UNIVERSITY

UTRECHT

DEPARTMENT

OF

MATHEMATICS

ARGUMENTS ON MOTIVATION IN THE RISE AND DECLINE OF A MATHEMATICAL THEORY; THE "CONSTRUCTION OF EQUATIONS", 1637 - CA. 1750.

BY

H.J.M. Bos

PREPRINT

NR. 246

AUGUST 1982

ARGUMENTS ON MOTIVATION IN THE RISE AND DECLINE OF A MATHEMATICAL THEORY; THE "CONSTRUCTION OF EQUATIONS", 1637 - ca. 1750.

by

H.J.M. Bos

I Introduction

I-1

Seventeenth-century mathematics saw the creation of two analytical methods whose descendants still dominate mathematics at the present day: the analytic geometry of Fermat and Descartes and the calculus of Newton and Leibniz. These methods were developed to solve problems in geometry, and in particular problems about curves. The foundations of that part of mathematics were exclusively geometrical. If a proof were to be convincing, it had to be in the geometrical style of Euclid and Archimedes. If a new mathematical object (as for instance a curve) had to be introduced, it was necessary to explain how it could be geometrically constructed. If a problem had to be solved, the ultimate answer had to have the form of a geometrical construction. Mathematicians often preferred not to work out the ultimate proofs or constructions in detail, but they were aware that the certainty of their arguments could be secured only by basing them on geometrical foundations.

Not only were the foundations of the field in which the new methods were developed geometrical, also the material studied in that field was geometrical. In fact a whole new world of geometrical objects was laid open to mathematical study in the seventeenth century, namely higher-order algebraic curves and many new transcendental curves.

However, the new methods, analytic geometry and calculus, were algebraic, or analytic, rather than geometrical. That is, they consisted in the manipulation of formulas. In the earlier phases of the development it was still felt that the equations of analytic geometry were representations of curves that had to be constructed geometrically, and that the theorems or the solutions of problems found by means of the analytical formulas of the calculus, had still to be proved correct by geometrical proofs. But the interest of mathematicians turned more and more to the analytical methods as such, whereby the attention for the geometrical origin and foundation of the entities, methods and proofs was gradually lost. Thus occurred a process which I shall call "de-geometrization". De-geometrization had strong effects; it implied deep changes in the mathematicians' conception of the objects they were dealing with and the aims they set for their research. Hence programmes and directions in research changed, old questions became meaningless and other questions acquired new meaning.

I am convinced that many developments in seventeenth- and eighteenth-century mathematics cannot adequately be understood without taking into account the deep changes in the conception of mathematical objects, aims, methods and proofs that were caused by the loss of attention for the geometrical foundations.

In the present article I shall discuss an example of the effects of de-geometrization in a separate branch of mathematics, namely the so-called "construction of equations". In this case the effect was extreme: in losing its geometrical basis the subject also lost its sense, and, after a flourishing period in the seventeenth and early eighteenth century, it fell into oblivion.

I-2

The construction of equations was a well defined area of research from the publication of Descartes' Géométrie (1637) until some time after 1750. For Descartes it was a central part of the technique of solving geometrical construction problems. He taught that in order to solve a construction problem one had to derive the algebraic equation which the length of the required linesegment had to satisfy. The problem was thereby reduced to finding a geometrical construction of the roots of a given equation. This was called "to construct the equation". Descartes gave general rules for constructing equations up to the sixth degree, and he gave suggestions about how to proceed for higher-degree equations.

The roots of equations of degree higher than two cannot, in general, be constructed by ruler and compass. Descartes, therefore, was confronted with the question of how to construct geometrically those linesegments that cannot be constructed by ruler and compass. In working out the theory of constructing equations Descartes gave a well-considered answer to that question.

Thus the construction of equation emerged as a crucial technique in geometry. Soon the subject acquired a more independent status and became a standard approach to the solution of equations, whether or not these equations had their origin in a geometrical construction problem.
In a process of further de-geometrization, the subject developed into a collection of algebraic techniques, whose geometrical origin was taken less seriously or ignored.

For Descartes, the construction of equations was a central part of the programme for geometry which he presented in his Géométrie. The subject did not keep the prominent a place in mathematics that it had for Descartes. Nevertheless, many mathematicians, among them Fermat, Sluse, Wallis, Newton, Jakob Bernoulli, l'Hôpital, Euler and Cramer, wrote about it, devised new constructions and debated the criteria with such constructions had to satisfy. Thus in the seventeenth century the construction of equations was a field of serious interest for mathematical research, and until well into the eighteenth century it was a respectable standard part of textbooks on algebra and analytic geometry. Still, after 1750 the subject fell into oblivion. Indeed, so marked was the loss of interest in the construction of equations that one may speak here of the death of a mathematical theory.

I-3

There is few secondary literature about the construction of equations. Zeuthen[1)] mentioned it very briefly as a method which was losing its meaning. Wieleitner[2)] and (in somewhat more detail) Boyer[3)] discussed it, stressing mainly the emergence of the idea to consider the roots of an equation as the points of intersection of the graph of the polynomial with the X-axis. These writers hardly touched the arguments on method and motivation which guided the research in this field.

The construction of equations deserves more serious attention from historians for several reasons. Construction was a crucial concept in 17th-century mathematics. In solving problems or introducing new mathematical objects, the solutins and objects had to be constructed. Indeed the words "construction" and "solution" were almost synomynous - Euler still spoke of the "construction of differential equations" to mean finding the solution of differential equations[4)]. The construction of algebraic equations was the prototype of constructional practice; it covered those cases in which the object to be constructed was a point in the plane, or equivalently a length determining that point. The construction of curves and the construction of differential equations, which were the objective of most of the newly developed techniques of the calculus in the later 17th century, cannot be adequately understood without knowledge of this prototype theory of construction.

Moreover, the construction of equations concerned a mathematical problem of evident importance: how to construct objects (lengths in particular) which cannot be constructed by ruler and compass. The development and decline of the construction of equations was in fact the story of serious but ultimately unsuccessful attempts to find convincing solutions to this problem. In view of the great importance which ruler-and-compass constructions have had in mathematics from antiquity to the present day, the discussions about this complementary question deserve more attention than they have received until now.

A third reason for studying the subject was already mentioned above; it concerns the process of de-geometrization which occurred in mathematics in the 17th and 18th centuries. The historical development of the construction of equations presents an illuminating example of this important process.

Finally, it is possible to give a fairly complete description of the development of the construction of equations in the somewhat more than hundred years of its existence as a distinct mathematical theory. The development of separate sectors of science, designated as paradigms, research programmes or otherwise, has attracted much interest from philosophers of science in recent decades. The programmatic aspects and the processes of degeneration that sometimes occur, have been especially discussed. Both features are clearly present in the case of the construction of equations. A study of its development may therefore contribute to the understanding of these processes by providing a fairly fully documented case-study and by an assessment of how well this particular case conforms to the patterns which philosophers of science have suggested for the development of branches of science.

I-4

The arguments of my present study can be summarised as follows. I shall be interested in the development of the theory of construction of equations from 1637 till ca. 1750, and particularly in the causes of its initial flourishing and later decline. I shall argue that the principal factors determining the developments of the theory did not lie in the sphere of mathematical technique, but rather in the sphere of method and motivation. Consequently I shall be brief in explaining the technical aspects of the subject and I shall devote attention especially to the arguments on method and motivation. These arguments concerned the acceptability of algebraic methods in geometry and the criteria of adequacy for constructions of equations.

Many mathematicians involved in textbook writing or research about the construction of equations expressed opinions on the motivation of the subject and on the reasons for preferring certain constructions over others. I shall claim that these arguments, and the ensuing discussions, were an essential part of the mathematical activity concerning the construction of equations, and that, indeed, the subject declined because on this level of motivation and method the arguments ultimately failed to carry conviction.

