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Calculus in the Eighteenth Century - the Role of Applications.

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[This is the slightly revised and annotated text of an invited talk on the influence of applications on the development of the calculus in the eighteenth century. It was delivered on december 14th, 1976, at a one day conference on the history of applied mathematics, organised jointly by the Institute of Mathematics and its Applications (Northern Home Counties Branch) and the British Society for the History of Mathematics.]

1. The influence of applications on the development of the calculus in the eighteenth century is a most complex and also a rather elusive process. The elusiveness of the process lies at least partly in the vagueness of the idea of "applications". This idea concerns boundaries of disciplines, which are crossed if knowledge from one discipline is "applied" in another. Such boundaries are never clearly marked. Moreover, the subdivisions of science, the names and boundaries of the various disciplines, have varied much in the course of the last four centuries. Consequently, the idea of "applications" is not only vague, but also strongly time-dependent.
I shall not be able to give more than a very global survey of the applications of the calculus in the eighteenth century and a discussion of some aspects of their influence. In connection with the time-dependence of the idea of "applications" I have found it useful to take contemporary eighteenth century accounts of the calculus and its applications as starting points of my survey. Two accounts, which appeared in prestigeous encyclopedias in the 1730's and the 1780's respectively, are useful for this purpose.

The first account I take from the Grosses Vollständiges Universallexikon[1] which Johann Heinrich Zedler saw through the press in the years 1732-1750. The 64 well sized volumes of this Lexikon are still much valued by historians, especially for biographical information. It is also a valuable source for the history of mathematics. For instance the Lexikon contains a remarkable early

definition of the concept of function (1735)[2] which is not mentioned in the usual accounts of the history of that concept.
The articles on differential and integral calculus are in volume 6, which appeared in 1733. In Zedler's Lexikon we have, therefore, a view of the calculus and its applications as seen in the 1730's. The other enclyclopedia is from the 1780's. It is the Encyclopédie Méthodique[3], a regrouping and extension of the articles contained in the famous Encyclopédie of d'Alembert and Diderot. The Encyclopédie Méthodique started to appear in 1782 and was completed in 1832, comprising 166 volumes. The articles on mathematics were grouped together in three volumes Mathématiques which appeared in 1784-1789.

The dates of these encyclopedias are very conveniently spaced in the period we are concerned with. If we put the beginning of the story of the calculus at the first publications, by Leibniz ([1684], [1686]) and by Newton (the Principia [1687]), we find Zedler's Lexikon 50 years later, at the close of the period of Newton, Leibniz, Jakob and Johann Bernoulli and l'Hôpital. In these first fifty years the differential calculus of Leibniz and the fluxional calculus of Newton became recognized as the most powerful tool for the study of curves and of problems relatable to curves. At the end of the period it had also become clear that Newton's fluxional version of the calculus was less versatile than the differential calculus, so that the further developments in analysis occurred on the Continent rather than in England. On the Continent here means mainly in Germany and Switzerland, so that it is fitting that we gain our view of the period from a German encyclopedia. The 1730's are also the beginning of a new period, lasting till the 1780's. This is first and foremost the period of Euler. He, together with mathematicians as Daniel Bernoulli, d'Alembert and Clairaut, consolidated analysis as an organised body of mathematical knowledge - especially Euler's great textbooks ([1748], [1755], [1768]) had this function.
With Euler first working at St. Peterburg, then at Berlin and then back at St. Petersburg, mathematical activity had these two towns as main centres, but at the end of the period France, or to be precise Paris, was in the rise as as mathematical centre. The

1780's indeed are the beginning of the great French half century in mathematics (to be succeeded by a German hegemony in the 19th century) with mathematicians as Lagrange, Legendre, Monge and Laplace.

2. So the dates of the encyclopedias coincide nicely with a periodization of the history of the calculus in the eighteenth century. Let us now look how these encyclopedias presented the calculus and its applications to an informed but not specialist public.

The article on the differential calculus in the Lexikon[4] gives the rules of differentiation of algebraic expressions; the inverses of these rules are mentioned under integral calculus. There is a not very illuminating explanation of differentials and higher order differentials. There are examples of a quadrature (calculation of the area under a curve), a rectification and a cubature. Also there is a short note on inverse tangent problems, that is, problems in which it is required to find a curve from a given property of its tangents. These problems lead to first order differential equations, but that term does not occur in the articles.[5]

In addition the articles deal with the utility of the calculus, with its history, in particular with the question of priority between Newton and Leibniz, and with the most important writers and books on the subject. The calculus is

> *"one of the most magnificent discoveries in mathematics, which has not only brought geometry to its highest summits but also has extended the other disciplines so far that one would have to write whole books if one wanted to specify the utility of this calculus".*[6]

