UNIVERSITY

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MATHEMATICS

THE STRUCTURE OF DESCARTES' GÉOMÉTRIE

by

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PREPRINT

NR. 506

Maart 1988

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To be published in:

Atti del Convegno Internazionale

"Descartes: il Discorso sul Metodo

e i Saggi di questo Metodo"

(Lecce, 21-24 Ottobre 1987)

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H.J.M. Bos

I. INTRODUCTION

There is no doubt about the significance of Descartes' Géométrie[1]: It gave us analytic geometry. From it mathematicians learned that curves have equations (in two unknowns) and that, conversely, such equations define curves; curve and equation are largely equivalent notions. A shape like that of the Cartesian folium (see Figure 1) is essentially the same thing as the equation $x^3 + y^3 = axy$; one can study a curve by means of its equation and an equation by means of the pertaining curve. Of course the story[2] is more complicated, other names have to be mentioned and nuances added. Nevertheless one may say that in the Géométrie of 1637 one of the themes that was to become a leitmotiv in seventeenth century mathematics resounded clearly for the first time: the link of formula and figure, the interconnexion of algebra and geometry.

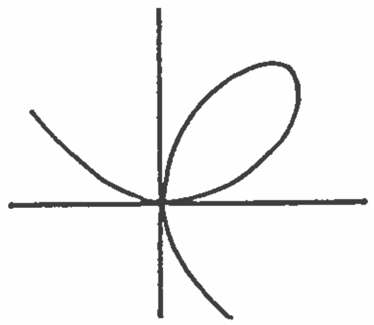


Figure 1

However, the reader who opens the Géométrie expecting to read a book on analytic geometry, is likely to have a confusing experience. And even if one approaches the book without specific expectations one comes up against many questions and puzzles. For instance, the equivalence of curve and equation, which is the core of analytic geometry, appears to be rather a side issue in the <u>Géométrie</u>. Had that equivalence been central one would have expected Descartes to deal with the curves according to their degrees, starting with the straight line, moving on to the conics and so on. In fact an equation of a straight line occurs only once in the text[3], more or less in passing, and Descartes discussed several curves without giving the equation at all[4]. Furthermore the book contains much algebraic theory about equations in one unknown, which at first sight seems unrelated to the theme of expressing curves by means of equations. And then there is the issue of the <u>demarcation</u> of geometry about which Descartes felt strongly. For Descartes algebra concerned addition, subtraction, multiplication, division and extracting roots. At that time logarithms, sines, cosines, exponentials and the like had not yet entered the arsenal of algebraic formulas. This meant that not all curves could be represented by means of an equation. Descartes emphatically demarcated the science of geometry as having to do only with those curves that have algebraic equations (in coordinates along straight axes); that is, equations involving +, -, x, : and roots only. A curious point indeed, for why should algebra be a criterion for the demarcation of geometry?

These enigmatic aspects have to do with the <u>structure</u> of Descartes' theory and with the resulting structure of his book. The separate passages of the <u>Géométrie</u> are understandable enough, but one often wonders why Descartes chose to include them, why he treated them at their particular places in the book and how they relate to the other parts. Still one never feels that Descartes himself was insecure about the structure he gave to the book; his transitions between sections are secure and confident and he usually states explicitly why he treats the subject at hand.

This suggests a special way to approach the questions relating to the <u>Géométrie</u>, namely to study the <u>structure</u> of the book; and that is what I intend to do in this paper. I shall look in particular at the order in which Descartes presented his topics, and at the reasons that he gave for dealing with the separate topics. There are three themes that are crucial for understanding the structure of the book: <u>problems</u>, <u>constructions</u> and <u>rules beyond mathematical correctness</u>. These themes define some guiding questions which I shall now mention briefly.

We open the book and we read:

> "Tous les Problesmes de Geometrie se peuvent facilement reduire a tels termes, qu'il n'est besoin par aprés que de connoistre la longeur de quelques lignes droites, pour les construire."[5]

It was indeed Descartes' objective to solve all geometrical problems. So we shall have to answer the question what these <u>problems</u> of geometry were. In the same opening sentence we read what solving problems meant; it meant construction, which is the second theme. When Descartes summed up his achievement at the end of his work, he wrote:

> "Mais mon dessein n'est pas de faire un gros livre, et ie tasche plutost de comprendre beaucoup en peu de mots: comme on iugera peutestre que iay fait, si on considere, qu'ayant reduit à une mesme construction tous les Problesmes d'un mesme genre, iay tout ensemble donné la façon de les reduire à une infinité d'autres diverses; et ainsi de resoudre chascun deux en une infinité de façons. Puis outre cela qu'ayant construit tous ceux qui sont plans, en coupant d'un cercle une ligne droite; et tous ceux qui sont solides, en coupant aussy d'un cercle une Parabole; et enfin tous ceux qui sont d'un degré plus composés, en coupant tout de mesme d'un cercle une ligne qui n'est que d'un degré plus composée que la Parabole; il ne faut que suivre la mesme voye pour construire tous ceux qui sont plus composés a l'infini."[6]

Clearly Descartes saw the constructions he had given as the final and concluding results of his study. We shall have to ask what was meant by <u>construction</u>? Finally there is a curious theme which occurs particularly in the third book. Descartes explained there that he was presenting a method to enable geometers to avoid what he called a "faute", a mistake, an error.

For instance:

> "Et ce n'est pas une moindre faute aprés cela, de tascher a le [sc. a problem, HB] construire sans y employer que des cercles et des lignes droites, que ce seroit d'employer des sections coniques a construire ceux ausquels on n'a besoin que de cercles. car enfin tout ce qui tesmoigne quelque ignorance s'appele faute."[7]

What was this "faute"? It was to construct the solution of a problem with improper means, particularly with means more complicated than necessary. According to Descartes such a procedure was improper, it showed ignorance. However, a mathematician who commits this error is not doing something which is mathematically incorrect. Descartes' insistence on the "faute" shows the importance he attached to certain <u>rules beyond mathematical correctness</u> and we shall have to inquire into the nature and effect of these rules.