The causes of the decline of the subject were not primarily a lack of success as a theory or a lack of usefulness in applications. Rather, as the arguments on motivation and method reveal, the subject declined because it lost sense and meaning. As their geometrical origin was less and less understood, the constructional procedures came to appear as meaningless. Old arguments in support of the sense of the subject proved unconvincing in the long run, better arguments failed to turn up, and so the construction of equations lost the interest of mathematicians.

I shall support this analysis of the development of the construction of equations by a detailed study of the arguments on method and motivation, and I shall deal particularly with the question why the arguments ultimately failed to be convincing. I shall find an explanation of this failure in the effects of the process of de-geometrization.

Thus my study will concentrate on a kind of question which until now, perhaps, has been too little studied by historians of mathematics, namely the role of motivational arguments in the development of mathematical theories.

I-5

I structure my further argument as follows: Section II deals with Descartes' Géométrie, the programme for geometry which he expounded in it, the origins of that programme and the role of the construction of equations within it, the results Descartes reached and the questions he left open. In section III, I formulate explicitly the central problem of the construction of equations, I explain the techniques that were elaborated to solve that problem and I discuss a central result on which mathematicians came to agree. Section IV is a rapid survey of the relevant primary sources, serving as a sketch of the factual development of the subject. Section V concerns the arguments on motivation and method that were put forth in connection with the construction of equations. I return in Section VI to the reasons of the initial flourishing and later decline of the subject. I add a few remarks about the implications of this development for general theories about the development of research programmes and on the role of constructions in general in seventeenth- and eighteenth-century mathematics.

II The construction of equations in Descartes' Géométrie

II-1

Descartes' Géométrie (1637) is a book with a programme[5]. Descartes wanted to reform geometry by providing it with a new and powerful method, and by clarifying its aims. The method was algebra, the use of equations to represent relations between known and unknown linesegments. The aim of geometry was to solve geometrical problems, primarily construction problems. Descartes thought that aim needed clarification because it was not clear what precisely was required for the construction of a problem.

In classical Greek mathematics the conviction emerged that constructions should preferably be performed by ruler and compass[6]. But it was also known (although not proved) in antiquity that certain problems cannot be constructed by ruler and compass. Mathematicians did not feel that geometry should renounce such problems and restrict itself to ruler and compass; consequently they were confronted with the question: What are the means of construction that we are to allow in geometry in addition to ruler and compass? According to Descartes this question had not been satisfactorily answered and hence the aims to geometry were insufficiently clear; in his Géométrie he provided an answer to the question.

In order to understand Descartes' answer it is necessary to mention earlier practices of constructing in geometry. Constructions by other means that ruler and compass were given in classical Greek mathematics, and two directions can be discerned in the choice of the additional means of construction. One was to accept a new class of curves as constructing curves in addition to the straight line and the circle: the conic sections. We find an example of their use (in fact probably the earliest example) in Menaechmus' (ca. 350 B.C.) construction of two mean proportionals between two given linesegments[7]. Let the given lengths be a and b; it is required to construct two lengths x and y such that

$$a : x = x : y = y : b.$$

Menaechmus' construction is as follows (see figure 1):

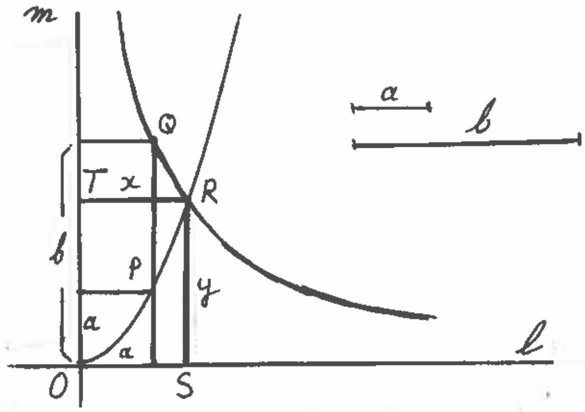


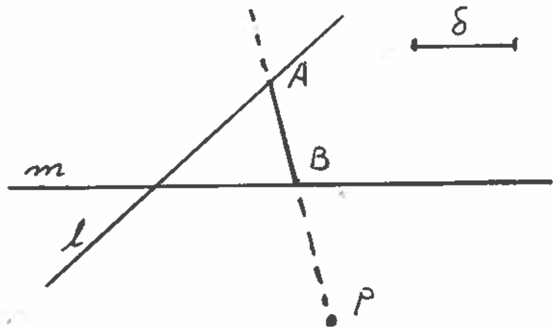
figure 1

Construct two perpendicular lines ℓ and m intersecting in O; construct points P and Q in one quadrant such that the distances of P to ℓ, P to m and Q to m are a, and the distance of Q to ℓ is b. Draw a parabola through P and O with main axis m. Draw a hyperbola through Q with m and ℓ as asymptotes. Determine the point of intersection R of the two conic sections. Draw perpendiculars RS and RT to ℓ and m respectively. Then $x = RT$ and $y = RS$ are the two required mean proportionals. (The construction is easily checked by analytic geometry: the parabola has equation $ay = x^2$, the hyperbola $xy = ab$, hence $a : x = x : y = y : b$.)

It is essential in this construction that Menaechmus, to overcome the restrictions of ruler and compass, allowed new means of construction; he allowed to draw conic sections and to determine their points of intersection. After Menaechmus, the most common approach to constructions beyond ruler and compass seems to have been this use of conic sections and their intersections. At any rate this approach is the basis of the classification of construction porblems which Pappus explained as standard and which was accepted, with some terminological modifications, until well into the seventeenth century. According to this classification[8)], problems constructible by ruler and compass were called "plane" and those constructible by conic sections (and not by ruler and compass) "solid". All other problems were called "linear", because for their constructions new and more complicated curved lines had to be introduced.

However, there was also another way of introducing additional means for solving construction problems, namely to accept a certain standard construction as possible, and to reduce the problems to that construction. This approach is evident in the so-called "neusis" constructions that were developed by Greek geometers[9)]. In a neusis construction it is assumed that it is possible (see figure 2) to construct a segment AB of given length δ

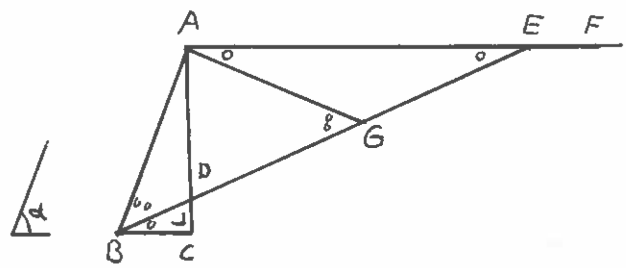
figure 2



such that A is on a given straight line ℓ and B on a given straight line m and the line AB (extended if necessary) passes through a given point P. In other words, it is assumed that a segment AB of given length can be placed between ℓ and m such that it "verges" towards P. This also explains the name of the construction, neusis derives from the Greek verb for verging. There were variants of neusis constructions in which the segment was placed between a straight line and a circle or between two circles.