Not having the length of books at his disposal, the writer had to be rather concise on the utility of the calculus, but he still takes a full column, and it is instructive to list here the topics which he mentions.[7]

First there are the applications of the calculus in higher geometry the study of curves, their tangents, extreme values, quadratures, cubatures, rectifications, the inverse method of tangents, exponential calculus and higher order differentiation. Then follows mechanics, where "the most hidden secrets of nature have been revealed"[8] by the calculus, because a great number of curves like catenaries, velaries and elastics are found (I shall return to these below). In the study of motion,[9] the knowledge of the differential calculus has made it possible to find curves with the "wunderbarsten" properties, like tautochrones, brachystochrones, and the motion of compound pendulums. In optics the caustics and the refraction of lightrays in air are found. Finally in astronomy "we can sufficiently show the utility of the calculus if we mention only the work of the great Newton, the Principia".

What these various curves are, is explained in the pertaining articles elsewhere in the Lexikon. I shall give a few particulars about each of the problems listed, in order to make clear what sort of applications the writer had in mind.

The catenary is the form of a chain or flexible rope suspended on two points. The problem to determine this form has a complicated history culminating in 1690/91 when it was publicy proposed and led to a series of articles by Leibniz, Johann Bernoulli and Huygens. Mastering the catenary problem was the first great public success of the differential calculus. The problem was generalised in the eighteenth century to catenaries with non uniform load.

The velary is the form of a sail blown by the wind. The problem leads to a second order differential equation; the brothers Bernoulli studied it in the early 1690's.

The elastica is the form of an elastic beam, fixed at one end and bent by a perpendicular force at the other end. It was studied in a grand article by Jakob Bernoulli, which was most important for the mechanics of elasticity as well as for analysis, because it contained a root of the theory of elliptic integrals.

Tautochronous curves have to do with pendulum motion: if the pendulum bob is forced to move along a path such that its period does not depend on its amplitude, that path is called a tautochronous path of curve. The problem arose through Huygens' interest in pendulum clocks; Huygens found the tautochrone in vacuo to be the cycloid.

The brachystochrone is the curve through two given points A and B along which a body, under the influence of gravity, falls in shortest time from A to B. The problem was publicly proposed in the 1690's

and it led to most important studies which contained the roots of the later calculus of variations. Many variants of the last two problems were studied in the eighteenth century as for instance tautochrones and brachystochrones in resisting media, or in relation to other than the Galilean law of fall.
The motion of simple and compound pendulums, a subject of research since Huygens' work on the pendulum in the 17th century, involved integration in order to find the centre of oscillation of pendulums.
Caustics in optics are the envelopes of families of reflected or refracted rays. They were an inspiration for the study of envelopes of families of curves in general.
Finally, the reference to Newton's Principia in the case of astronomy, should suffice indeed, because I can refer here to dr. Whiteside's paper.

As to the literature of the calculus, the Lexikon mentions l'Hôpital's Analyse des infiniment Petits ([1696]) (which only treats differation) and a number of other sources[10] on quadratures and integration, but the list of those sources makes clear that by the 1730's the calculus was still in need of comprehensive advanced textbooks.

To summarise, we get the following picture of the calculus as seen in the 1730's: It was recognized as the summit of mathematical knowledge. Its greatest achievements were seen in its application to singular problems in mechanics, optcs and celestial mechanics (most of these problems yielded curves as solutions). As to its methods (especially of integration and differential equations), however, the calculus had not yet found a unified presentation.

Let us now turn to the 1780's. The Encyclopédie Méthodique does tell us more about the calculus and its applications than the Lexikon. This first of all because there is more to tell in the 1780's, but also because the Encyclopédie Méthodique gives more space to it - three volumes mathematics, with the articles relevant for our subject mainly by d'Alembert[11] (whose habit it was not to spare words) and with a long historical introduction by the Abbé Bossut.
In the articles Calcul Différentiel and Calcul Intégral,[12] we find the mathematics of the calculus explained. Here a significant difference with respect to the Lexikon is that the problem of the foundation of the calculus, the then so-called "metaphysics of the calculus",

is recognised. There are substantial separate articles on differential equations and partial differential equations. There is also a separate article "Maximum" which contains an explanation of the problems and methods of the calculus of variations (though it does not use this term).

Furthermore, the lack of textbooks, still evident in the Lexikon, is now remedied; there are Euler's textbooks, and those for whom these may be too far reaching may turn to others, like the textbooks of Bougainville [1754] and Agnesi [1748].