II EARLY MODERN GEOMETRY

The passages quoted above and the themes to which they point, indicate that Descartes wrote his book from a particular view of geometry. He saw geometry as the art of solving geometrical problems. This is a very important point, if merely because such a vision of geometry, although very common in the early modern period, is no longer familiar to us today. So let me contrast it with some other views of geometry. Descartes did not see geometry as the axiomatic deductive science which derives theorems about geometrical objects. That is, he did not follow the style of Euclid's <u>Elements</u>. In fact many geometers of his time appear to have had the idea that with the <u>Elements</u> geometry had sufficient theorems at its disposal, and that now it was time to use these theorems for solving problems. Nor did Descartes see geometry as the investigation of properties of geometrical objects or configurations - which is the view that fits the modern conception of analytic geometry.

I am aware that, if we choose to argue along strictly mathematical lines, it makes no difference whether we see geometry as the activity of solving problems, proving theorems or investigating properties. The solution of a problem can be formulated as a theorem or as a property of a geometrical configuration. Still I maintain that for the practice of geometrical research it makes a great difference whether one adopts the one view or the other. The mathematician's vision of geometry determines to which goals he directs his research, what he finds important and how he structures his writings.

So Descartes saw geometry as the art of problem solving and his goal was to solve "tous les problesmes de geometrie". What were these problems? And what did solving, that is, constructing such problems mean? To answer that question I shall discuss here a problem which, together with its construction, is characteristic for what can be called the early modern tradition of geometrical problem solving. I take the example from the <u>Geometria Practica</u> (1604) of Clavius, a work that Descartes probably knew well.[8] It is as follows.

<u>Problem</u>

Given a triangle ABC and a point D (see Figure 2), it is required to draw a straight line through D dividing the triangle into two equal parts.

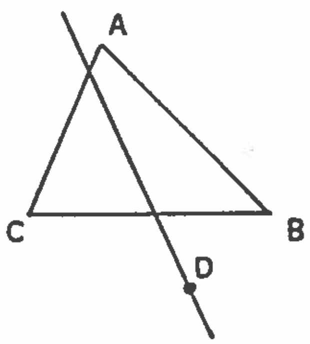


Figure 2

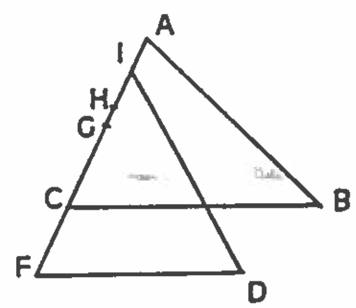


Figure 3

Clavius gives the following

<u>Construction</u>

(1) Extend AC (see Figure 3) and draw DF parallel to CB intersecting AC in F; take G on AC such that AG = CG.

(2) Take H on CA such that CH is the fourth proportional of DF, BC and CG, that is

$$DF : BC = CG : CH .$$

Clavius assumes that his reader knows how to construct a fourth proportional. A standard construction was (see Figure 4): Mark off the lines DF and BC along the arms of an arbitrary angle; connect the endpoints; mark off CG along the same arm as DF, draw a parallel through its endpoint; the resulting segment on the other arm is the required fourth proportional, as is evident because of the similarity of the triangles.

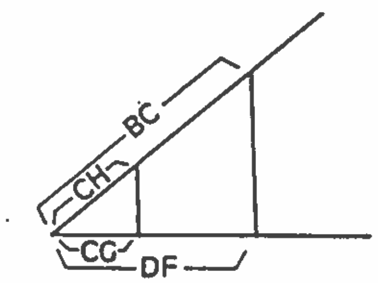


Figure 4

(3) Construct the mean proportional L of FC and CH, that is, a line L satisfying

$$FC : L = L : CH .$$

Again Clavius does not explain this construction; the standard procedure was (see Figure 5) to mark off FC and CH along a line, draw a semicircle on FH, draw a perpendicular in C intersecting the semicircle in Q; CQ is the required

mean proportional, because the triangles FCQ and QCH are similar.

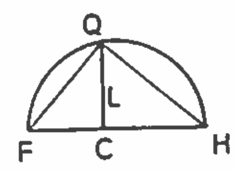


Figure 5

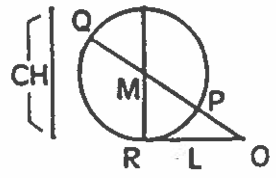


Figure 6

(4) Construct I on CA such that:

$$CI \times HI = L^2 ,$$

or, as Clavius put it, such that the rectangle with sides CI and CH is equal to the square with side L. Here Clavius refers to a construction he had explained in a comment on Theorem III-36 in his edition of Euclid's <u>Elements</u>[9]. That construction is as follows (see Figure 6): Draw a circle with diameter CH; mark off RO = L along a tangent; draw a line through O and the centre M of the circle, it intersects the circle in P and Q. Then (by Euclid III-36)

$$OR^2 = OP \times OQ ,$$

so that if we take CI = QO and HI = PO (which fits, because QP = CH) we have

$$CI \times HI = L^2$$

as required.

(5) Draw a straight line through D and I (see Figure 3); that line divides the triangle into equal parts.

Clavius then proves that DI is indeed the required line[10].

This construction requires some comments. First of all, Clavius did not explain how he found the construction. Perhaps insiders could infer that from his proof (for there is a system behind his construction[11]), but not easily. Clavius did not present a method for finding constructions; he did not give an analysis. Secondly, all the steps in the construction can be performed by ruler and compass. Problems that admitted such constructions were called "plane" problems. Thirdly, Clavius apparently did not expect his reader actually to take compass and ruler and perform the construction. Construction was a mental operation; Clavius' text helped the mind to see that the construction could be done and (if one knew the standard constructions and checked the references) how it could be done. But, fourthly, however formalized and remote from actual execution the presentation was, the terminology of the construction did refer to actual execution of the construction. The process was expressed as a task, a procedure, almost a ritual which the geometer had to perform. A task implies that rules have to be obeyed and criteria satisfied. What were they in this case? The rules were that every step in the construction had to be performed by ruler and compass. The criterion of adequacy was simplicity: Constructions should not be more complicated than necessary. Actually Clavius' construction was very adequate; the job cannot be done much more simply.

