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THE CLOSURE THEOREM OF PONCELET

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SUNTO. — Si dà conto di uno studio effettuato congiuntamente con C. Kers, F. Oort e D. W. Raven sugli aspetti storici e matematici del teorema di chiusura di Poncelet.

Sono discusse le dimostrazioni di Griffiths (1976), Jacobi (1828) e dello stesso Poncelet (1822), e si riporta un nuovo risultato concernente una certa famiglia di curve dipendenti da un parametro.

Questa famiglia di curve scaturisce in modo naturale dagli argomenti usati da Poncelet nella dimostrazione originale ed offre un caso interessante di non-commutatività forte di dualizzare e specializzare.

1. - INTRODUCTION.

In this note I report on joint work done at Utrecht together with F. Oort and two students, C. Kers and D. Raven. The work is a combined historical and mathematical study of the closure theorem of Poncelet. It was inspired by the article of Griffiths [1976] in which the author gave a modern proof of the closure theorem and made some remarks about its history. The full results of our work are now available as preprint [Bos, Kers, Oort, Raven, 1984]; they will be published in the journal *Expositiones Mathematicae*. Here I shall mainly sketch the proofs of the theorem by Poncelet, Jacobi and Griffiths, and I shall mention some historical and mathematical aspects that we have worked out in our joint study.

2. THE THEOREM.

In its modern form the closure theorem is as follows. Let (see Figure 2.1) C and D be two smooth conics in $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$. Let P_1 be a point of C and L_1 a tangent to D through P_1. From (P_1, L_1) we construct a « Poncelet traverse between C and D », that is, a

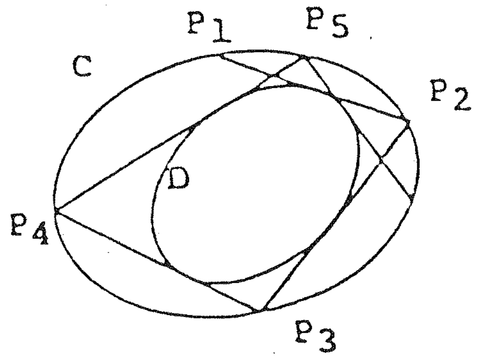

Fig. 2.1.

sequence P_1, L_1, P_2, L_2, P_3, L_3, ... with $P_i \in C$, L_i tangent to to D, $P_i = L_{i-1} \cap L_i$. We say that the traverse closes after n steps of $P_{n+1} = P_1$. There are trivial cases of closure that occur if, for some i, $P_i \in (C \cap D)$. In that case there is only one tangent to D through P_i, so that $L_i = L_{i-1}$, $P_{i+1} = P_{i-1}$; the traverse as it were returns in itself, and $P_{2i-1} = P_1$. Similarly if L_i is a common tangent to C and D, then $P_{i+1} = P_i$, $L_{i+1} = L_{i-1}$ and ultimately $P_{2i} = P_1$. We now have (see Figure 2.2):

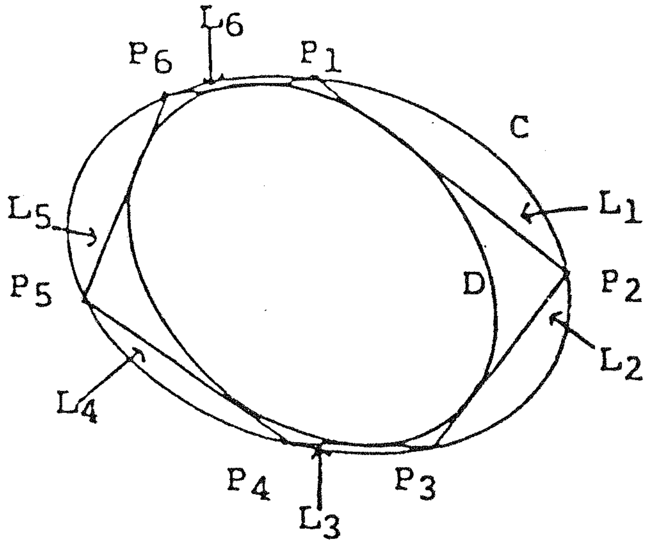

Fig. 2.2.

