. I have explained elsewhere (36) that after 1700 the theory of constructing equations did indeed enter a degenerating phase and that by 1750 in the opinion of most mathematicians the theory had lost its sense. There is no evidence that Sluse's infinities of constructions directly caused mathematicians to lose interest in the theory. Still the greater insight which he provided in the variety of constructions must have had a role in this respect. For one of the factors causing the decline of the theory was that mathematicians realized that construction by the intersection of curves left too much freedom of choice, and that there were no obvious criteria for choosing the best or the most appropriate ones among the many possible constructions.

By keeping to the classical style of Viète, Sluse's *Mesolabum* already stood somewhat outside of the main developments in mathematics at the time of its publication. It also belonged to a tradition, the construction of equations, that later was forgotten. Not surprisingly, therefore, the book has attracted little interest, even from historians of mathematics. But there is a certain injustice in that lack of interest because, as I hope to have shown, despite the differences in style and objective that separate the book from the modern interests in mathematics, it is possible to appreciate Sluse's fine, careful and stylish mathematical mind through the results published in the two editions of his *Mesolabum*.

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Notes

(1) Sluse, 1659 .

(2) Review, 1669 , p. 909.

(3) Wolff, Elementa , vol. 1, p. 389. (All translations from non-English sources are mine.)

(4) Reimer, 1798 , p. 216.

(5) For instance Zedler, 1732 , vol. 6, col. 1098, and Montucla, 1799 , vol. 2, p. 159.

(6) Cf. "Mesolabium est ars inveniendae mediae" (Salignac, 1574 p. 5). Occasionally, however, the term was used for instruments that had nothing to do with mean proportionals ; an example is Bedwell's *Mesolabium architectonicum* (Bedwell, 1631).

(7) Archimedes, 1921 pp. 588-620.

(8) Notably Heron, cf. Vereecke's note 2 in Archimedes, 1921 , p. 590.

(9) Barbour (1961 p. 6) quotes Salinas (1577 p. 173) on equal temperament : "the placing of the frets may be made regular, namely that the octave must be devided into twelve parts equally proportional, which twelve will be equal semitones" ; Salinas suggested to use a mesolabum in this construction.

(10) Cf. Bos, 1984 , pp. 332-337.

(11) E.g. Descartes, 1637 , pp. 315-319.

(12) Notably in his *Supplementum geometriae* (1593). In the general corollary which concludes that treatise Viète remarked that cubic equations without quadratic terms either correspond to trisecting an angle or to finding two mean proportionals between two given segments. He also remarked that fourth degree equations can be reduced, by the method now usually called after Cardano, to cubic equations, and that these in turn can be reduced to cubic equations without quadratic term. These reductions, if geometrically interpreted, correspond to constructions which can be performed by straight lines and circles (algebraically they involve the solution of linear and quadratic equations only). Thus Viète had indeed shown that any third or fourth degree equation is reducible to either a trisection or a construction of two mean proportionals, and that in both cases a neusis construction is possible. However, his result was rather abstract ; the geometrical interpretation of the Cardano reduction is very cumbersome (I know of no example where these constructions are actually spelled out), so that Viète's result did not provide a practicable construction for fourth degree equations. It was, indeed, one of the important achievements of Descartes that he gave a relatively simple construction for third and fourth degree equations independently of Cardano's reduction (cf. Section 7).

(13) On the question of curves, instruments and construction in the *Géométrie* see Bos, 1981 passim, and Bos, 1984 pp. 337-342.

(14) Pappus, Collectio , vol. 1, pp. 54-57 and 270-273 (i.e. lib. III 20-22 and lib. IV 57-59).

(15) "...one can always find its (sc. an equation of third or fourth degree) root by means of one of the three conic sections, whichever it may be, and even

by some part of one of them, however small, further using straight lines and circles only." (Descartes, 1637 , pp. 389-390).

(16) Cf. Bos, 1984 , pp. 345-346.