III A THESIS

After this example of the practice that formed the background to Descartes' geometrical essay, I can formulate the thesis that I want to defend in this paper about the structure of Descartes' Géométrie. It is this:

> The aim of the Géométrie was to provide a method for the art of geometrical problem solving as outlined above. That aim involved two levels of problems, a technical level and a methodological one; consequently the book had a twofold programme. The structure of the Géométrie is appropriate to and understandable from its aim and its twofold programme.

On the technical level, the programme was to provide an analysis, that is, a universal method of finding the constructions for any problem that could occur within the tradition of geometrical problem solving. That method was: to use algebra in analysing geometrical problems.

On the level of methodology the programme concerned a crucial question within the tradition of geometrical problem solving, namely:

How to construct when ruler and compass are insufficient?

As the classical Greek geometers had already realized, not all problems can be constructed by ruler and compass; the "classical problems" for instance, duplication of the cube, trisection of the angle, quadrature of the circle, cannot be constructed in this way. Geometers still wanted to solve such problems. Which other means of constructions were acceptable and which were not; what should be the criterion of simplicity in deciding whether constructions were good enough? These questions had to be answered; in the second, methodological part of his programme Descartes provided answers.

IV THE GEOMETRIE; BOOK I

I shall now discuss how the technical and methodological questions mentioned above determined the structure of the Géométrie. The work consists of three books. Descartes provided marginal titles for subsections within these books; book I contains nine such subsections, Book I 19 and Book III 32. Thematically the books can be split up into a smaller number of sections. I have given this division in Table I; the characterisation of the contents is mine.

TABLE I
The structure of the Géométrie:

Book I: Analysis of plane problems

I-A Geometrical interpretation of the operations of arithmetic
G. pp. 297-300; A.T. pp. 369-372
I-B Problems, equations, construction of plane problems
G. pp. 300-304: A.T. pp. 372-376
I-C Pappus' problem; deriving the equation, cases when the problem is plane
G. pp. 304-315: A.T. pp. 377-387

Book II: Acceptability of curves

II-A Acceptable curves, their classification
G. pp. 315-323: A.T. pp. 388-396
II-B Pappus' problem continued, solution in the case where there are 3 or 4 given lines, plane and solid loci, simplest case of five given lines
G. pp. 323-339: A.T. pp. 396-411
II-C Acceptability of pointwise construction of curves and construction by cords
G. pp. 339-341: A.T. pp. 411-412
II-D Equations of curves, their use in finding normals
G. pp. 341-352: A.T. pp. 412-424
II-E Ovals for optics
G. pp. 352-368: A.T. pp. 424-440
II-F Curves on non-plane surfaces
G. pp. 368-369: A.T. pp. 440-441

Book III: Simplicity of curves and of constructions

III-A Acceptability of curves in constructions, simplicity
G. pp. 369-371: A.T. pp. 442-444
III-B Equations and their roots
G. pp. 371-380: A.T. pp. 444-454
III-C Reduction of equations
G. pp. 380-389: A.T. pp. 454-464
III-D Construction of roots of third and fourth degree equations, solid problems
G. pp. 389-402: A.T. pp. 464-476
III-E Construction of roots of fifth and sixth degree equations, "supersolid" problems
G. pp. 402-413: A.T. pp. 476-485

Book I, which I shall deal with briefly, can be characterised as algebraic technique in the methodologically unproblematical case where the problems can be solved by means of ruler and compass. Descartes first (I-A, see table I) shows how the operations of arithmetic, addition, subtraction, multiplication, division and extracting square roots, can be interpreted in geometry. He then (I-B) explains how the geometer, in dealing with a problem, should apply these operations and derive an algebraic equation. The solution of this equation will provide the solution of the problem. In the usual case, as with the Clavius problem, it will be an equation in one unknown. And in the cases to which Descartes restricts himself in the first book, that equation will be of first or second degree. Descartes explains how the roots of such an equation can be constructed by ruler and compass, thereby providing the geometrical solution to the original problem, namely the construction. It is worth noting that if one duly applies Descartes' method to the problem of Clavius, one gets precisely the same construction as Clavius gave[12].

Sometimes the problem involves one degree of freedom. In that case the resulting equation has two unknowns; the solution is a locus or a curve. One such problem is the famous problem of Pappus[13] which Descartes uses in the first two books to illustrate his methods and ideas. In the last part of book I (I-C) Descartes starts his discussion of the problem. Here his primary interest is not in the locus as curve, but in the constructibility of points on the locus. In particular he determines in which case these points can be constructed by ruler and compass.

V METHODOLOGY

Now I have to return to the methodological question which determines the structure of books II and III. The question was: How to construct when ruler and compass cannot do the job? The first thing to note is that algebra does not provide the answer. If we apply algebra to a geometrical problem the general situation is as follows: a configuration is given; it is required to find a point or a line segment within that configuration. That is: there is a length, as yet unknown, and we are required to construct a segment of that length. We call that length x and we derive an equation for x. We must then find the root or roots of that equation. In some cases algebra provides a formula expressing the roots in terms of the coefficients. But if the degree of the equation is greater than four there is no such formula, so in that case algebra is no help at all. If the degree is 3 or 4 there are such formulas, but they are very complicated, and, what is worse, they involve cubic roots. Cubic roots cannot be constructed by ruler and compass, so if we want to construct the solution - and we want that because we are doing geometry - we are still left with the problem of how to construct the cubic roots. If the degree is one or two, the case is unproblematical, a construction of the root by ruler and compass can then be found and that is indeed what Descartes explained in the methodologically unproblematical book I.