THEOREM 2. 1 (*Closure theorem*). - If a Poncelet traverse, starting at $P_1 \in C$, closes non-trivially after n steps then a Poncelet traverse from any point on C will close after n steps.

If a traverse closes after n steps we have an « interscribed n-gon » between C and D, that is, an n-gon with vertices on C and sides tangent to D. So we may formulate Theorem 2.1 also as: If there is one interscribed n-gon between C and D, then there are infinitely many such n-gons. Or: two conics have either no interscribed n-gon, or infinitely many.

3. - History of the theorem.

Jean-Victor Poncelet (1788-1867) found and proved the theorem in 1813-1814 while in captivity as a prisoner of war at Saratov on the river Wolga in Russia. He published the theorem, with a new proof, in his *Traité des propriétés projectives des figures* in 1822 [Poncelet 1822]. Poncelet only considered conics in the real plane. His proofs involve an argument which he called the « principle of continuity ». This principle is not acceptable in modern mathematics; it was already considered doubtful by some contemporaries of Poncelet. Still in this case (as in many other cases) it led him to correct results.

In 1828 Carl Gustav Jacob Jacobi (1804-1851) published a proof of the theorem by means of elliptic functions in *Crelle's Journal* [Jacobi 1828]. He proved the theorem for pairs of circles in the real plane, the one circle lying within the other; he noted that, by a projection, it can be generalized to pairs of real ellipses, the one lying inside the other.

During the period 1830-1930 many mathematicians studied the theorem. Interest focussed in particular on the conditions for C and D to have an interscribed n-gon, on finding purely algebraic proofs (avoiding Jacobi's elliptic functions), on using invariant theory and on generalizing the theorem. (The problem under which conditions two circles admit an interscribed triangle or n-gon had already been studied earlier, for instance by Fuss, Steiner and others.) It seems that after 1930 the interest in the theorem faded. It was revived by P. A. Griffiths who published a new proof, using elliptic curves, in 1976 [Griffiths 1976]. Griffiths proved the theorem for smooth conics in complex projective space; he did not study the special case that the two conics are tangent to each other.

In the next sections I shall sketch the three proofs mentioned above, but in inverse chronological order.

4. Griffiths' proof.

For smooth conics C and D in $\mathbb{P}^2(\mathbb{C})$, $\#(C \cap D) = 4$, consider (cf. Figure 4.1)

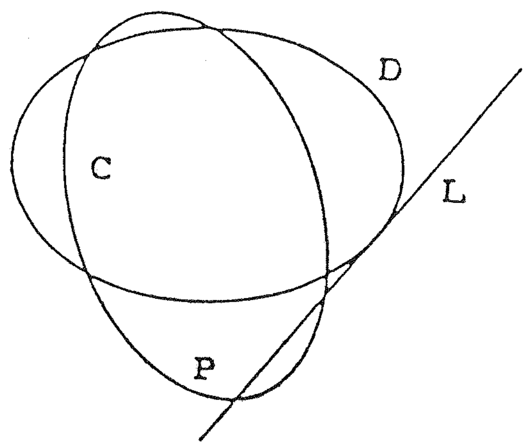

Fig. 4.1.

$$E = \{(P, L) \mid P \in C,\ L \in D^*,\ P \in L\} \subset C \times D^*$$

(* denotes the dual). E is an algebraic curve, it is a double covering of C and it branches over $C \cap D$; hence E is an elliptic curve. The construction of a Poncelet traverse can now be described by two mappings γ and δ, $E \to E$, defined (cf. Figure 4.2) as