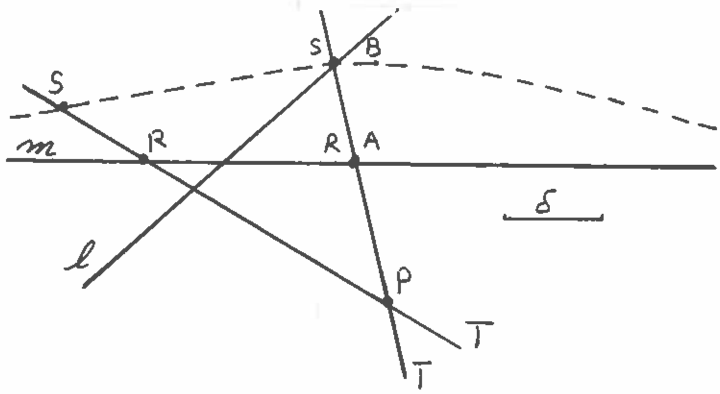
Pappus records a neusis construction for the trisection of an arbitrary angle[10]; it may serve here as example of such constructions. Let (see figure 3) α be the given angle. Make a triangle ABC with $\angle ABC = \alpha$ and $\angle BCA = 90°$. Draw AF through A parallel to BC. Insert a segment DE of length 2AB between AF and AC such that it verges toward B. Then $\angle EBC = \frac{1}{3}\angle ABC$. (To prove this, let G be the middle of DE, and draw AG. Then $\angle ABG = \angle BGA = 2\angle AEG = 2\angle DBC$, hence $\angle DBC = \frac{1}{3}\angle ABC$.

figure 3



These neusis constructions cannot, in general, be performed by ruler and compass. Pappus describes an instrument, invented by Nicomedes, for neusis constructions[11]. The instrument (see figure 4) consists of a ruler ST with

figure 4



an adjustable pin R ot it. The pin is adjusted such that RS = δ. Then the ruler is made to slide along the given point P while the pin R is guided along the line m. The movement is stopped the moment that S coincides with ℓ; SR is then the required position for the segment AB.

The instrument can be considered as a generalized compass. In that sense the neusis constructions are based on allowing a new constructing instrument in addition to ruler and compass, rather than allowing new constructing curves (namely the conics) in addition to straight line and circle, as exemplified in Menaechmus' construction. However, neusis constructions can also be interpreted in terms of constructing curves. In the process of using Nicomedes' instrument the point S describes a curve and the construction is performed by intersecting this curve with ℓ. Hence accepting neusis constructions is equivalent to accepting this particular curve as constructing curve in addition to circles and straight lines. The curve is a fourth-degree curve, its equation (with respect to m as X-axis and the Y-axis perpendicular to m through P) is

$$y^2x^2 = (a + y)^2(\delta^2 - y^2),$$

where a is the distance from P to m, and δ is the length of the segment. Nicomedes discussed the curve (which came to be called the "conchoid of Nicomedes") in connection with his neusis instrument and proposed that it should be used for performing neusis constructions.

Pappus explained how neusis constructions can be performed by intersection of conics[12)] (a circle and a hyperbola in this case). It seems likely that this was known to classical geometers. Still, construction problems were often reduced to a neusis without further reducing this neusis to a construction by intersection of conics - this occurs for instance in Archimedes' works. A neusis apparently was a sufficient construction, perhaps because its reduction to construction by conics was standard and uninteresting, or because a neusis was considered equally acceptable as, or even better than construction by conics.

Besides these two types of construction, by conics and neusis, others were considered as well in antiquity. They used special curves, such as the quadratrix, the cissoid and others, which were often specifically created for solving a particular construction problem.

Thus the sources about classical mathematics, such as they became known in the sixteenth and seventeenth centuries, did not provide an unequivocal answer to the question how to construct when ruler and compass are insufficient; but they did suggest that the question was an important one and that ancient mathematicians had considered it seriously.

II-2

Let me stress that the question which means of construction should be accepted in addition to ruler and compass is a non-trivial methodological question. Accepting a new means of construction implies that one singles out one particular construction problem (as for instance to determine the intersection of two conics, or to perform a neusis) and postulates that that problem is already "solved". There is a danger in doing so; by postulating too many problems as solved, the solution of the others may become trivial and uninteresting[12a)]. Hence in Greek mathematics there was a clear tendency to introduce as few means of construction as possible. Moreover, mathematicians required that problems should be constructed by the simplest possible means. In cases where construction by ruler and compass was possible it was a considerable error[12b)] to use conics; again, if construction by conics was possible it was not permitted to use more complicated curves. The requirement of using the simplest possible means of construction underlies Pappus' classification of problems mentioned in the previous section.

There is another reason why accepting new means of construction in geometry is a non-trivial matter: the arguments for or against accepting certain means rather than others are "meta-mathematical", they cannot be based on axioms or proved results. For instance (to quote some arguments from the seventeenth-century debates) one may prefer neusis constructions because one considers the relevant instrument simpler than instruments for tracing conic sections. Or one may prefer them because one considers that the motion involved in tracing the conchoid is simpler than motions that trace conic sections. Conversely, one may prefer conic sections because they have a lower degree than the conchoid. Whatever the choice, the arguments do not <u>prove</u> that it is right; the correctness of the choice cannot be deduced from geometrical axioms. Even to accept ruler and compass as fundamental constructing instruments is a choice; it cannot be proved correct, rather it is a matter of tradition and general consent among mathematicians. So, in working out theories of construction, mathematicians are confronted with questions that have to be answered but cannot be answered by deductive arguments. Hence those mathematicians who are not content with merely following tradition are forced to consider the "meta-mathematical" motivations for their procedures.

II-3

For Descartes, geometry was the art of solving geometrical problems, in particular construction problems. That art, as Descartes could find it in classical and contemporary works, was in some state on confusion: there were different opinions on what means of construction were to be allowed beyond ruler and compass, and also there was no general method to find best possible constructions. Descartes saw that a programme for geometry was needed which should clarify its aims and provide general methods. In his <u>Géométrie</u> he presented such a programme[13)].

As to the aims of geometry, the programme laid down which constructions were acceptable and which were not, and it gave an order of simplicity (and thereby of preference) among the acceptable means of construction. This clarified the aim of geometrical construction: it was to find, for any proposed problem, a construction using the simplest possible means.

As to method, Descartes' programme provided a unification through algebraical techniques. By these techniques all construction problems could be reduced to a set of standard problems, and for these Descartes set out to provide standard constructions.

Working out his method further, Descartes explained that, if a construction problem was proposed, one should first give names (letters) to both the given and the unknown quantities involved. Then the data and the requirements of the problem should be translated into equations expressed algebraically in terms of the assigned letters. (Descartes was convinced that in all truly geometrical problems the resulting equations would be algebraical, that is, they would involve only the operations +, −, x, :, and $\sqrt[k]{\ }$ ($k > 1$, integer). (All mathematicians using analytic geometry in the seventeenth and eighteenth centuries seem to have assumed as a matter of experience that radicals can always be removed from equations by reordering the terms and raising to a suitable power. Hence the equations in Cartesian geometry were in principle always polynomial equations.) Then the unknowns should be successively eliminated from these equations; this procedure should ultimately result in one equation, involving one unknown only[13a)]. The problem was thereby reduced to solving this equation; that solution would provide the value of the one unknown, from which the values of the other unknowns, if necessary, were to be obtained by solving further equations.

This translation of geometrical properties into algebraic equations has received much attention from historians of mathematics. The emphasis on this part of Descartes' programme has tended to obscure the importance of the other parts. However, merely translating into algebra does not solve a geometrical problem. For instance, the problem to construct two mean proportionals between given lengths a and b (cf II-1) is readily reduced to the equation

$$x^3 = a^2b$$

which can be solved algebraically:

$$x = \sqrt[3]{a^2b}.$$

But that solution is insufficient for the geometrical problem because it does not tell how the length $x = \sqrt[3]{a^2b}$ should be geometrically constructed. Claiming to do geometry, Descartes could not leave it at algebraic solution of equations. The application of algebra had provided a reduction of problems to the solution of equations in one unknown; the next step had to be to give rules how the roots of such equations should be constructed. This procedure was the "construction of equations". Descartes devoted a considerable part of his Géométrie to this construction of equations, and in doing so he had to explain which constructions were acceptable and how they should be ordered as to simplicity. Descartes stated that constructions by intersection of algebraic curves were acceptable in geometry and that the curves used in such constructions should have lowest possible degree. That is, he introduced algebraic curves as means of construction beyond ruler and compass, and he ordered these curves as to simplicity by means of their degree. I shall illustrate his construction procedure by reviewing his standard constructions for equations of degree ≤ 6 in the next sdctions, and in section III-4 I shall return to Descartes' opinions on the aims of geometrical construction and on the proper way to construct equations.