As to applications of the calculus, Bossut's introduction to the volumes is illuminating.[13] Bossut is very concerned about applications. He writes, at the beginning of a long survey of the development of the calculus after Leibniz and Newton:

> *"Of all the discoveries that have ever been made in the sciences there is none as important and as fertile in applications as that of the infinitesimal analysis."*[14]

He also stresses the dual role of these applications, namely that they serve the calculus itself as well as the fields in which they occur:

> *"the new geometry (i.e. the calculus) was applied in all the other parts of mathematics and all these have forced it to perfect itself, by continually offering problems, which eventually become problems of pure analysis."*[15]

It is revealing to look in some more detail into Bossut's rendering of the progress of analysis in the 18th century. First of all it is noteworthy that he orders his account not according to subjects within analysis proper, but according to the fields which supplied problems for analysis. After giving about the same list as the Lexikon for the applications in the period till ca. 1730, Bossut deals successively with mechanics, dynamics, hydrodynamics, hydrodynamics applied to navigation, astronomy, optics, and lastly analysis itself, or rather the study of curves. Under each of these headings he mentions special problems which analysis helped to solve, or at least to deal with successfully. Rather than giving here the full list,[16] I shall mention some remarkable aspects of Bossut's presentation of the calculus.

The first striking aspect of the list is the increasing importance and scope of the problems over the period. In the early period we have for instance catenaries and tautochronous motion in resisting media. More exciting is the problem of the figure of the earth, a problem within hydrodynamics, as the earth is considered a fluid under influence of its own gravity and the dynamics of rotation. This problem was much discussed in the first half of the century, it had philosophical implications, and expeditions were sent out to far countries in order to check the results of theories by direct measurements. Then, directly relevant to astronomy and navigation, there are the problems of planetary motion, especially the calculation of the perturbances caused by other planets. The theory of the moon's motion, to which Euler contributed the decisive mathematical tools, belongs under this heading.

Secondly, there is the greater generality of the topics treated. This was made possible by the formulation, in terms of analysis, of "general laws" and "fundamental principles" of mechanics and hydromechanics. Many separate problems could thereby be recognised and treated as special cases of a more general problem situation. Put otherwise: the field of rational mechanics, of general, strongly mathematical theories of mechanics, hydromechanics, elasticity and celestial mechanics, expressing the fundamental principles in terms of differential equations, was created in that period.[17)]

Thirdly, the sections on the applications are more informative on the mathematics involved, than the last section "Analysis". Partial differential equations, for instance, are mentioned under "Analysis", but Bossut gives more information about them (especially on the role of the arbitrary functions) in connection with the problem of the vibrating string (under Mechanics) and the problem of the cause of the winds (under celestial physics).[18)]

Finally, it should be remarked that, however exciting the list of applications is, the list of names in this story is utterly tedious: it consists essentially of four: Johann Bernoulli, Daniel Bernoulli, Euler and d'Alembert.

3. This last point in fact, presents us with a problem. We looked for applications of the calculus. We have found long lists of such applications but it appears that these applications were performed by the same people (and a very small group of people too) who created the analysis that was applied.

No doubt these people saw their own work as a unity, not as a realm of activity with a boundary in the middle dividing pure analysis from its "applications". So we must ask the question: are these "applications"? If, with the term applications, we think about crossings of boundaries between disciplines, we should hesitate to call them applications. Within eighteenth century science, rational mechanics and analysis were seen as one whole; the interaction between the problems was seen as normal and quite self-evident.

This is by no means a new observation; it is well recognized in the more elaborate accounts of the history of mathematics. It should, however, be stressed, because the shorter presentations of the history of the calculus usually take as their organising principles an interest in the foundations of the calculus,[19] or at least they are organised according to subdivisions of analysis itself. This tends sometimes to obscure the roots which analysis has in these fields of rational mechanics which we now consider as separate, but which, at the time, were interwoven with analysis.

Though natural and much to be expected, the interrelation of analysis and rational mechanics did determine the development of analysis in a special way, which it is instructive to characterise a bit further.

I want to mention three aspects of this interrelation.

First, rational mechanics provided the language to formulate challenging problems for the new methods in analysis. Especially in the decades around 1700 the traditional stock of problems of the geometry of curves, as tangents, areas, cubatures, rectifications, curvature, etc. was too narrow to supply enough problems.

So it was enriched with mechanical conceptions: centres of gravity, motion, centres of oscillation, elasticity, motion, gravity, central forces, resisting media etc. Indeed the term "applications" does not suggest the most significant direction of influence in this case. Mechanics suggested problems and analysis worked out new theories and methods to solve them. These methods and theories are the lasting results, not the solution of the original mechanical problems. Bossut said it quite clearly: the problems eventually become problems of pure analysis.