$$\gamma(P, L) = (P, L')$$
$$\delta(P, L) = (P', L)$$

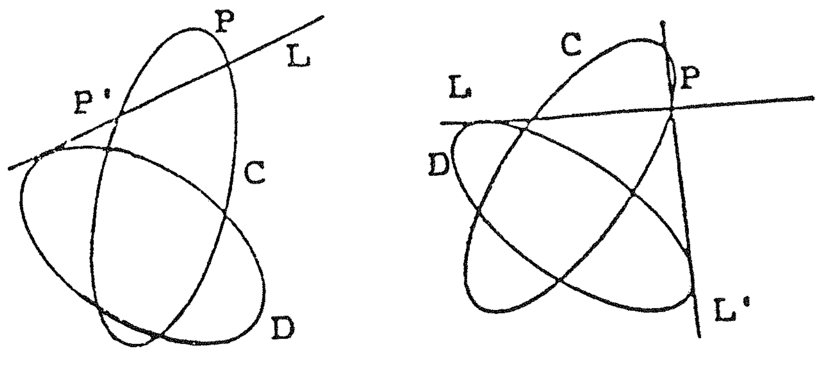

Fig. 4.2.

Clearly $\gamma^2 = \delta^2 = \text{id}$. Cal $\gamma\delta = \sigma$. A Poncelet traverse from (P, L) consists of the successive images $\sigma(P, L)$, $\sigma^2(P, L)$ etc. The traverse closes if

$$\sigma^n (P, L) = (P, L).$$

Now use general theory about algebraic curves and the group structure on E (having choosen a point $0 \in E$) to prove that there is a point $c \in E$ such that

$$\sigma (x) = x + c$$

for all $x \in E$. If a Poncelet traverse from $(P_0, L_0) = x_0 \in E$ closes after n steps, we have

$$\sigma^n (x_0) = x_0 + nc = x_0 ;$$

so

$$nc = 0.$$

But then for every $x \in E$

$$\sigma^n (x) = x + nc = x,$$

hence from any $(P, L) \in E$ a Poncelet traverse between C and D will close after n steps; this proves the theorem.

5. - JACOBI'S PROOF.

Consider (Figure 5.1) two real circles C and D, D within C, and a Poncelet traverse between them with vertices P_1, P_2, P_3 etc. Let m and M be the centres of D and C respectively; $mM = a$ and the line mM cuts C in O. For each P_i let

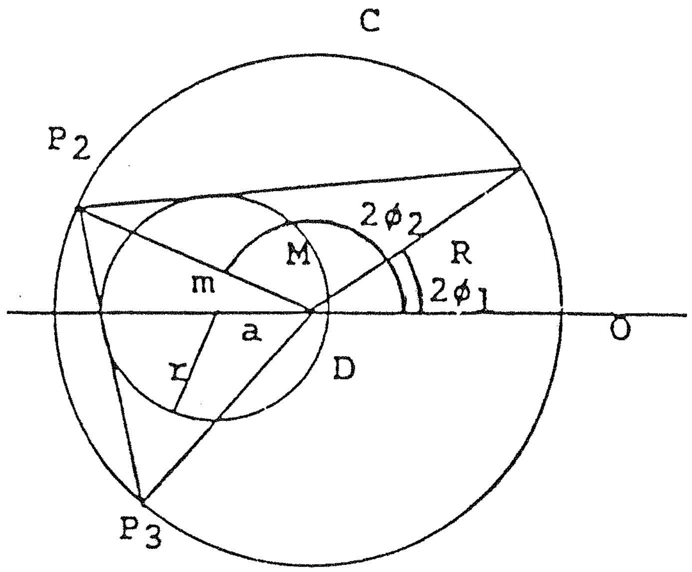

Fig. 5.1.

$$\angle OMP_i = 2\,\varphi_i$$

measured counterclockwise. Use elementary trigonometry to prove that, for all i,

$$tan\ [(\varphi_{i+2} + \varphi_i)/2] = [(R - a)/(R + a)]\ tan\ \varphi_{i+1}.$$