II-4

Let us suppose that a construction problem has been reduced, by using algebra as explained above, to an equation in one unknown. Descartes taught that then first it should be checked whether this equation could be reduced, that is, whether the equation, say

$$H(x) = 0,$$

could be written as

$$H(x) = U(x).V(x) = 0,$$

in which the coefficients of U and V could be constructed by ruler and compass from those of H. Descartes provided some methods for checking this[14]. (The methods in fact only cover the cases that the coefficients in U and V are rational in those of H.) If such a reduction was possible the problem was solved by solving either $U(x) = 0$ or $V(x) = 0$. Proceeding this way the problem is ultimately reduced to an irreducible equation.

Descartes now classified problems according to the degree of that irreducible equation. If that degree was 1 or 2, the roots could be constructed by ruler and compass. Descartes gave such a construction[15] and explained (taking over the classification mentioned in II-2) that such problems were "plane".

If the degree of the equation was 3 or 4, Descartes asserted that its roots could not be constructed by ruler and compass. He gave a general rule by which the roots of any third- or fouth-degree equation could be constructed by the intersection of a circle and a parabola[16]. Such problems therefore were "solid"; they required a conic section for their construction.

To illustrate Descartes' procedure, let

$$x^3 + px + q = 0$$

be the equation to be constructed. Applying Descartes' rule leads to the following construction (see figure 5):

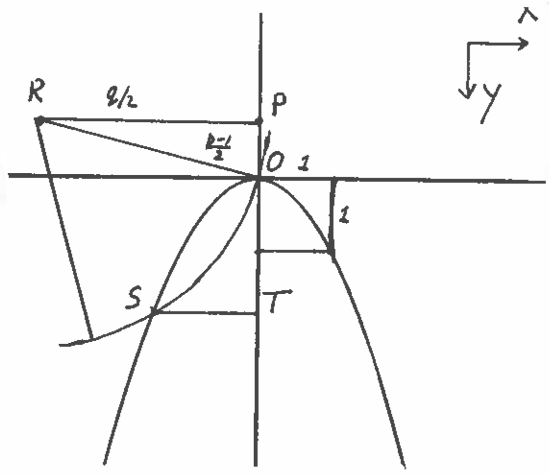


figure 5

Draw a pair of perpendicular axes intersecting in 0. Draw a parabola with the vertical line as main axis and with its top in 0. Adjust the unit length such that the parabola has equation $y = x^2$.[16]. Take $OP = \frac{p-1}{2}$ along the vertical axis, upwards if $\frac{p-1}{2}$ is positive, downwards otherwise. Take $PR = \frac{q}{2}$ horizontal, to the left if $\frac{q}{2}$ is positive, otherwise to the right. Draw a circle with centre R and going through 0. The circle intersects the parabola in another point (or points) S. Draw ST horizontal with T on the vertical axis. Then ST is the required root, to be taken negative if S lies on the left of T, positive otherwise. (The proof that the construction is correct is easy; I leave it to the reader; an analysis by van Schooten of the construction will be discussed in III-5.) In the case of a fouth-degree equation Descartes' construction was a bit more complicated as the circle no longer passes through O.

It was known that solid problems could be constructed by conic sections, but Descartes was the first to show that all problems leading to third- or fourth-degree equations could be constructed by circle and parabola only.

If the equation was of degree 5 or 6 Descartes called the problem "supersolid". For these also he gave a general construction rule[17)], introducing as additional means of construction a new curve, namely the (later so-called) "Cartesian parabola". He considered this curve as described by a special kind of tracing movement (which I shall discuss in III-4). It is a third-degree curve; its equation is

$$axy = y^3 - 2ay^2 - a^2y + 2a^3.$$

Descartes explained how, given any fifth- or sixth-degree equation, its roots could be constructed by the intersection of a circle and this curve. The construction is complicated but basically correct.

II-5

These constructions of equations of degree up to six are discussed in the third and last book of the Géométrie, and they form the conclusive results of the treatise. Descartes thought that these constructions showed sufficiently how one should proceed in constructing equations of higher degree that 6. He wrote:

> "... having constructed all plane problems by intersection of a circle and a straight line, and all solid problems by the intersection of a circle and a parabola, and, finally, all that are but one degree more complex by intersecting a circle by a curve but one degree higher than the parabola, it is only necessary to follow the same general method to construct all problems, more and more complex, ad infinitum for in the matter of mathematical progressions, whenever the first two or three terms are given it is easy to find the rest. I hope that posterity will judge me kindly, not only as to the things which I have explained, but also as to those which I have intentionally omitted so as to leave to others the pleasure of discovery" (1637 p. 413, tr. Smith & Latham).

In fact, Descartes' legacy was more problematical than he suggested here. I shall deal with the problems he left open in the next chapter.

III Construction of Equations: the problem and the techniques

III-1

At this point it is necessary to formulate more precisely what was the problem of constructing equations. This will be done in sections III-2 and III-3. On the basis of that formulation I shall discuss Descartes' opinions on the proper ways of constructing equations (III-4), the techniques that were developed after Descartes for finding constructions of equations (III-5, 6, 7), and a central result concerning the construction of higher-order equations on which mathematicians came to agree in the course of about 75 years after the publication of the Géométrie (III-8).

III-2

In modern terminology and notation the problem of constructing an equation can be formulated as follows: Let

$$H(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

be the equation which has to be constructed. This means that two constructing curves $\mathcal{F}$ and $\mathcal{G}$ have to be found such that the roots of the equation $H(x) = 0$ occur among the x-coordinates of the points of intersection of $\mathcal{F}$ and $\mathcal{G}$. Let the equations of $\mathcal{F}$ and $\mathcal{G}$ be

$$F(x,y) = 0 \text{ and } G(x,y) = 0$$

respectively. The x-coordinates of the points of intersection of $\mathcal{F}$ and $\mathcal{G}$ are the roots of an equation

$$R_{F,G}(x) = 0,$$

which is formed by eliminating y form $F(x,y) = 0$ and $G(x,y) = 0$. $R_{F,G}$ is called the resultant of F and G.

There are now two requirements for the curves $\mathcal{F}$ and $\mathcal{G}$.

Requirement 1

The roots of $H(x) = 0$ should be roots of $R_{F,G}(x) = 0$, that is,

$$R_{F,G}(x) = AH(x),$$

in which A may be a constant or a polynomial in x.

Requirement 2

The curves $\mathcal{F}$ and $\mathcal{G}$ should be acceptable as constructing curves in geometrical constructions, and they should be the simplest possible for the construction of $H(x) = 0$.

Descartes' construction of

$$x^3 + px + q = 0,$$

discussed in II-4, provides a good illustration of the role of the two requirements. We have there

$H(x) = x^3 + px + q;$

$\mathcal{F}$: the parabola $F(x,y) = y - x^2 = 0;$

$\mathcal{G}$: the circle $G(x,y) = y^2 + (p-1)y + x^2 + qx = 0;$

$R_{F,G} = x^4 + px^2 + qx = x(x^3 + px + q) = xH(x);$

so that requirement 1 is satisfied. According to Descartes, the circle and the parabola are acceptable mean of construction, so the first part of requirement 2 is also satisfied. The sense in which Descartes considered these curves as the simplest possible for constructing the equation will be discussed in III-4.