Secondly, in addition to providing a language for the formulation of problems, rational mechanics supplied image and prestige for the new methods in the calculus. For, as we have seen in the encyclopedias, the progress of the calculus is shown and explained by means of the problems it can solve. And the problems in rational mechanics can be explained and their importance conveyed to a general public, whereas such explanation is much less feasible for the mathematics itself.

Finally, the fact that analysis was developed in the 18th century in close relation with problems from rational mechanics determined in a very deep way the concepts of analysis in that century. This is especially the case with the concept of variable. Throughout the eighteenth century, analysis was primarily the science of variable quantities. This concept is natural for a theory of physical problem situations. It is less natural for a purely mathematical theory for a purely mathematical theory of analysis; there the concept of function, which is quite something different, is the natural one.[20] The rise of the concept of function occurred in the eighteenth century precisely for purely mathematical reasons - especially in connection with Euler's shaping of analysis as a unitary structure. But the function concept took over very slowly, just because of the use of analysis in physical problem situations. The tension between the two concepts explains much of the story of the foundation of analysis in the eighteenth and nineteenth centuries, as for instance why it took so long for the limit concept to be recognised as the appropriate foundation.[21]

The same tension between the concepts of variable and function accompanies the drifting apart of pure and applied mathematics in the nineteenth and twentiest centuries - in which the function concept became the fundamental concept of pure mathematics (and consequently the derivative function became the fundamental concept in analysis) whereas the applications in science and engineering still needed and used variables and their differentials.

4. So far about the aspects of the interrelation between analysis and its "applications" within rational mechanics. But we have noted that these involved no clear crossings of boundaries, that in fact they are called "applications" with some doubt. So the question arises: were there any "real" applications of the calculus in the

eighteenth century?
There were indeed such crossings of boundaries. The boundary in question was the boundary between rational mechanics and the rest of what was called "mixed mathematics":[22] architecture, navigation, geography, warfare, machines and technology in general.
Clear examples of such crossings of boundaries, of applications or uses of calculus methods, combined with mechnanical theory, in the fields of mixed mathematics, are not numerous. This is not surprising because the simplifications and abstractions which were used in rational mechanics to arrive at mathematically tractable problems were so considerable that the application of the results on practice was often not feasible. For instance, the current hydrodynamical theories disregarded viscosity and internal friction of the fluids. Still, crossings there were. The most spectacular one occurred in connection with the problem of longitude, that is, the problem of determining a ship's position at sea. For this, accurate tables predicting the position of the moon were needed. To calculate such tables required good observations of the basic parameters and a good theory of the very complicated motion of the moon. It was the calculus, combined with Newton's mechanics, that eventually supplied such a theory. Many mathematicians worked at it but it was especially Euler's theory of the moon, combined with most accurate observations by Tobias Mayer, which provided, in the 1750's, tables accurate enough for effective determinations of ships positions at sea - a very spectacular result recognized as such when they came in general use in the 1760's.[23]
A second example is ballistics, where the theory of projectile motion under resistance proportional to the square of the velocity supplied, in combination with experimental data on the velocity of projectiles shot from canon, the so-called quadratic theory of ballistics, which in the second half of the eighteenth century was adopted by artillery. How much this theory, and the artillery tables that could be calculated with help of it contributed to the effectiveness of artillery in the 18th century is difficult to assess, but certainly quadratic ballistics provided the beginning of theoretical ballistics which in the nineteenth century was one of the bases of greater effectiveness in artillery.[24]

5. I want to say something more about the crossings of boundaries in the case of hydrodynamics. At first sight this may seem not a very appropriate example because hydromechanics was notorious in the eighteenth century for the lack of accord of its theories with experimental results. Newton's Principia [1687], Daniel Bernoulli's Hydrodynamics [1738], Lagrange's Analytical Mechanics [1788] and a number of other books and articles in the learned journals promoted theoretical hydromechanics in a magnificent way, but the results were of little use in practice.
Especially the theory of resistance of bodies moved in water (and hopefully ships moving in the sea) proved of little value for the practitioners in the shipbuilding trade. Still there were crossings of boundaries in the case of hydrodynamics.
To illustrate this I follow an argument by professor R. Hahn, presented in a lecture "Hydrodynamics in the eighteenth century - scientific and sociological aspects" [1964b]. Professor Hahn notes that before the appearance of the above mentioned great books on hydrodynamics there was already a discipline called hydraulics. This was the knowledge of the crafts of the builders of waterworks, of shipbuilders and of navigators.
There was treatises in which this knowledge was comprised. In the first half of the seventeenth century the treatises on hydraulics were organised as books of reference, often alphabetically organised, surveying the known techniques within the trades.[25] They reflected a static knowledge, the combined experience of shipbuilders, navigators and constructors of waterworks.
But after the middle of the seventeenth century a change occurred in the style of these treatises, they began to be organised as a theory, starting with basic principles, from which - it was claimed - the rules of the craft could be deduced. This change of style,[26] which occurs also in the treatises on architecture, was related to the requirements of education; it may also have been related to changes in philosophy. In the second half of the seventeenth century there arose military and civil engineering schools. However, not surprisingly, the principles professed in the new treatises were very feeble. Here was a topic of dispute, and thus of contact, between the teachers in the craft of hydraulics and the scientists. Thus Huygens entered a controversy with Renau d'Eliçagaray on the