So algebra does not provide constructions. I wish to emphasise this point. Too often, I think, Descartes' application of algebra to geometry is seen as a brilliant trick for doing away with a mess of cumbersome earlier methods by simply applying algebra. One forgets that algebra did only half the business, namely the analysis. The geometrical construction had still to be done and here algebra gave no help or guidance. This remark is in fact the key to my understanding of the structure of the Géométrie[14]; it underlines the importance of the methodological questions which Descartes had to answer.

Of course Descartes was not the first to ask how constructions beyond ruler and compass should be performed. Indeed that question had been discussed from the classical beginnings of deductive geometry. The difficulty was that geometers had not reached a communis opinio on the matter. There were three alternative approaches in Descartes' time. The first was to use other instruments in addition to ruler and compass, and the second to use curves other than circles and straight lines. These two possibilities are closely

linked because the instruments would usually trace curves, in the same way as the compass and the ruler trace circles and straight lines. A third alternative was simply to postulate, without further explanation, that certain standard higher constructions were possible. That approach was in fact an extension of the way in which the Euclidean constructions are based on the first three postulates of the Elements[15].

Descartes made a choice. He chose construction by curves. That is, he accepted constructions whereby points are found by tracing a curve and intersecting it with a straight line, a circle or another curve. This choice, crucial for Descartes' methodological programme and for the structure of the book, led to two further questions.

The first question was: which curves may be used in constructions? Not just any curves. There were two curves in particular which Descartes could not accept as means of construction. These were the Spiral and the Quadratrix. Geometers had realized that if a spiral or a quadratrix is given, several problems, even difficult ones, could be constructed in a simple way. Too simple in fact. Using the spiral or the quadratrix, the trisection of the angle (which cannot be done by ruler and compass) would be as simple as the bisection (which can be done by ruler and compass). In fact the division of an angle into any number of equal parts would be a simple matter[16]. Geometers had felt uneasy about that: clearly, if one accepted constructions such as those with the quadratrix, the game would lose its interest. Descartes wanted to exclude these curves. So he had to formulate criteria of acceptability, he had to fix a demarcation between geometrical and ungeometrical curves.

The second question related to the "faute" mentioned above. Geometrical construction had to be effected with the simplest possible means. Obviously this meant that the curves used in the construction should be as simple as possible, but when is a curve simple? Here too choices had to be made; criteria of simplicity had to be formulated.

Thus we see that Descartes' programme of construction by means of curves naturally led him to two further issues. These issues characterise the remaining books of the Géométrie; book II is about acceptability and book III about simplicity.

VI ACCEPTABILITY AND DEMARCATION; BOOK II

What criterion does Descartes choose for the acceptability of curves? In the first section of the second book (II-A) he explains that acceptable curves are those that are traced by acceptable combinations of motions:

> ... et considerant la Geometrie comme une science, qui enseigne generalement a connoistre les mesures de tous les cors, on n'en doit pas plutost exclure les lignes les plus composées que les plus simples, pourvu qu'on les puisse imaginer estre descrites par un mouvement continu, ou par plusieurs qui s'entresuivent et dont les derniers soient entierement reglés par ceux qui les precedent. car par ce moyen on peut tousiours avoir une connoissance exacte de leur mesure.[17]

Motions are acceptable if they are continuous. Combinations of motions are acceptable if one primary motion completely determines the other motions that follow it. Descartes describes various examples. I shall explain one[18]. Descartes considers (see Figure 7) a parabola, which, he has earlier explained, is an acceptable curve. This parabola moves vertically and carries with it the point P. There is also a ruler which connects a fixed point D and the moving point P. If the parabola moves, the ruler follows; its motion is determined by that of the parabola. The combined motions of the ruler and the parabola in their turn determine the motion of their points of intersection I; during that motion, the intersections trace a new curve DEFGH. The new curve, according to Descartes, is traced by an acceptable combination of motions; it is therefore a geometrical curve. It is, in fact, the curve that later came to

be called the "Cartesian Parabola"; it plays an important role in the Géométrie. We will meet it again later on.

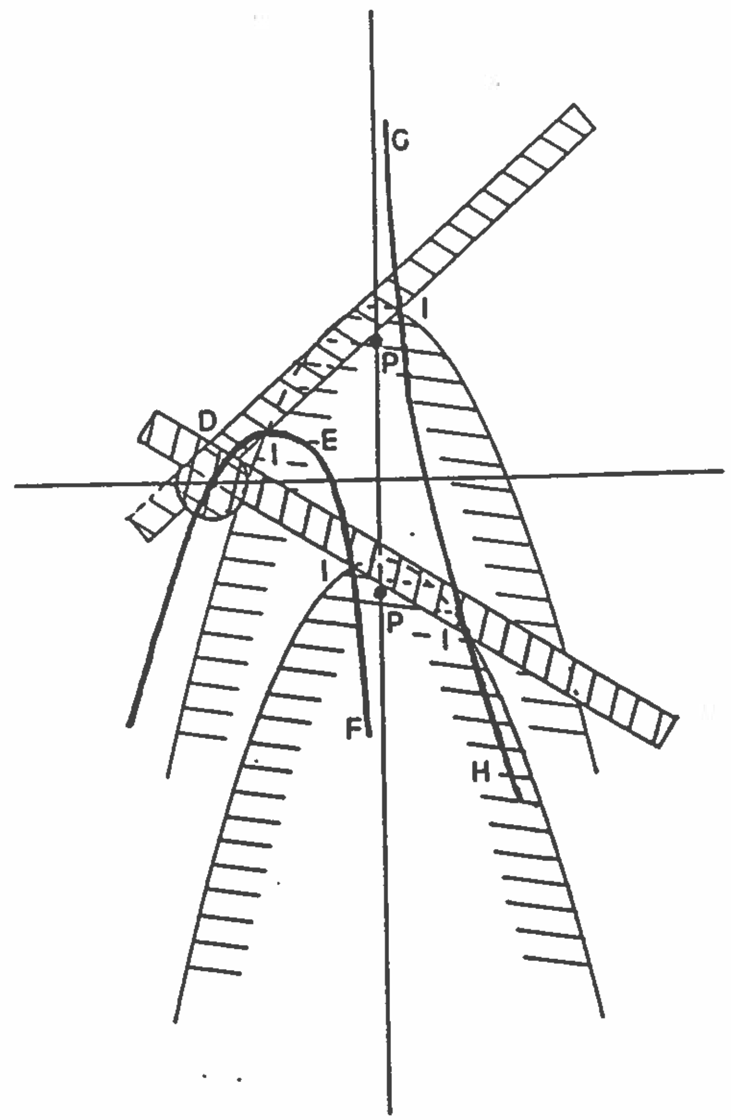


Figure 7

Why did Descartes choose the motion criterion for acceptability? To our modern eyes it does not seem very convincing or clear. But, if we recall Descartes' views on mathematical certainty as expressed in the