The formulation of requirement 1 shows that, algebraically, the problem of constructing equations is an inverse elimination problem. In a direct elimination problem the equations $F(x,y) = 0$ and $G(x,y) = 0$ are given and it is required to eliminate y, that is to determine $R_{F,G}$. Here $H(x)$ is given and $F(x,y)$ and $G(x,y)$ have to be found such that $R_{F,G}(x) = H(x)$, or $R_{F,G}(x)$ has $H(x)$ as a factor. Algebraically, such an inverse elimination problem seems to have little sense. The required equations $F(x,y) = 0$, and $G(x,y) = 0$ are more complicated than the given equation $H(x) = 0$ because they involve two unknowns. Moreover the problem is trivially solvable. A first solution that suggests itself is to choose

$F(x,y) = y - H(x)$

and

$G(x,y) = y,$

that is, to take the graph of $H(x)$ and the X-axis as constructing curves. There are many other choices possible.

Because requirement 1 leaves so much freedom, requirement 2 becomes crucial. Its function is to restrict this freedom and to give sense to a problem which purely algebraically has little sense. But the requirement is not clear: what are acceptable curves and when is a curve simple enough? It was about these questions that mathematicians argued when debating the construction of equations. Before discussing requirement 2 further, however, something must be said about an additional requirement namely that $H(x)$ should be irreducible.

III-3

There are two ways in which the construction of an equation can sometimes be reduced to that of two equations of lower degree. One of these was discussed by Descartes (cf II-4); it occurs in the case that $H(x)$ can be written as

$H(x) = U(x).V(x)$,

in which the coefficients of the polynomials $U(x)$ and $V(x)$ can be constructed by ruler and compass from those of $H(x)$. The construction of $H(x) = 0$ is thereby reduced to that of $U(x) = 0$ and $V(x) = 0$. In connection with the construction of third- and fourth-degree equations Descartes suggested methods to check if equations are reducible in this sense[18]. Later mathematicians took up this topic; Hudde, for instance, wrote a long treatise (1659) about it.

However, these studies were not pursued in direct connection with the construction of equations, and, conversely, writers on the construction of equations after Descartes did not devote explicit attention to the reducibility of the equations they dealt with[19].

The second kind of reducibility occurs when $H(x) = 0$ can be written as

$H(x) = U(y)$,

with

$y = V(x)$,

in which U and V are polynomials of lower degree than H. In that case a construction of $H(x) = 0$ can be performed in two steps: first construct $U(y) = 0$, then, inserting the value of y thus found, construct $y = V(x)$. This kind of reducibility occurs for instance in connection with Cardano's rule for solving fourth-degree equations, which leads to a sixth-degree equation in y which is a third-degree equation in y^2. In his discussion of this case[20] Descartes is obviously aware that the sixth-degree equation is solved in two steps. He does not, however, develop tests for such stepwise reducibility of equations, nor does he state explicitly that before constructing an equation it should be checked whether such a reduction is possible.

Fermat dealt explicitly with stepwise reducibility in a particular case. In his Dissertatio he discussed construction of equations of the form

$x^n = a^{n-1}b$

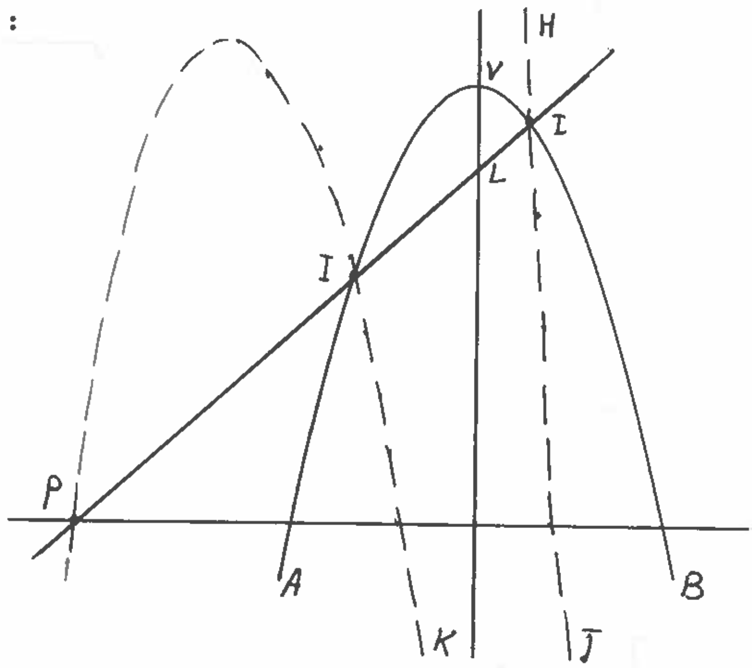
for certain prime values of n. He chose prime degrees because he wanted to show his method of construction in the case of an irreducible equation and he realized that for n not prime the equation would be reducible and constructible stepwise. (The primes are in fact the first five "Fermat numbers" $2^{2^k} + 1$, $k = 0,\dots,4$; cf IV-2 where I discuss these arguments in more detail.).

De la Hire gave a passing reference to stepwise reducibility[20a]. All other writers on the construction of equations did not separately consider the case in which the equations were reducible in any of the two senses discussed above. I shall quote this fact in V-1 as a sign that from the very beginning the motivation of the subject in the actual practice of geometrical construction was not fully understood.

III-4

I now return to requirement 2, on the acceptability and the simplicity of the constructing curves. In their discussions of this requirement most later writers based themselves on what Descartes had written in the Géométrie. According to Descartes, curves were acceptable in geomety if they could be traced by certain continuous motions. In particular he claimed that, if two acceptable curves are made to move with respect to each other, their motions being connected by certain linkage mechanisms, then their intersections trace new curves that are also acceptable in geometry. For instance, the "Cartesian parabola", used for the construction of fifth- and sixth-degree equations, is traced by the combined motion of a straight line and a parabola as follows[21] (see figure 6):

figure 6



The parabola AVB moves in vertical direction along its axis. The straight line PL turns around the fixed point P and is linked to the parabola in such a way that its intersection L with the axis keeps constant distance from the vertex V. The intersections I then trace the "Cartesian parabola" PIK,HIJ. As the parabola itself is acceptable in geometry, the "Cartesian parabola" is also acceptable.

Which curves are traceable in this way and thereby acceptable in geometry? In the Géométrie Descartes came to the conclusion that all algebraic curves can be so traced. That conclusion is by no means evident. I have discussed Descartes' arguments on this issue, and their relation to his overall programme for geometry, in my 1981. For my present purpose it suffices to state that the readers of the Géométrie could conclude that all algebraic curves were acceptable in geometry as constructing curves, and that this acceptability had somehow to do with the possibility of tracing these curves by a continuous motion. Most mathematicians followed Descartes in accepting all algebraic curves as geometrical and did not think much about the reason for this. The first part of requirement 2 was therefore considered satisfied if F and G were algebraic, i.e. if $F(x,y)$ and $G(x,y)$ were polynomials in x and y. The result of this was that, algebraically, the construction of equations became a theory about polynomial equations.

On the second part of requirement 2 Descartes had been more explicit: a constructing curve is as simple as possible if it has lowest possible degree[22]. As I shall show (cf V-8), this interpretation was not generally accepted, but it did guide most of the studies on the construction of equations.