mechanical principles which the [1689] latter had professed in his textbook on the manoevring of ships. The same discussion is taken up by Johann Bernoulli is his Essay of a new theory of the manoevring of ships [1714]. Later writers of textbooks - like Bélidor and Bouguer - offered little higher mathematics and mechanics in their treatises, but they were keenly interested in the development of the theoretical side of hydrodynamics. Euler's Naval Science [1749] was translated [1773] from the latin and adapted to the level of the well educated engineer (which meant leaving out most of the calculus).
These textbooks in hydraulics were used at the military and civil engineering schools which, especially in the eighteenth century France, had become very prestigeous centres of education;[27] they were the forerunners of the famous École Polytechnique.
Around these institutions there arose the scientifico-technical professions of teachers and, more important, examiners.[28] Bossut, whom we have met, was such an examiner. These people formed a link between the analytical science of hydrodynamics, or rational mechanics in general, and technology.

In 1775 the discrepancy between the theoretical hydrodynamics and the experimental results led to an effort, especially supported by the French minister Turgot, to close the gap, through a theoretical and experimental research program. It is not surprising that it was again Bossut who was asked to carry out the program.[29] This resulted in his theoretical and experimental treatise on hydrodynamics [1786], meant for an engineering public, in which he combined the mathematical theory and the experimental side of the subject. As it happened, the experiments made the gap between theory and practice only much clearer, but work in the same style was continued and in the nineteenth century the gap began to close, through the work of Navier, Coriolis, Saint Venant and others.

What occurred in the case of hydrodynamics occurred to rational mechanics in general. It acquired an increasing role in education at the French technological schools of artillery, shipbuilding, engineering, architecture etc. Over the century the knowledge of pure mathematics required for entry in these schools, and the mathematics

taught there was extended. By the 1750's there were occasional courses in the calculus at some engineering schools;[30] in the École Polytechnique of the 1790's calculus had become an established part of the curriculum.[31]
The prestige of teachers and examiners increased - and these people were interested in the theoretical sciences. Thus there was created a public of engineers with at least some knowledge of the techniques of the calculus and of their use in the abstract theories of mechanics.

6. To conclude: the example of hydrodynamics shows that once applications are seens as crossing of boundaries, there is much more to it than simply taking a piece of mathematics and using it somewhere. There is more to fertilizing relations of fields of scientific and technological activity than the too restricted term of "applications" suggests. There are the social relations, the sociology of the groups involved, professional functions, jobs, styles, education etc. This implies that even if in the eighteenth century the tangible applications of the calculus over the boundary between rational mechanics and mixed mathematics or technology were not very great in number, these other aspects were important enough, both for the development of the calculus itself as for the fields in which it was, with more or less success, applied.

For analysis itself the importance of these crossings of boundaries lay primarily in education: in the second half of the eighteenth century analysis became a teachable subject, teachable to engineering students. This implied an additional interest, for didactical reasons, in questions of foundation and inner organisation of the subject. The great tradition of the french "Cours d'Analyse", courses of analysis,originated in the eighteenth century awareness that students of engineering can usefully be taught calculus.
For technology the eighteenth century crossings of boundaries were important because they prepared the ground for an increasing role of the calculus in technological applications in the subsequent centuries.

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Notes

1) Zedler [Lexikon]; according to Zischka ([1959] p. XL), the writing of the Lexikon was supervised by Johann Christoph Gottsched, Johann Heinrich Rother and (later) Carl Günther Ludovici; but the names of the nine editors for special disciplines are not known.

2) "Functio Lineae heisset in der Analysi Mathematicorum eine Grösse, welche die Beschaffenheit einer Linie ausdrücket, so entweder aus ermeldeter Linie oder einer Potenz von ihr bestehet, oder aus derselben Linie mit andern beständigen Grössen auf alle mögliche Art zusammengesetzet ist. (...) Die allgemeine Expression der Function einer Linie A ist

$$mA^a + nA^b \text{ etc:} + \overline{pA^d + qA^e \text{ etc:}}^{\,\alpha} + \overline{sA^g + tA^h \text{ etc:}}^{\,\gamma} + \overline{xA^i + yA^l \text{ etc: etc:}}^{\,\delta}$$

$\pm$ etc. " (Zedler [Lexikon] vol. 9 (1735), col. 2308-2309).