Regulae[19], we can appreciate his choice: He would have held that this kind of motion can be clearly and distinctly intuited, seen by the mind and that the combinations of motions are a case in which deduction, in the sense of an ordered sequence of consecutive intuitions, can retain the certainty of the first motion all the way through the series of linked motions to the last one. Indeed, if we look in the Géométrie for the "long chains of reasoning" mentioned in the Discours[om], we find them not in logical deduction from axiom to theorem but in the chains of motions that combine to trace the curves that are acceptable in geometrical constructions.

So the motion criterion was a natural choice. But it was not an easy one to work with because it left a lot of questions unanswered. For instance: Did the criterion exclude the spiral and the quadratrix? Can one really trace in this way all the curves that one would like to include? There are many other ways of tracing or constructing curves: point-wise, by means of instruments, by means of cords, etc. What is their relation to the motion criterion? And finally, methodologically a crucial question: How is one to define simplicity? Can one define simplicity of curves by the simplicity of the motions that trace them?

Descartes dealt extensively with all these questions and reached a final position, which was: Acceptable curves are precisely those that have algebraic equations, that is, equations involving only addition, subtraction, multiplication, division and roots. Later on in the 17th century these curves came to be called "geometrical curves".

At first sight this result is odd, to say the least, for why should the motion criterion coincide precisely with algebraicity? I cannot go into details here[20]; sufficient to say that Descartes took the issue very seriously indeed; he did not just equate the geometrical with the algebraical. His arguments were, however, not completely conclusive or convincing, and very few people after him bothered about these matters. Most of Descartes' followers took the outcome as dogma and did not think about the relation between acceptability of motion and algebraicity of the curve.

Most of the arguments on demarcation and acceptability can be found at the beginning of book II, notably in sections II-A, which is about acceptable motions, and II-C, which is about the acceptability of other means of constructing curves (point-wise and with cords). In section II-B Descartes completed his treatment of the Pappus problem, since he now could discuss the curves that occurred as solutions to that problem[21].

Section II-D is about the determination of normals to a curve (that is: lines that intersect a curve at a right angle). This section has been very influential in the history of infinitesimal methods. However, it stands somewhat apart as far as the structure of the Géométrie is concerned. I shall return to it below. The subsequent sections, on certain ovals which provide optically interesting shapes for lenses (II-E), and a very short remark (II-F) on three-dimensional geometry, also seem to be side issues within the general structure of the book.

VII SIMPLICITY OF CURVES AND CONSTRUCTIONS; BOOK III

I now turn to book III whose structure is determined by the question of simplicity. I referred earlier to the "faute" which Descartes enjoined geometers to avoid, the error of either constructing a problem with too complicated means or of trying, vainly, to construct a problem with simpler means than it requires. Simplicity is the key word here, and since constructions in Cartesian geometry are performed with curves, there should be a criterion by which it can be decided which curves are simple and when one curve is simpler than another. Descartes provided such a criterion. It was:

A curve is simpler in as much as the degree of its equation is lower.

Thus second degree curves (conic sections) are simpler than third degree curves (such as the Cartesian parabola) etc.[22]

This choice of criterion was not obvious and Descartes realized that. In fact in section III-A, where Descartes discussed the simplicity of curves, he first mentioned an alternative criterion of simplicity, namely the simplicity of the tracing motion. This is a more likely criterion because, after all, Descartes accepted curves only if they were traced by acceptable motions. But Descartes decided against that criterion and accepted the algebraic degree instead. It is likely that Descartes did so because he could not formulate a generally applicable criterion of simplicity of curve tracing. The fact that he chose the algebraic degree as a criterion for simplicity led to a certain inconsistency; the degree was not an obviously geometrical criterion. But the criterion had the advantage of being clear, and with it, finally, the business of construction beyond ruler and compass could be settled completely. The result was a clear-cut canon for dealing with geometrical problems. It was as follows.

(1) When a geometer was confronted with a problem, he should first translate it into its algebraic equivalent, that is, an equation.

(2) If the equation involved one unknown the problem was a normal construction problem. In order to get the simplest construction, the geometer should make sure that that equation had the lowest possible degree; that is, he had to check whether the equation was reducible, and if so, he had to perform the reduction and arrive at an irreducible equation.

(3) Once convinced that the equation was irreducible, he had to rewrite it in a certain standard form.

(4) Then he could read up in book III the standard construction for the roots of that standard equation by which the geometrical solution of the problem, that is a construction, was reached.

(5) If the equation contained two unknowns, this meant that the solutions formed a locus. In that case the geometer could construct points on the locus by choosing an arbitrary value for one of the unknowns and dealing with the resulting equation (in which there was only one unknown left) according to numbers (2)-(4).

By the method in (5) the locus was constructed "point-wise", that is, arbitrarily many points could be constructed on it. In section II-B, Descartes showed how in the case of second degree equations, occurring as solutions in a special case of the problem of Pappus (the so-called four line case), the loci, which in that case are conics, could be constructed as curves. However, he did not explain analogous procedures for higher order curves. In section II-D he claimed that the equation of a curve implied all its properties but he gave no general rules on how to derive these properties from the equation; he only treated the determination of normals to the curve.