Recognize this relation as a functional equation, satisfied by the elliptic function *am* in the following sense: Define

$$am\ (u, k) = \varphi\ (= am\mathrm{u}\ \text{for short})$$

if

$$u = F(\varphi, k) = \int_0^{\varphi} \frac{dt}{\sqrt{1 - k^2\, sin^2\, t}};$$

($F(\varphi, k)$ is the elliptic integral of the first kind). If now, for some u and c, we write

$$\chi_1 = 2\ am\mathrm{u}$$

$$\chi_2 = 2\ am\ (u + c)$$

$$\vdots$$

$$\chi_i = 2\ am\ (u + (i - 1)\ c)$$

then it follows from Jacobi's theory about the elliptic functions *am* that

$$tan\ [(\chi_i + \chi_{i+2})/2] = \Delta\ tan\ \chi_{i+1}$$

with

$$\Delta = [1 - k^2\ sin^2\ (amc)]^{1/2}.$$

Hence it is possible to adjust k, c and u such that

$$\Delta = [(R - a)/(R + a)]\ \#\text{and}\#\ 2\ am = \varphi_1,$$

whereby $\varphi_i = \chi_i$ for all i; so the construction of a Poncelet traverse is described by the sequence amu, $am(u + c)$, $am(u + 2c)$, etc. (In fact Jacobi adjusted k and c in such a way that k is constant for all inner circles D' which belong to the pencil of circles defined by C and D. This procedure was advantageous in further arguments of Jacobi in connection with the theorem which I shall not discuss.)

Now suppose the Poncelet traverse from P closes after n steps, and $\measuredangle OMP = 2\varphi_1$. Then $P_1 = P_{n+1}$, so

$$\varphi_{n+1} = \varphi_1 + r\pi,$$

for certain integer r. But also

$$\varphi_{n+1} = am\,(u + nc)$$

for certain c and $am u = \varphi_0$. Now use, from the theory of the function am, the relation

$$am\,(u + 2rK) = am u + r\pi$$

where

$$K = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin t^2}} = F(\pi/2, k).$$

This gives

$$am\,(u + nc) = \varphi_{n+1} = \varphi_1 + r\pi = am\,u + r\pi = am\,(u + 2rK),$$

hence

$$u + nc = u + 2rK,$$

so that

$$nc = 2rK.$$

But now, for a traverse starting from arbitrary $Q_1 \in C$ (setting $\measuredangle OMQ_1 = 2\psi_1 = amv$, $\measuredangle OMQ_{n+1} = 2\psi_{n+1}$) we have

$$\psi_{n+1} = am\,(v + nc) = am\,(v + 2rK) = am\,v + r\pi = \psi_1 + r\pi,$$

hence $Q_{n+1} = Q_1$, so the traverse from Q_1 also closes after n steps.

6. - Poncelet's proof.

Poncelet's proof of the closure theorem is based on the following

LEMMA 6. 1. - Let (see Figure 6.1) C, D_1 and D_2 be circles from one pencil. Let P, R_1 and R_2 be points on C with the chords PR_i tangent to D_i at points Q_i. Each point P thus defines a chord R_1R_2 of C. These chords envelope a circle D belonging to the same pencil of circles as C, D_1 and D_2. Moreover, the points G in which the chords $R_1 R_2$ touch the envelope D are constructed as follows: Draw