3) [Encycl. méth.]; this reworking and systematic rearrangement of the articles in the [Encyclopédie] of d'Alembert and Diderot was organised by Ch.F. Pancoucke.

4) Zedler [Lexikon] vol. 6 (1733), col. 185-195 ("calculus differentialis") and col. 199-204 ("calculus integralis").

5) There is a short article "Differentialgleichung" in Zedler [Lexikon] vol. 7 (1734), col. 892, which mainly refers to the article on integral calculus.

6) "...einer der allerherrlichsten Erfindungen in der Mathematic, welche nicht nur die Geometrie fast bis auf den höchsten Gipffel erhoben, sondern auch die andern Disciplinen dergestalt erweitert hat, dass man ganze Bücher schreiben müsste, wenn man den Nutzen von diesem Calculo specificiren wollte." (Zedler [Lexikon] vol. 6 (1733), col. 190).

7) Zedler [Lexikon] vol. 6 (1733), col. 191.

8) "In der Mechanic sind uns dadurch die verborgensten Geheimnisse der Natur eröffnet worden." (Zedler [Lexikon] vol. 6 (1733), col. 191).

9) There is an apt description of the method used in this study of motion: "In der Abhandlung de motu variato ist man durch die Erkäntnis derer Differential-Grössen in den Stand gerathen, die theorie desselbigen auf das höchste zu poussiren, und curvas von denen wunderbarsten Eigenschaften zu finden; indem man eine solche veränderliche Bewegung in motus elementares resolviret, solche alsdenn als uniformes betrachtet, und aus denen conditionibus virium, velocitatum, temporum &spatiorum, ihre mutationes entdecket, und die curvas darinnen sich dergleicehn Bewegungen zutragen ausfündig gemacht" (Zedler [Lexikon] vol. 6 (1733), col. 191).

10) The textbooks on the calculus mentioned in the Lexikon articles on differential and integral calculus are: Carré [1700], Cheyne [1703], Craig [1685], [1693], l'Hôpital [1696], Manfredi [1707], Wolff [1704], [Elem. Math.]. In addition to these, there are references in general to the articles of Newton, Leibniz, Tschirnhaus, l'Hôpital, Huygens, the Bernoullis, Craig, Varignon and Hermann in the journals Acta Eruditorum, Philosophical Transactions, Histoires et Mémoires de l'Académie Royale des Sciences, and Commentarii Academiae Petropolitanae.

11) It is not quite clear how far d'Alembert, who died in 1784, has himself updated his encyclopédie articles for the encyclopédie méthodique. Such updating has taken place, but the fact that most articles were originally written in the 1750's is apparent in the encyclopédie méthodique; it explains, for instance, why there are fewer references to Euler's textbooks than one would expect in the 1780's.

12) [Encycl. méth.] Math. vol. 1, article "différentiel - Calcul Différentiel" pp. 520-526; ibid vol. 2, article "Intégral - Calcul Intégral" pp. 214-228.
13) The articles on differential and integral calculus do list a considerable number of applications, but Bossut's list is more extensive and detailed.
14) "De toutes les découvertes qui se sont jamais faites dans les Sciences, il n'y en a point d'aussi importante, ni d'aussi féconde en applications, que celle de l'analyse infinitésimale." ([Encycl. méth.] Math. vol. 1, p. LXXII).
15) "La nouvelle Géometrie a été appliquée à toutes les autres parties des Mathématiques, & toutes l'ont forcés de se perfectionner elle-même, en offrant sans cesse des problêmes qui finissent par se reduire à de pures questions d'Analyse."([Enc. méth.] Math. vol. 1, p. XCVI).
16) The full list (for which a footnote may be an appropriate place) of the topics which Bossut mentions in his historical survey of analysis since Leibniz and Newton is as follows:
Fundamental concepts of the calculus; Leibniz on tangents and extreme values; Newton on motion in conic sections, central forces, $1/r^2$, perturbation of planetary motion, tides, precession of the equinoxes; isochronous curve; paracentric isochrone; catenary; rectification of parabolic and logarithmic spirals, quadratures and cubatures of these; loxodromes; catenary with non uniform weight; sailcurve; tended arc, elastic beam; quadrable parts of a sphere; Bernoulli's inverse tangent problem (curve OP through origin, whose tangent PQ meets X-axis in Q, PQ:OQ given); exponential calculus; l'Hôpital on tangents, extreme values, points of inflexion, cusps, radii of curvature, evolutes, caustics by reflection and refraction; brachystochrone; isoperimeters; brachystochrones towards a given curve; synchronous curve; differentiation with respect to a parameter; curve of equal pression, solid of least resistance; orthogonal trajectories, reciprocal trajectories; vibrating strings; path of a projectile in medium with resistance $:v$, $:v^2$, $:v^n$; probability; the "equations of condition" (cf. note 18); homogeneous differential equations. Mechanics: principles of mechanics; general catenaries; elastica under influence of its own gravity; tautochronism in medium with resistance; rotation among a variable axis; oscillations of a hanging chain; motion resulting from excentric percussion; momentaneous centre of rotation; general principles of dynamics; vibrating strings and arbitrary functions; oscillations of a column of air, sound in organ pipes. Hydrodynamics: efflux from a vessel through a hole; general equations of hydrodynamics; form of the earth. Applications of hydrodynamics to navigation: stability of floating bodies; motion of ships in water. Astronomy: catalogue of visible stars; determination of fundamental constants; aberration of stars; nutation of the earth's axis. Celestial Physics: universal gravitation; perturbation of planetary motion; tides; tidal effects on the atmosphere; general theory of planetary motions; lunar theory; comets; precession of the equinoxes; nutation of the earth's axis. Optics: chromatic and spherical aberration; achromatic lense systems. Analysis: equations of higher than 4th degree; binomial equations; series; reduction of equations; analytic geometry; third order curves; interpolations; quadratures; algebraic curves; rectifiable curves on surfaces; reciprocal trajectories; curvature of surfaces; calculus of sines and cosines; general solution of isoperimetric problems (i.e. variational calculus); rectifications