To restrict the constructing curves for an equation to algebraic ones of lowest possible degree still leaves freedom for the choice of the curves. Descartes himself had made definite choices: he constructed third- and fourth-degree equations by cicle and parabola (any pair of conics would have satisfied the requirements), and fifth- and sixth-degree equations by circle and Cartesian parabola. He had written that "it is only necessary to follow the same general method to construct all problems, more and more complex, ad infinitum"[22a] (cf. II-5). It seems likely that Descartes had the following general method in mind: Equations of degree 2n-1 and 2n belong to the same class; they should be constructed by the intersection of a curve F_n of degree n and a circle. F_2 is the parabola, F_3 is the Cartesian parabola. F_n is generated from F_{n-1} by a motion analogous to the motion of F_2 which generates F_3.

This interpretation of Descartes' programme can be found for instance with Kinckhuysen, de la Hire and Jakob Bernoulli[23]. It seems that none of the mathematicians involved actually tried to work out this programme. The reason of this was probably that one felt that the requirement of lowest degree was better satisfied it F and G were of approximately the same degee, than if, as Descartes suggested F was of degree n and G (the circle) of degree 2. Also Descartes had not explained why he choose the Cartesian parabola, with its particular generation from the parabola, as F_3, so that choice remained unconvincing because of its arbitrariness.

In his Dissertatio Fermat suggested that to construct equations of degree 2n-1 Descartes needed a curve of degree 2n-1, and that in the general case he could not do better[24]. This interpretation is clearly wrong

(Descartes' construction of fifth- and sixth degree equations contradicts it).

These, then, were Descartes' opinions on the requirements for constructing curves as they were formulated in the Géométrie and interpreted by later mathematicians. I shall return to the further debates on these requirements in III-8 and in V.

III-5

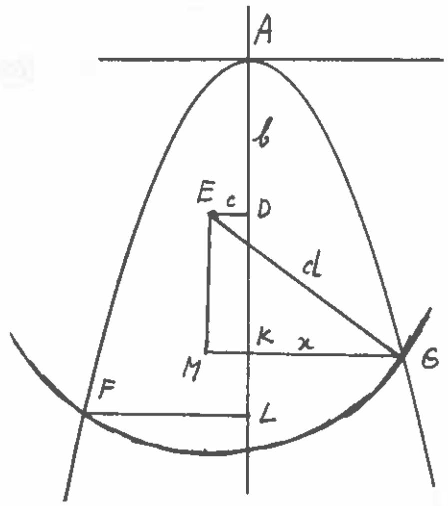
Descartes did not explain in his Géométrie how he had found the constructions which he presented for equations of degree up to six. He did give proofs that they were correct, but from these proofs the method of finding the constructions was not obvious. Later writers developed and published techniques to find, for a given equation H(x) = 0, the equations F(x,y) = 0 and G(x,y) = 0 of the constructing curves F and G. It will be useful to explain these techniques separately.

The techniques can be distinguished into three types, which I shall call "undetermined coefficients techniques", "insertion techniques" and "geometrical techniques" respectively.

The undetermined coefficients technique starts with choosing the two constructing curves F and G, while leaving the parameters of these curves undetermined. Then, either algebraically or by geometrical arguments with respect to a figure, one derives the equation for the x coordinates of the points of intersection of these curves. The coefficients in this equation depend on the parameters. These parameters are then adjusted such that the coefficients coincide with the coefficients in the proposed equation H(x) = 0. The values of the parameters thus found determine the constructing curves F and G.

As an example of this technique I paraphrase the note which van Schooten, added to Descartes' construction of third- and fourth-degree equations by parabola and circle in the 1659 Latin edition of the Géométrie. Explaining how such constructions can be found, van Schooten argued as follows[25]:
Let a fourth degree equation be given which we want to construct by a circle and a parabola (see figure 7). The parameters involved are: the latus rectum a of the parabola, the coordinates AD = b and BE = c of the centre E of the circle, and the radius d of the circle. Let x = GK be the ordinate of a

figure 7



point of intersection G; then $AK = x^2/a$. We have

$$d^2 = EG^2 = EM^2 + MG^2 = (x^2/a - b)^2 + (x + c)^2.$$

This gives the following equation for x:

$$x^4 + (a^2 - 2ab)x^2 + 2a^2cx + a^2(b^2 + c^2 - d^2) = 0.$$

Hence to construct a fourth-degree equation in this way we must first remove its second term (which can be done) and write the equation as

$$x^4 - apx^2 + a^2qx - a^3r = 0,$$

(a is used as unit). We then adjust the coefficients:

$$b = \tfrac{1}{2}(a + p),\ c = \tfrac{1}{2}q,\ d = \sqrt{(\tfrac{1}{2}a + \tfrac{1}{2}p)^2 + (\tfrac{1}{2}q)^2 + ar}.$$

These values determine the constructing curves (the parabola and the circle); in particular they show how the centre and the radius of the circle can be constructed. Van Schooten stated (correctly) that Descartes' rules for constructing third- and fourth-degree equations conform to these formulas.

III-6

The second type of technique, that by insertion, can be described as follows: Given $H(x) = 0$, one chooses for F a curve whose equation is of the form

$$f_1(x) = f_2(x,y)$$

(the most common approach is to choose $x^k = y$). Then one "inserts" this equation in $H(x) = 0$ by locating terms or factors $f_1(x)$ in $H(x)$ and replacing these by $f_2(x,y)$. The result is an equation

$$g(x,y) = 0$$

of a curve G. F and G are constructing curves for $H(x) = 0$. In case F or G are not satisfactory, one may perform the insertion in a different way to arrive at new curves G_i and one may form combinations

$$G(x,y) = \alpha F(x,y) + \Sigma\beta_i G_i(x,y) = 0.$$

Then one chooses any two of the curves thus found which have the required form.

This technique was first described by de la Hire in his 1679. One example he gave there[26] is as follows: Let the proposed equation be

$$x^6 + a^2x^4 - a^2bx^3 - a^2bcx^2 - a^2bcd^2 = 0.$$

De la Hire takes for F the equation

$$ay = x^2,$$

and inserts in two ways, finding equations $G_1 = 0$ and $G_2 = 0$:

$$y^3 + ay^2 - \frac{b}{a}x^3 - \frac{bc}{a}x^2 - \frac{bc}{a}d^2 = 0$$

$$y^3 + ay^2 - bxy - \frac{bc}{a}x^2 - \frac{bc}{a}d^2 = 0,$$

these are curves of the "second kind" (i.e. third degree curves). De la Hire notes that by subtracting $G_2 = 0$ from $G_1 = 0$ one gets an equation $G_3 = 0$, namely

$$ay - x^2 = 0,$$

which is in fact the original F, the equation of a parabola. The proposed sixth-degree equation can now be constructed by any two of the curves G_i; if one wants to have lowest possible degree one should use G_3 and one of the others.

III-7

In the third, geometrical, technique for constructing equations the equation is interpreted as a geometrical problem. For a given equation in x, the second unknown y is introduced in such a way that the equation becomes equivalent to two or three simultaineously valid proportionalities between linesegments that are linear in x and y. Each proportionality then defines a conic section and these can serve as constructing curves. The method was devised by Sluse who used it in his Mesolabum (1659) and explained it in the second edition (1668) of that book. He saw the method as a generalization of determining mean proportionals (Mesolabum was the term used in antiquity to designate instruments for constructing mean proportionals.) In the case of two mean proportionals x and y between a and b[27], we have (cf II-1)

$$a : x = x : y = y : b.$$

The three proportionalities each define a conic section:

The three porportionalities each define a conic section:

$a : x = x : y$	$ay = x^2$	a parabola
$a : x = y : b$	$xy = ab$	a hyperbola
$x : y = y : b$	$y^2 = xb$	a parabola,

any two of these can be used as constructing curves.