This canon (especially numbers (2)-(4)) determines the structure of book III. After the short section on acceptability and simplicity of curves used in constructions (III-A) a large part (III-B,C) is devoted to the theory of equations and their roots. At first sight, as I mentioned above, this theory seems totally unrelated to geometry. But in fact it is not. All the themes dealt with in that part refer either to the reducibility of the equation, or to its transformation in standard form; both factors are necessary ingredients of the programme[23].

After this the remaining part of the book (III-D,E) gives the natural conclusion of the whole work: the standard construction of roots of equations. Descartes deals first with equations of degrees 3 and 4; for these he gives a standard construction by the intersection of a parabola and a circle, the combination of simplest possible curves for that case. He proceeds to degrees 5 and 6 and presents a construction by intersection of the "Cartesian Parabola" and a circle. Then he claims that the principle should be clear and leaves it to the reader to go on to higher degrees - an optimistic attitude; Descartes certainly underestimated the difficulties of this "etcetera"[24].

VIII BEYOND THE STRUCTURE

We have seen that Descartes adopted a particular view of geometry, quite common in his time, but unfamiliar nowadays. He took the consequences of that point of view, both the technical and the methodological ones. The resulting approach largely determined the structure of the Géométrie. There are, of course, many questions left. I shall touch briefly upon two, namely: Was he successful in his programme? and: What lies beyond the structure?

On the technical side the success of the Géométrie was immediate and lasting. The application of algebra proved a most powerful tool and mathematicians took it over quickly, eagerly and with great profit. But what about the methodological side? How were the main elements of Descartes' methodological position received, namely the demarcation of geometry, the criterion of simplicity and the geometrical construction of roots of equations? The demarcation between geometrical and non-geometrical curves in terms of motion proved convincing for a time, but soon broke down. Descartes' rejection of the motions that produced the quadratrix or the spiral, for instance, was based on his conviction that the length of curved lines could never be found exactly. Shortly afterwards the first rectifications of curves were found, undermining this belief[25].

Descartes argued that acceptable geometrical curves are precisely the ones that have algebraic equations. In a few examples he calculated the

equations of curves described by acceptable motions[26]; from these examples it was obvious that the curves traced by such motions had algebraic equations. But the other way round the matter is not so easy. Can each curve defined by an algebraic equation be traced by such combinations of motions as Descartes envisaged? They can indeed, as was proved in the nineteenth century[27], but Descartes' arguments here are vague and unconvincing[28]. Very few people were interested in the matter anyway; generally, Descartes' followers accepted as dogma that geometrical curves are precisely the algebraic ones. Descartes' criterion for simplicity of curves, namely the algebraic degree, was less easily accepted. Several mathematicians criticised it and tried to substitute other, more directly geometrical criteria. However, no feasible alternative criteria were found[29]. As we have seen, the crucial counterpart to Descartes' use of algebra as an analytical tool was the geometrical construction of roots of equations. These constructions roused considerable interest among later mathematicians; a separate discipline even emerged, called the "construction of equations"; in this discipline variants were studied of the constructions that Descartes had given and methods were worked out to extend the constructions beyond equations of sixth degree, where Descartes had stopped. It was an active discipline for some time, but during the first half of the eighteenth century the interest in it faded and the theory died without having provided a satisfactory solution of the general problem of constructing roots of equations[30].

Thus the factors that determined the structure of the Géométrie, in particular Descartes' methodological choices, had very little influence on later mathematics. The book exerted its influence as it were in spite of its structure. In its structure, Descartes book was not modern; it fitted into the view of geometry at the time. But that view was soon superseded, mainly, curiously enough, as a result of the influence of the Géométrie itself.

So what were the really influential ideas of the Géométrie? First of all there was certainly the relation between curve and equation, the key idea of analytic geometry[31]. Although that idea turned out to be very fruitful, it did not have a predominant place in the structure of the Géométrie. Then there was the double-root method for determining normals (and tangents) to curves (II-D). In the book it is very much a side issue, but in the subsequent history of infinitesimal methods it was to be a very influential idea. The third most influential part of the Géométrie was the theory of equations and their roots (III-B,C). This theory did fit into the structure: it helped the geometer to avoid the "faute" of constructing improperly. That context was soon dropped, but the theory itself attracted much interest and was developed further. In summary one might say that the lasting elements of the book defied its structure and broke through with a strength of their own.

What about Descartes himself? Did he feel restricted by the structure he chose? I think, given the state of mathematics at the time and Descartes' awareness of the philosophical questions concerning geometry, the structure was more or less imposed upon him. But, reading the Géométrie, one does get the impression that Descartes occasionally felt impeded by his self-imposed cadre. There are some expressions of boredom and irritation in the book ("et ie tascheray d'en mettre la demonstration en peu de mots. car il m'ennuie desia d'en tant escrire"[32]) which seem to reflect his frustration at having to explain uninteresting details. Also it should be noted that the topic that Descartes valued most in the Géométrie in fact falls outside the structure. This is the determination of normals to curves by means of his double-root method (II-D), about which he writes :

> Et i'ose dire que c'est cecy le problesme le plus utile, et le plus general non seulement que ie sçache, mais mesme que i'aye iamais desiré de sçavoir en Geometrie.[33]

Perhaps we can say that the Muse of algebra was trying to tempt Descartes away from his adherence to the traditional geometrical framework - and occasionally succeeded.

IX CONCLUSION

I have touched upon many things and I have often had to omit important details and explanation. I do not want to leave the reader with the impression that the structure of the Géométrie is very clear-cut. The whole question of acceptable curves, for instance, is rather complicated, and so is the role of the curves within the theory, because they occur not only as means of construction but also as objects of study and as solutions of problems. Also the question of how far algebra guided (rather than was subservient to) Descartes' approach to geometry needs more careful consideration.