of ellipses and hyperbolas (i.e. elliptic integrals); partial differential calculus; integral calculus with finite differences. ([Encycl. méth.] Math. vol. 1, pp. LXXII-CXIV).

17) In illustration, I list chronologically the principal works which appeared between 1735 and 1755 shaping rational mechanics into a recognizable discipline: Euler's Mechanics [1736], D. Bernoulli's Hydrodynamics [1738], Clairaut's Theory of the figure of the earth [1743], d'Alembert's Treatise of dynamics [1743], Johann Bernoulli's Hydraulics [1743], d'Alembert's Treatise on the equilibrium and motion of fluids [1744], Euler's Naval science [1749], d'Alembert's The resistance of fluids [1752].

18) Partial differential equations are a good example of the close interaction between analysis and rational mechanics. The first article to deal with the solution of partial differential equations is usually considered to be Euler [1740b]. Here, and in [1740a], the problem is indeed not directly connected with mechanics. Euler treats families of curves in whose equation an integral occurs; he wants to find an equation in finite terms. This leads him to the problem of finding solutions (in finite terms) of partial differential equations of the form $\frac{\partial z}{\partial x} = f(x,y)$. Euler does not seem to be aware here of the special nature of such differential equations; in particular he is not aware of the great variety of possible solutions, arbitrary functions do not occur in the article.
There are several instances of the occurrence of partial differential equations in rational mechanics studies around 1740. Thus for instance the "equations of condition" $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$, for a differential $Adx + Bdy$ to be exact, are used in Clairaut's study [1743] on the figure of the earth, and in d'Alembert's treatise of dynamics [1743] partial differential equations occur as expressions for conditions of motion. However, in these cases only special solutions of the partial differential equations are involved. The first general solutions of partial differential equations, clearly exhibiting the role of arbitrary functions, occur in two treatises of d'Alembert, [1747a] on the general cause of the winds (in which this cause is sought in tidal movements in the atmosphere), and [1747b] on the vibrating string. In both cases d'Alembert reduces a physical problem to the mathematical problem of chosing two functions α and β, occurring in two differential forms, such that these forms become simultaneously exact. In the case of the vibrating string these forms are such that the problem is equivalend to solving the partial differential equation $\frac{\partial^2 y}{\partial x^2} = c^2\frac{\partial^2 y}{\partial t^2}$; in the other case he gives a more general approach equivalent to solving a (hyperbolic) second order homogeneous partial differential equation with constant coefficients.
There evolved a long discussion on the nature and acceptability of the arbitrary functions introduced by d'Alembert; in this discussion Euler stressed the acceptability of all functions, regardless of their representability by means of analytical formulas or the occurrence of singularities.
Meanwhile d'Alembert, Euler, Daniel Bernoulli and Lagrange developed the theory of partial differential equations further

in connection with the many examples of such equations which they encountered in mechanical studies. Indeed, in Truesdell's words: "... the great gift of continuum mechanics to analysis is the theory of partial differential equations" ([1960], p. 418). Euler's textbook [1768] has an extensive section on partial differential equations; Cousin's lectures [1777] deal with them and afterwards a number of textbooks devoted more or less attention to the subject. Still, throughout the 18th century it remained a subject which only few mathematicians could master. (I am indebted to Mr S.B. Engelsman for calling my attention to a number of the references cited above.)