$$H = R_2 Q_1 \cap R_1 Q_2,$$

then

$$G = P H \cap R_1 R_2.$$

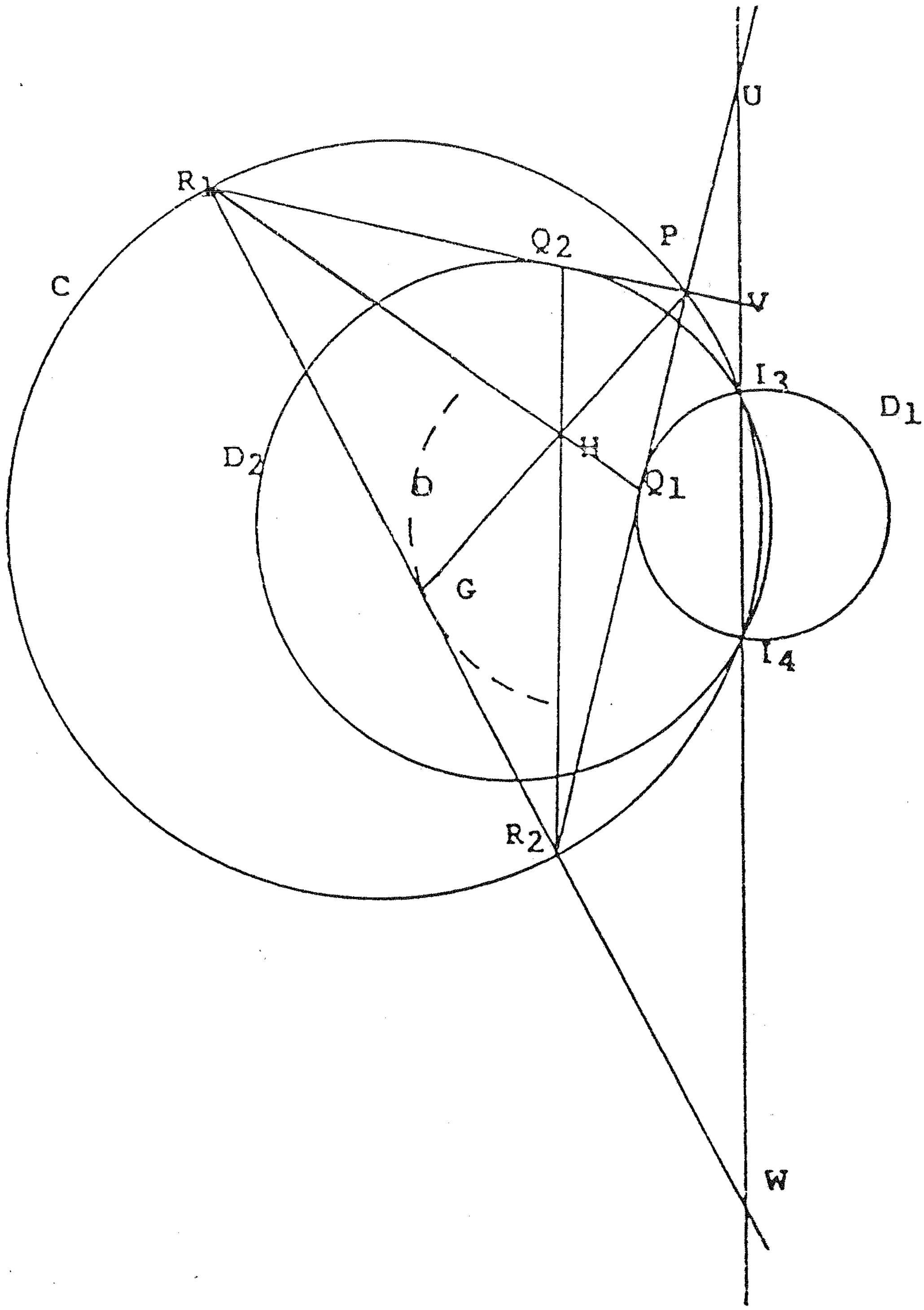

Fig. 6.1.

Remarks 6.2. - I shall not discuss in how far Poncelet dealt with exceptional cases arising for certain positions of the three circles (e.g. cases in which D reduces to a single point).

Poncelet considered real circles. His definition of pencils of circles and conics involved the concept of « ideal chords » which I shall not discuss here; in effect, Poncelet's pencils coincide with those defined nowadays, but restricted to the real case. Thus conics belong to one pencil if they have four (real, or pairwise complex conjugate) points in common; circles belong to one pencil if they have two (real or complex conjugate) points in common other than the isotropic points.

In fact, each point P on C, as in the Lemma, defines four chords $R_1 R_2$, because there are two tangents through P to each D_i. The Lemma applies to each of these chords; thus there arise four circles as envelopes, but these circles coincide pairwise. Poncelet was aware of that property of the envelope.