In conclusion, let me say first of all that the Géométrie is a great book. Much of its content proved important and influential despite the book's restrictive structure. In this lecture I have concentrated on aspects of the structure and the content of the book that did not last. I have done so because these aspects of the Géométrie are historically interesting and essential for understanding the book as a whole. I also feel that by studying the structure of the Géométrie and by interpreting the answers that Descartes gave to the difficult methodological questions in geometry, we can appreciate the workings of a great mind. Despite the absence of ultimate success, Descartes' treatment of these questions was an outstanding intellectual achievement.

NOTES

[1] Descartes (1637 b), henceforth abbreviated to "G."; my references follow the page numbers of the original edition; these page numbers are indicated in the A.T. edition.

[2] For further details and different opinions see e.g. Bos (1981), Bos (1984), Boyer (1956), pp. 74-102, Costabel (1982) in particular pp. 27-37, Dhombres (1978) pp. 134-143, Grosholz (1980), Hofmann (1951), Itard (1956), Lenoir (1979), Mahoney (1980), Milhaud (1921) pp. 124-148, Molland (1976), Scott (1952) pp 84-157, Vuillemin (1960).

[3] G. pp. 327-328; it is the line $y = m - \frac{n}{z}x$, which Descartes draws as a preliminary to finding the conics $y = m - \frac{n}{z}x + \sqrt{mm+ox-\frac{p}{m}xx}$; these conics are the loci in the Pappus problem in four lines (cf. notes [13] and [21]).

[4] For instance the curve that solves a particular case of the Pappus problem in 5 lines (G. p. 339, cf. also Bos (1981) p. 316, note 21). Descartes does not use equations to represent the optical ovals in section II-E.

[5] "Any problem in geometry can easily be reduced to such terms that a knowledge of the length of certain straight lines is sufficient for its construction." (G. p. 297) (Here and below I have taken over the translations by Smith and Latham of passages from the Géométrie (cf. (Descartes 1637b)); I have, however, adapted these translations when I considered them too imprecise or disagreed with the implicit interpretation.)

[6] "But it is not my purpose to write a large book. I am trying rather to include much in few words, as will perhaps be inferred from what I have done, if it is considered that, while reducing to a single construction all the problems of one class, I have at the same time given a method of reducing them to an infinity of others, and thus of solving each in an infinite number of

ways; that, furthermore, having constructed all plane problems by cutting a straight line by a circle, and all solid problems by cutting a parabola also by a circle, and, finally, all that are only one degree more complex by similarly cutting a curve only one degree higher than the parabola by a circle, it is only necessary to follow the same method to construct all problems, more and more complex, ad infinitum." (G. pp. 412-413)

[7] "And it is then as great a mistake to try to construct it [sc. a problem] by using only circles and straight lines as it is to use the conic sections to construct those [problems] that require only circles; for after all any evidence of ignorance is termed a mistake." (G. p. 383)

[8] Clavius (1604), book VI, Prop. 12, Probl. 2, pp. 294-295, also in (Clavius 1611-1612) vol. 2 pp. 159-160. On Descartes' acquaintance with Clavius' work see (Milhaud 1921), p. 235.

[9] Euclid (1589).

[10] The proof can be summarised as follows (cf. Figure 3): $CI \times HI = L^2 = FC \times CH$ by (3) and (4); hence $CF : CI = IH : CH$, from which it follows that $IF : CI = CI : CH$. Furthermore $IF : CI = FD : CK$ (by similar triangles), hence $FD : CK = CI : CH$, so $FD \times CH = CK \times CI$. Now by (2) $FD \times CH = CG \times CH = \frac{1}{2}AC \times CB$, so $\frac{1}{2}AC \times CB = CK \times CI$. Hence $\Delta\ CKI = \frac{1}{2}\ \Delta\ CBA$, because the areas of triangles which have an angle in common are as the products of the sides around that angle.

[11] One notes that Clavius has translated all given and required relations into proportionalities that apply to line segments along one line in the figure, in this case along AC. The problem is then solved by using standard constructions to find line segments satisfying these proportionalities.

[12] See section II and Figure 3. First the given and unknown segments should be denoted; therefore call the given segments $CA = b$, $CB = a$, $AB = c$, $FD = p$, $CF = q$, and the unknown segments $CI = z$ and $CK = u$. Then the given and the

CF = q, and the unknown segments CI = z and CK = u. Then the given and the required relations have to be translated into equations. The similarity of triangles CKI and FDI yields pz = u(q+z). The division into equal parts leads to uz = $\frac{1}{2}$ba. Eliminating u from these two equations yields

$$z^2 = \frac{\frac{1}{2}ba}{p}z + \frac{\frac{1}{2}baq}{p}$$

Descartes gives (G. pp. 302-303) a standard construction for the roots of the equation

$$z^2 = Fz + G^2 ;$$

that construction is in fact exactly the same as item (4) from Clavius' construction with RM = $\frac{1}{2}$F and RO = G. Therefore, in order to use this standard construction, the Cartesian geometer has to construct line segments F and G such that

$$F = \frac{\frac{1}{2}ba}{p}$$

and

$$G^2 = \frac{\frac{1}{2}baq}{p} = Fq ,$$

so

$$G = \sqrt{Fq}$$

This is done by the constructions for multiplication, division and extracting square roots which Descartes explained on the first pages of his book (G. pp. 297-280). If one performs these constructions (taking p as unit line segment) one gets precisely items (2) and (3) of Clavius' construction. (Actually Descartes takes an arbitrary unit segment; if one takes that unit unequal to p the construction becomes more involved than Clavius' construction, although it leads to the same result.)

[13] The problem is the following: Let n straight lines L_i be given in the plane, as well as n angles ϕ_i and a constant segment a. For any point P in the plane one defines oblique distances d_i to the lines; these are the lengths of segments that are drawn from P towards L_i making with Λ_i the

angle ϕ_i. It is required to find the locus of points P for which a certain proportion involving the d_i and depending on the number of lines is constant. The relevant proportions are:

For three lines: $d_1^2 : d_2d_3$

For four lines: $d_1d_2 : d_3d_4$

For five lines: $d_1d_2d_3 : ad_4d_5$

For six lines: $d_1d_2d_3 : d_4d_5d_6$

Etc.