19) This applies especially to Boyer [1949], whose original title (the concepts of the calculus) stresses that it does not pretend to deal with other aspects of the calculus than its fundamental concepts. However, as Boyer's book is now the standard reference for the history of the calculus, its special point of view should not be overlooked.

20) On the tension between the two concepts see Bos [1974], p. 6, and Freudenthal [1973], pp. 553-559.

21) Cf Bos [1975], pp. 28-30.

22) Cf: "Mathematicks is commonly distinguished into pure and speculative, which consider quantity abstractedly; and mixed, which treat of magnitude as subsisting in material bodies, and consequently are interwoven everywhere with physical considerations. Mixed mathematics are very comprehensive; since to them may de referred Astronomy, Optics, Geography, Hydrostatics, Mechanics, Fortification, Navigation, &c...", ([Encycl. Britt.], article "mathematics").

The "système figuré des connoissances humaines" in vol. 1 of d'Alembert's and Diderot's Encyclopédie arranges under "mathématiques mixtes": mechanics (with further subdivision), acoustics, pneumatics, art of conjecture, analysis of chance, geometric astronomy (with further subdivision) and optics (with further subdivision).

Another illustration of the extent of mathematics in the eighteenth century is the contents of a standard compilation as Wolff's [Elem. Math.], whose edition Geneva 1735-1743 deals with: arithmetic, geometry, plane trigonometry, analysis (of finite and infinite quantities), mechanics, statics, hydrostatics, aerometrics, hydraulics, optics, perspective, catoptrics, dioptrics, theory of the sphere and spherical trigonometry, astronomy (both "spherical" and "theoretical"), geography, hydrography, chronology, gnomonics, pyrotechnics, and architecture (both military and civil).

23) The tables came in use at sea in the 1760's. The Nautical Almanac was published since 1766 to reduce the calculations involved in using the tables. In 1765 the british parliament rewarded £3000 to Mayer's widow (Mayer had died in 1762) for his contribution to solving the longitude problem; £300 went to Euler for his part in the theoretical foundation of Mayer's work. £5000 was given to Harrison, for the invention of the chronometer, which provided an alternative method to determine longitude. See Forbes [1973].

24) Cf. Charbonnier [1928] and McShane [1953].

25) Cf Hahn [1964b], pp. 7-8.

26) Hahn ([1964b], p. 11) mentions as an example Hoste [1697].
27) On the history of these schools see Taton [1964].
28) The list of mathematicians who in some way or other served the french system of technical schools as "examinateurs" includes Camus, Bézout, Bossut, Monge and Laplace.
29) On this research program, and on the chair of hydrodynamics which formed its institutional base, see Hahn [1964a] and [1964b], pp. 22-25. Bossut got the chair, d'Alembert and Condorcet were connected with the program, which included extensive experiments on resistance of bodies moving in water.
30) At the École Royale des Ponts et Chaussées, founded in the 1740's, "on enseigna aussi, et dès les premiers temps, le calcul différentiel et intégral, mais il n'est pas certain que les élèves aient été dans l'obligation de suivre les leçons" (Taton [1964], p. 358). When Bossut started teaching at the École Royale du Genie at Mézières in 1753, he improved the mathematics program by, among other things, introducing the elements of the calculus. His reforms, however, were hampered by the more conservative requirements set by Camus as examiner (cf. Taton [1964], p. 587). Cf. also: "Dans certaines écoles royales militaires, on peut apercevoir quelques nouvelles tendances qui démontrent que l'enseignement n'était pas trop en retard sur les progrès scientifiques de l'époque. D'abord, il faut signaler la présence des questions de calcul intégral et différentiel dans plusieurs exercises publics, tels que ceux de Brienne en 1782 et de Sorèze en 1784. Pour cette étude, le livre de Mlle Agnesi (i.e. [1748]) semble avoir été utilisé. Mais il faut remarquer que peu d'élèves étaient en état de répondre à ces questions qui, pour l'époque étaient de haute mathématique." (Taton [1964], pp. 534-535).
31) From its foundation in 1794 the École Polytechnique planned the calculus as part of its curriculum. Lagrange taught analysis at the École from 1795 til 1799; he was succeeded by Lacroix. Lagrange's teaching lent much prestige to the school. It was advertised for instance in Prony [1795], where it is proudly announced that Lagrange will present his own demonstrations of the fundamental principles of the differential and integral calculus, avoiding the disadvantages of both fluxions and limits. (p. 208). It seems, however, that his lessons were rather too abstract and did not fit in well into the curriculum (cf. Fourcy [1828], pp. 190-191).

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