Poncelet's proof of the Lemma is a brilliant, complicated and idiosyncratic piece of geometric argument. It involves all the special concepts which Poncelet developed in his programme of a synthetic geometry, in particular the « ideal chords » and the « principle of continuity ». The proof also uses classical theorems as those of Ceva and Menelaus, infinitesimal motions and envelopes of pencils. A considerable part of the joint work on which I am reporting was devoted to translating Poncelet's arguments into the language of modern algebraic geometry and checking them. Although Poncelet's style of mathematical inference, in particular his use of the « principle of continuity », is not acceptable from the modern point of view, it appeared that his results are essentially correct. The scope of this report does not allow further discussion of the proof of the Lemma; but I shall return (see Section 7) to one aspect of it that led us to an interesting new result.

Poncelet used the lemma to prove

Main Theorem 6.3 (circles). - Let $C, D_1, \ldots, D_n$ be circles from one pencil. Consider a traverse $P_1, L_1, P_2, L_2, \ldots, P_n, L_n, P_{n+1}$, with $P_i \in C$ and $L_i = P_i P_{i+1}$ tangent to D_i. Let P_1 vary along C. Then the chord $L = P_1 P_{n+1}$ will envelope a circle from the pencil.

Proof by induction on n. For $n = 2$ this is the Lemma. By induction, the chord $L' = P_n P_1$ will envelope a circle D' from the pencil. Now consider the traverse $P_1, L', P_n, L_n, P_{n+1}$. As L' and

L are tangent to circles from the pencil, the Lemma implies that $L = P_1 P_{n+1}$ also envelopes a circle from the pencil.

In order to generalize the Main Theorem to the case of conics, Poncelet invoked a *projection theorem* which states that every pair of conics can be considered as the projective image of a pair of circles. He actually proved this for pairs of conics which have no more that two intersections (recall that Poncelet worked with real conics), and he knew that when there are more intersections the conics cannot be the images of circles under a real projection. Nevertheless Poncelet claimed the projection theorem valid in general, arguing that the validity of the theorem for pairs with no more than two intersections implies its general validity by the « principle of continuity ». This argument is a good example of the use he made of this principle throughout the *Traité*. Now the properties in the Main Theorem for circles are projectively invariant, so, by the projection theorem, Poncelet concluded:

Main Theorem 6.4 (conics). - Theorem 6.3. holds for conics.

The closure theorem now appears as a special case of the main theorem:

Proof of the closure theorem: For conics C and D admitting one interscribed polygon $P = P_1, L_1, P_2, L_2, P_3, \dots, L_{n-1}, P_n, L, P_1$, apply the Main Theorem, taking $D_i = D$ for all i. Conclude that, if P varies along C, L envelopes a conic D' from the pencil defined by C and D. In the positions $P = P_1$, $P = P_2, \dots, P = P_n$ the traverse coincides with the interscribed n-gon, hence the corresponding L's touch both D' and D. So D' and D have at least three tangents in common. As they belong to the same pencil, they coincide. Hence for all positions of P on C the Poncelet traverse closes after n steps.

7. - History.

The historical part of our study comprised, apart from the proofs of Poncelet and Jacobi, a survey of the « prehistory » of the theorem (notably in connection with interscribed triangles and n-gons between circles), an analysis of Poncelet's synthetic geometrical style and a discussion of the relation of the three proofs sketched above. In the modern studies on the theorem one finds (e.g.

[Griffiths 1976], p. 345) statements to the effect that Poncelet's and Jacobi's proofs are « essentially the same » as the modern one. These statements led us to a detailed comparison of the proofs. We came to the conclusion that, although the proofs are clearly related as to subject matter and result, the differences in style, method, extension of the result and conception of the objects, are so marked that they cannot be called the same. We experienced that the comparison of the proofs involved a kind of culture barrier between styles in mathematics. But such comparisons are central in historical research and they are challenging. Especially the question in how far the elliptic curve of Griffiths' proof is already present in Jacobi's argument proved very delicate and intriguing.

8. - Poncelet's Lemma generalized, dualizing.