Sluse was able to work out similar methods for all third- and fourth-degree equations. To illustrate them I give his construction[28)] of the third degree equation

$$x^3 = px + q.$$

He rewrites the equation as a proportionality

$$\sqrt{p} : x^2 = x : (x + \frac{p}{q}).$$

Introducing y as the mean proportional between x and $x + \frac{q}{p}$, we can write

$$p : x = x : y = y : (x + \frac{q}{p}),$$

which gives three conic sections

$x^2 = \sqrt{p}y$	a parabola
$y^2 = x(x + \frac{q}{p})$	a hyperbola
$xy = \sqrt{p}(x + \frac{q}{p})$	a hyperbola,

any two of which can be used as constructing curves for the equation $x^3 = px + q$.

The method has the drawback that it cannot be extended so as to serve for equations of degree greater than four.

III-8

The techniques mentioned above were used to find the best constructing curves for a given equation. This meant, for most of the mathematicians involved, the constructing curves with lowest possible degree. This led to a general question, namely, given an equation, what are the lowest possible degrees for its constructing curves? In the course of the development of the subject there grew a consensus among mathematicians about this question. For the sake of clarity I shall state the essence of that consensus in the form of a theorem, and refer to it as the "main result". It is as follows:

<u>"Main result"</u>

Let $H(x) = 0$ be an equation of degree n. Let k be the smallest integer such that $(k - 1)^2 < n \leqslant k^2$. Distinguish two cases: $(k - 1)^2 < n \leqslant k(k - 1)$ (case a), and $k(k - 1) < n \leqslant k^2$ (case b). It is possible to construct the equation $H(x) = 0$ with curves F and G, of degrees k and k - 1 (in case a) or k and k (in case b). Moreover, as far as the degrees of F and G are concerned, this is the best possible choice.

The degrees in the best possible constructions, according to this "main result" can therefore be tabulated thus:

degree of $H(x) = 0$	degree of F	degree of G
2	2	1
3, 4	2	2
5, 6	3	2
7, 8, 9	3	3
10, 11, 12	4	3
13,...,16	4	4
17,...,20	5	4

etc.

Thus in the best construction of an equation of degree n, the degrees of the constructing curves are integer approximations of $\sqrt{n}$.

The "main result" occurs explicitly in print for the first time in 1707 in l'Hôpital's Traité[29]. Newton had formulated it as early as 1665 in a manuscript[30] (to be discussed in more detail in V-4), which, however, remained unpublished at the time.

The acceptance of the "main result" was based on a mixture of argument, conviction and experience which was usually not stated explicitly. For my later discussion of the development of the theory of construction of equations it will be useful to make these arguments, convictions and experience explicit here. I shall separate them in different steps and comment upon them.

Step 1

If $F(x,y)$ and $G(x,y)$ are polynomials of degree p and q respectively, the equation $R_{F,G}(x) = 0$ resulting from the elimination of y from $F = 0$ and $G = 0$, has degree pq.

This statement has later become known as the "theorem of Bezout", after E. Bezout who, in his 1779 gave the first reasonably satisfactory proof of it[31]. However, it was already known in the 17th century as a matter of experience[32]. Apparently it was considered so evident that only in the 18th century were the first attempts made to prove the theorem, for instance by MacLaurin, Euler and Cramer[33]. Earlier writers on the construction of equations all considered step 1 to be obvious.

Step 2

If $H(x) = 0$ has degree n it cannot be constructed by curves the product of whose degrees is smaller than n.

If one disregards (as most writers on the construction of equations did, cf III-3) the possibility that $H(x) = 0$ is reducible and thereby constructible by lower-degree curves, step 2 is an obvious consequence of step 1.

Step 3

If the degrees p and q of F and G have to be chosen lowest but such that $pq \geqslant n$, and if k is as in the "main result", then the best choice is, in case a: $p = k$, $q = k - 1$; and in case b: $p = k$, $q = k$.

Note that here a further interpretation of the requirement of lowest degree is introduced, namely that the degrees of F and G should be approximately equal. This interpretation underlies the studies of Fermat and de la Hire (cf IV-2). Although some mathematicians questioned this interpretation, it was mostly accepted.

Step 4

If the degrees p and q are chosen according to step 3, F and G must be chosen such that $R_{F,G} = x^{(pq - n)}H(x)$.

That is, if $pq > n$ the factor A(x) (cf III-2) is chosen to b a power of x. This technique of increasing the degree of H(x) by introducing new roots 0 was applied already by Descartes (cf II-4 and III-2); before construction, a third degree equation is transformed into a fourth -degree one, and a fifth-degree equation into a sixth-degree one. In fact, this technique is behind Descartes' classification of equations into classes each consisting of equations of degree $2n - 1$ and $2n$.

Step 5

With H, n, k, p and q as in step 3, it is actually possible to find constructing curves F and G, of degree p and q. That is, it is possible to find polynomials F(x,y) and G(x,y) of degree p and q such that their resultant $R_{F,G}$ is equal to $x^{(pq - n)}H(x)$.

Disregarding the special choice of p and q, this statement is a kind of inverse of Bezout's theorem (step 1). It says:

For every polynomial H(x) of degree pq there exist polynomials F(x,y) and G(x,y) of degrees p and q respectively such that $R_{F,G} = H$.

The conviction that this is true arose in the seventeenth century as a matter of experience; if the degree of H is not too high it is indeed very easy to find such F and G. l'Hôpital gave an attempt to prove the statement[35]. The idea of that proof is a dimension argument. He formulated it in the special case of the construction of a 20th-degree equation by curves of degrees four and five, but he certainly meant it more general. His argument can be summarised as follows:

In a polynomial F(x,y) of degree p there are $\frac{(p + 1)(p + 2)}{2}$ coefficients. Hence chosing constructing curves F and G of degree p and q respectively involves the free choice of $\frac{1}{2}\{(p + 1)(p + 2) + (q + 1)(q + 2)\}$ coefficients. l'Hôpital claimed that "by the rules of algebra" it is certain that the resultant of F and G is of degree p.q, and therefore involves pq + 1 coefficients. Now $pq + 1 < \frac{1}{2}\{(p + 1)(p + 2) + (q + 1)(q + 2)\}$, so if we want to adjust the coefficients in F and G such that the resultant becomes equal to a given equation H = 0, we have more coefficients free to choose than coefficients to adjust, hence, l'Hôpital stated without further argument, this will be possible.

The proof is remarkable for several reasons. First of all it is a general, non-constructive existence proof. This is a type of argument which at the turn of the seventeenth century was seldom made explicitly. Then it is notable that l'Hôpital took for granted both Bezout's theorem (step 1) and the solvability of a system of equations in which the number of unknowns is larger than the number of equation. Both apparently were facts of experience which, for the algebrists of the time were self-evident.

This proof was considered satisfactory by later writers on the subject (Euler, Cramer cf. V-11). It is, in fact, insufficient, because it is not at all clear that in this case the system of equations for determining the coefficients is actually solvable. In fact, the inverse of Bezout's theorem is a by no means trivial statement. I do not know if it is true or not; it seems that the question up to now has not been studied[36].

IV A survey of the sources

IV-1

Having summarised the main mathematical themes concerning the construction of equations in the previous chapter, I shall now give a rapid survey of the relevant publications from the 1640's to the 1750's. The survey is meant as a chronological sketch of the development of the subject, listing the writers and works involved, and the main ideas that were put forth. The sketch should serve as background information for my discussion in Chapter V of the arguments on motivation and method concerning the construction of equations.

I shall not give a complete list of works dealing with the construction of equations; in particular I have left out works that only treat linear and quadratic equations. The list includes at least the sources that were considered important at the time, as far as can be judged from contemporary references to them.