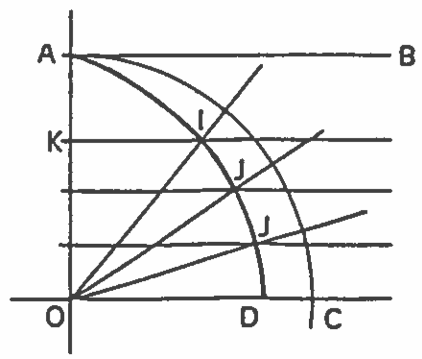
[14] Significantly the same issue marks the place where the Regulae (A.T. 10, Cf. Descartes 1977) break off, namely precisely at the point where Descartes would have to retranslate the result of the algebraic analysis, the equation, into geometrical constructions. Apparently around 1628 Descartes was not yet able to do this.

[15] Notably Viète, who advocated, in his Supplementum Geometriae (Viète 1593), the introduction of a new postulate to "supplement" geometry and make problems like the duplication of the cube or the trisection of the angle constructable. He postulated the possibility of the so-called neusis construction which had already been extensively used in classical Greek geometry. In a neusis construction a segment of given length is placed between two given straight lines or circles in such a way that the segment (or its prolongation) passes through a given point.

[16] The quadratrix (see the Figure) is the curve AD within the quadrant OAC which is traced by the intersection I of a horizontal line and a radius if both these lines move uniformly in the same time-span, the horizontal line from position AB to position OC, and the radius from position OA to position OC. It follows from that generation that for any point I on the quadratrix

angle COI : angle COA = OK : OA.

Hence, given a quadratrix, an arbitrary angle COI can be divided into 2, 3 or any number of equal aparts by dividing the line segment OK into as many equal parts (which can be done by ruler and compass) and drawing horizontals through the division points. These horizontals intersect the quadratrix in points J; the radii OJ divide the given angle in the required manner.



[17] "...and if we consider geometry as a science which in general teaches us how to know the measurements of all bodies, then we have no more right to exclude the more complex curves than the simpler ones, provided they can be imagined as described by a continuous motion or by several successive motions, each motion being completely determined by those which precede it; for in this way an exact knowledge of the measurements of each is always obtainable. (G. p. 316)

[18] G. pp. 335-338; other examples are on pp. 317-323.

[19] Notably rules 3 and 5-7, A.T. 10 pp. 366-370, 379-393; cf. (Descartes 1977) pp. 295-302; cf also (Hacking 1980)

[20] Descartes' arguments in this connection occur at several places in the Géométrie; one important passage is G. p. 319. I have collected and analysed the relevant statements and arguments in (Bos 1981) pp. 323-325.

[21] Section II-B contains in fact a complete solution of the Pappus problem in four lines (cf. note [im]); Descartes proves that the locus in that case is

a conic section. He also explains how in any given case the conic can be constructed (by constructions explained in Apollonius' treatise on conics). Descartes also deals with two special cases of the problem in five lines.

[22] G. p. 371

[23] To be more specific: Sections III-B,C contain 21 subsections for which Descartes gives separate titles in the margin. Almost all of these serve construction-related aims. These aims are: [1] the reduction of the equation (to avoid the "faute" of constructing by improper means) and [2] the transformation, mainly by substitutions $x \rightarrow x+a$, of the equation into standard forms. For third and fourth degree equations [2a] this standard form is a fourth degree equation in which the coefficient of x^3 is zero (Descartes gives the construction of the roots of this standard equation in III-D, G. pp 389-395; the construction proceeds by the intersection of a parabola and a circle). For fifth and sixth degree equations [2b] the standard form is a sixth degree equation in which the coefficients are alternately positive and negative (the construction of its roots, by the intersection of the "Cartesian parabola and a circle, is given in III-E, G. pp. 402-411). To indicate how these aims determine Descartes' theory of equations, I list ([a], [b],...) the topics of the subsections of III-B,C and indicate in brackets how they relate to one of the aims [1], [2a] and [2b]: [a] Number of roots of an equation (preliminary); [b] negative roots (preliminary); [c] lowering of the degree of an equation by division by $(x-x_0)$ ([1]); [d] checking whether x_0 is a root ([1] via [c]); [e] number of positive roots of an equation, "rule of signs" ([2b] via [j]); [f] transformation by $x \rightarrow -x$ (preliminary to [g]); [g] transformation $x \rightarrow x+a$ ([2]); [h] effect of that transformation on negative roots ([2b]); [i] to remove second term ([2a]); [j] use of $x \rightarrow x+a$ to make all real roots positive ([2b]); [k] idem to make all coefficients unequal to zero ([2b]); [l] transformation $x \rightarrow cx$ or $x \rightarrow x/c$ (simplifying coefficients,

useful for simplifying the constructions); [m] removing fractions from coefficients (idem); [n] making one coefficient equal to a given value (?); [o] real and imaginary roots ([2b] via [j]); [p] reducibility of cubic equations ([1]); [q] division by $x-x_0$ ([1]); [r] irreducibility of cubic equations ([1]); [s] reducibility and irreducibility of biquadratic equations ([1]); [t] example ([1]); [u] general method to test reducibility ([1]).

[24] The geometrical construction of roots of higher order degree equations became part of the theory called "construction of equations", on which cf. note [30].

[25] Van Heuraet, Fermat and Neile independently found such rectifications around 1658. Cf. e.g. (Baron 1969) pp. 223-228.

[26] Notably the hyperbola, G. p. 322 and the Cartesian parabola, G. p. 337.

[27] (Kempe 1876).

[28] For an analysis of Descartes' arguments see (Bos 1981) pp 323-324.

[29] Cf. (Bos 1984) pp. 355-371.

[30] Concerning this theory see (Bos 1984)

[31] G. p 341

[32] "and I will try and give the demonstration in a few words, for I am already wearied by so much writing." (G. p. 309)

[33] "And I dare say that this is not only the most useful and most general problem in geometry that I know, but even that I have ever desired to know." (G. p. 342)

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