(17) Sluse, 1659 , preface.

(18) They are in the first six propositions of the book ; Prop. 1 : construction by circle and infinitely many ellipses, prop. 2 : by circle and infinitely many hyperbolas, props 3, 5, 6 : special constructions by circle and hyperbola ; prop. 4 concerns duplication of the cube by circle and ellipse.

(19) Sluse, 1659 , p. 27.

(20) Notably in Viète, 1615 , Ch. III (pp. 164-167 in Viète, 1983).

(21) For clarity, I use modern notation to describe Sluse's (and Viète's) use of algebra ; on Sluse's own notation see Section 10.

(22) Loc. cit. note 20 . The idea of geometrical paradigm problems for equations remained alive in algebra for a long time, it occurs as late as 1702 in a work of Ozanam (1702 , p. 224).

(23) *Latus rectum* and *latus transversum* are the classical terms for certain line segments occurring in the defining properties of conic sections. If the vertex of the conic section is taken as origin and the X-axis is along the diameter, then the *latus rectum* a and the *latus transversum* b occur in the analytical formular for the conics in the following way:

$$y^2 = ax \text{ (parabola)};$$

$$y^2 = ax - ax^2/b \text{ (ellipse)}; \quad y^2 = ax + ax^2/b \text{ (hyperbole)}.$$

In the case of Sluse's construction we have:

$$a = L = FG \times HI/FA; \; b = HI.$$

(24) Cf. note 25 .

(25) Sluse, 1659 , end of preface. It should be noted that Sluse did not adopt the classical style in order to conceal his methods. He told Huygens that he had written "in such a way that I expounded the demonstration in an easier way and that I showed the method of analysis which I used as openly as it can be done in the geometry of the Ancients" (Sluse to Huygens 9-9-1659, Huygens, *Oeuvres* vol. 2, pp. 477-478).

(26) Cf. for example Oldenburg to Sluse 25-11-1667 Oldenburg, *Correspondence* vol. 2, pp. 615-617.

(27) Sluse to Oldenburg 26-9-1667 and 14-11-1667, Oldenburg, *Correspondence* vol. 2, pp. 488-490 and 594-598.

(28) Perhaps a study of Sluse's manuscripts could clarify this. It may well be that before 1659 Sluse (in accordance with Viète, cf. note 12) restricted himself to cubic equations because fourth degree equations can be reduced to cubic ones by the method of Cardano.

(29) Relevant passages are in the following letters : Huygens to Sluse 12-10-1657 Huygens, *Oeuvres* vol. 2, pp. 65-67), H. to S. July 1659 (ibid. pp. 442-443), S. to H. 9-9-1659 (ibid. pp. 477-478), S. to H. 13-10-1664 (ibid.

vol. 5, pp. 121-123), H. to S. 28-10-1664 (ibid., p. 127) and S. to H. 4-11-1664 (ibid. pp. 131-134).

(30) Sluse to Huygens 14-8-1657, Huygens, *Oeuvres* vol. 2, pp. 46-48, here p. 47.

(31) Sluse to Huygens 15-7-1659, Huygens, *Oeuvres* vol. 2, pp. 436-438, here p. 437.

(32) The Miscellanea (pp. 99-181) consist of 10 chapters ; I give a short list of the topics. I and III : quadratures of spirals ; II, VII, VIII and IX : quadratures and centres of gravity of curvilinear figures ; IV : maxima and minima ; V : points of inflexion of the conchoid of Nicomedes ; VI normals to the parabola ; X : numbertheoretical problems.

(33) Cf. Sluse to Huygens (5-8-1659, Huygens, *Oeuvres* vol. 2, pp. 449-451, esp. p. 450.

(34) As given by Eutocius, Archimedes, 1921 pp. 603-605.

(35) "Resolvatur in analogismo ut ratio Regulae evidentius appareat" (Sluse, 1668 , p. 91).

(36) Bos, 1984 , pp. 371-375.

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