Poncelet's Lemma suggested us to study the case that the three circles are not from one pencil. Generalizing to conics and taking into account the four possible positions of the chord $R_1 R_2$, we describe the situation by a structure akin to E in Griffiths' proof. Let C, D_1 and D_2 be conics. Consider

$$F = \{P, L_1, L_2) \mid P \in C, L_i \in D_i^*, P \in L_i\} \subset C \times D_1^* \times D_2^* .$$

The construction of the chords as in Poncelet's Lemma can be described by a mapping $f : F \to (\mathbb{P}^2)^*$ defined by

$$f(P, L_1, L_2) = R_1 R_2 .$$

This definition is valid except in a finite set S of points of F. Call $f(F - S) = \Gamma^0 \subset (\mathbb{P}^2)^*$ and let Γ be the closure of Γ^0. Γ is an algebraic curve, the required envelope X of the chords is the dual of Γ:

$$X = \Gamma^* .$$

Our study of F, Γ and X has yielded the following results:

Situation I 7.1: the conics C, D_1 and D_2 belong to the same pencil, i.e. $C \cap D_1 = C \cap D_2$ and $\#(C \cap D_i) = 4$. Then

$$F = F_1 \cup F_2 ,$$

where F_1 and F_2 are elliptic curves intersecting transversally in 4 points. Further

$$\Gamma = \Gamma_1 \cup \Gamma_2$$

and the components Γ_i have degree 2. Also

$$X = X_1 \cup X_2, \quad X_1 \neq X_2,$$

and the components X_i are conics from the same pencil. (This is the modern reformulation of Poncelet's Main Theorem).

Situation II 7.2: the conics C, D_1 and D_2 are in general position (and in particular $\#(C \cap D_1) = \#(C \cap D_2) = 4$, $C \cap D_1 \cap D_2 = \emptyset$. Then:

F is a smooth curve of genus 5;

Γ is a curve of degree 8 without cusps;

X has degree 24.

Moreover, in both Situations I and II Poncelet's construction for points on X applies (cf. 6.1); that is, for $x = (P, L_1, L_2) \in F$ take R_i as explained above, and $Q_i = L_i \cap D_i$. Call $R_1 Q_2 \cap R_2 Q_1 = H$, then the point

$$G = P H \cap R_1 R_2$$

is the point on X corresponding to $x \in F$.

These results are of interest because they furnish an example of complicated limit processes which, because of the availability of an explicit construction, can in this case be studied in depth. Let C^t, D_1^t, D_2^t be triples of conics, depending on a parameter t such that for $t \neq 0$ they are in general position (Situation II) but C^0, D_1^0, D_2^0 belong to one pencil (Situation I). We can now study the behaviour of Γ and X under specialization $t \rightarrow 0$. We note in particular that specializing and dualizing do not commute. On the one hand

$$(\lim_{t \to 0} \Gamma^t)^* = (\Gamma^0)^* = X_{(Sit.\ I)} = X_1 \cup X_2.$$

But we can compute that, on the other hand,

$$\lim_{t \to 0} ((\Gamma^t)^*) = \lim_{t \to 0} X^t = X^0$$

where

$$X^0 = 2\, X_1 \cup 2\, X_2 \cup \left(\bigcup_{j=1}^{8} T_j\right) \cup \left(\bigcup_{k=1}^{4} 2\, S_k\right),$$

in which X_i are the same conics as above (but occurring with multiplicity 2), T_j are the 2×4 tangents which C has in common with either D_1 or D_2, S_k are the common tangents of D_1 and D_2.

The strong non-commutativity in this case suggests that it may be rewarding to work out a « good theory » for dualizing plane curves such that (with a modified definition of a dual curve) specialization and dualizing do commute.

8. - CONCLUSION.

In conclusion I would like to mention that all four authors of the study have experienced the confrontation of old and new mathematics (with a time distance of over 150 years) as most inspiring and rewarding.

SUMMARY. — A report on a joint study together with C. Kers, F. Oort and D. W. Raven on historical and mathematical aspects of Poncelet's closure theorem. Proofs of the theorem by Griffiths (1976), Jacobi (1828) and Poncelet himself (1822) are discussed and a new result is reported concerning a certain one-parameter family of curves. This family of curves arises naturally from arguments in Poncelet's original proof and it offers an interesting case of strong non-commutativity of dualizing and specializing.

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