IV-2

Twelve years after the publication of the Géométrie, van Schooten brought out a Latin translation of Descartes' text with annotations. This was the Geometria (Descartes 1649). As to the construction of equations this edition did not offer more than the Géométrie itself had done; van Schooten stressed the importance and the novelty of Descartes' constructions, but he did not comment upon them, in particular he did not explain how they were found.

In 1657 Wallis published a treatise on proportions[37)] in which he also discussed the cubical parabola, i.e. the curve with equation $y = x^3$. He explained how third-degree equations can be constructed by the intersection of this curve and a straight line. (The idea is to construct $x^3 + px + q = 0$ by intersecting $y = x^3$ with $y + px + q = 0$.) Wallis discussed the merits of this construction as compared with construction by conic sections and he hinted that the approach could be extended to higher-order equations.

Two years later appeared the first work to devote considerable space to the construction of equations up to degree four by all sorts of combinations of conic sections. This was Sluse's Mesolabum (1659) in which many such constructions were geometrically presented and proved, but no explanation was offered of how they were found. Sluse promised to explain this later, and he did so in an appendix to the second edition of Mesolabum (1668).

In the same year 1659 the first volume of van Schooten's much enlarged edition of the Geometria appeared (Descartes 1659). Van Schooten added new edition of the Geometria appeared (Descartes 1659). Van Schooten added new notes, and the two-volume edition (the second volume appeared in 1661) contained supplementary treatises by van Schooten himself and other mathematicians. Volume 1 constained the first explicit analysis of a general method to construct equations up to degree four, namely in van Schooten's note in which he deduced Descartes' circle-and-parabola construction by an undetermined coefficient method; I have explained that method in III-5. Moreover, van Schooten added several variant constructions of equations, for instance one (due to Hudde) by a hyperbola and a circle[38], and a construction for third-degree equations by parabola and circle for which it was not necessary first to remove the quadratic term[39].

The 1659/1661 Geometria edition was in fact a survey of the most advanced and recent results in Cartesian geometry, together with introductory treatises. On a more elementary level Kinckhuysen published in the years 1660-1663 a series of textbooks in Dutch which formed an introduction to Cartesian geometry. They were Grondt der Meetkonst (Fundament of Geometry) (1660), Algebra (1661) and Geometria (1663). In the two geometrical works Kinckhuysen discussed construction of equations. In Grondt der Meetkonst the undetermined coefficient method is used to find constructions by parabola and circle, cubical parabola and circle, Cartesian parabola and circle and even higher order curves and circles[40]. Kinckhuysen here claims explicitly that Descartes had found his constructions by means of undetermined coefficients. In his Geometria Kinckhuysen treated some geometrical problems that lead to third degree equations which he constructed in Descartes' manner by parabola and circle or by another conic and a circle[41].

In 1679 Samuel de Fermat published the Varia Opera Mathematica of his father Pierre de Fermat, who had died in 1665. One of the studies contained in that volume is of special interest for the construction of equations, namely the Dissertatio. This short work (it takes six pages in the Varia Opera) had been circulating in manuscript before. Its date of writing is not known precisely; recent studies have put its origin in the years 1641-43[42]. In the Dissertatio Fermat gave a general method to find, for an equation of degree 2n, constructing curves of degree n. The method used the technique of undetermined coefficients. Fermat saw his results as an improvement on Descartes', who, he thought (wrongly, as I have noted in section III-4), would always need a curve of degree 2n - 2 for constructing an equation of degree 2n - 1 or 2n. Fermat went on to deal with special equations which he constructed with curves of even lower degree. In particular he considered equations

$$x^{n+1} = a^n b$$

(the root x is the first of n mean proportionals between a and b) and found, for certain values of n, constructing curves of degree approximately equal to n. In a sequel to this argument Fermat introduced the (later so-called) Fermat numbers $2^{2^k} + 1$, which he claimed to be prime (wrongly, as Euler was to find later), and considered equations

$$2^{2^{2^k}+1} = a^{2^{2^k}} b.$$

He showed that these equations can be constructed by curves of degree $2^{2^{k-1}} + 1$, and he claimed that they are irreducible because their degree is prime (cf III-3). He concluded that in this way

> "We can construct a problem whose degree has to the degree of the curves that serve its solution a ratio greater than any given ratio" (Dissertatio p. 131)

a result by which Fermat meant to show how feeble were Descartes' results on the construction of equations.

In the Dissertatio we find a number of ideas which later were incorporated in what I have called the "main result" (cf III-8): the degrees of the constructing curves should be approximately equal; the best result is to have these degrees approximately equal to the square root of the degree of the equation. Fermat showed little interest in the geometrical motivation of the subject; he accepted without question that the degree is the measure of the simplicity of a curve and presented his further arguments as purely algebraic manipulations.

In the year in which Fermat's Dissertatio appeared in print, de la Hire published his Nouveaux Elemens (1679). The book contained a substantial section called La construction des equations analytiques[43], which can be considered as the first textbook treatment of the construction of equations of arbitrary degree. De la Hire had earlier received, through Huygens, a copy of Fermat's Dissertatio, to which he referred. De la Hire used the method of insertion which he explained at considerable length. He derived by that method the constructions of equations of degree up to four with conic sections; he worked out the case in which one of the conic sections is prescribed, and he stressed that one of the constructing curves should preferably be a circle.

For equations of arbitrary degree de la Hire explicitly formulated the programme to find constructing curves of lowest possible degree. With the insertion method he found better results than Fermat (whose general method he mentioned), but the degrees in his construction were not yet as low as in the "main result". For the equations $x^{n+1} = a^n b$, which Fermat had constructed for special n, de la Hire presented a general construction in which the degrees do conform to the requirements of the "main result".

Rules for constructing third- and fourth-degree equations by a parabola and a circle were called "Baker's rules" in the eighteenth century. This was because Thomas Baker had published such rules in his The geometrical key or the gate to equations unlock'd (Baker 1684). These rules did not require that the second term of the equations should first be removed. Baker considered this a great advantage over Descartes' construction. Moreover, he treated separately all the different cases that arise as to whether the coefficients in the equations are positive, negative or zero. In each of these cases he spelt out the construction rule explicitly.

Baker's Geometrical key is a rather grotesque piece of mathematical writing. It is both in English and in Latin. Baker wrote with a strange mixture of modesty about his mathematical abilities and exaltation about his results. He was an amateur mathematician and in distinguishing all the different cases according to the signs of the coefficients he was decidly oldfashioned. Foreseeing critique on the prolixity of his treatment he wrote that his book was meant for beginners, and

> "Are not Homer's Iliads written in capital letters and enlarged into a Folio, better legible (and therefore the more intelligible) and John Tredescant's common silver house-spoons more useful, than when the one are crammed into a Nut-shell, and the other into a cherry-stone." (Baker 1684, preface)

The passage is a fair specimen of his style. Nevertheless the book acquired a prominent place in mathematical literature. Wallis referred to Baker's work already in his Algebra of 1685. Sturm and Harris explained and praised Baker's rules, Halley and an anonymous published proofs of the rules, Hermann criticised them, Wolf and Zedler mentioned them, and as late as 1748 Euler still wrote of the "well-known rule of Baker".[44)]

IV-3

After Sluse's Mesolabum, de la Hire's textbook-version of the construction of equations, and Baker's rules, the construction of equations of degree up to four became a standard topic in expository writings on algebra and analytic geometry. Sections on the subject can be found in the following books: Wallis' Algebra (1685 1693), Sturm's Mathesis enucleata (1689, tr. engl. 1700), Ozanam's Dictionaire Mathématique (1691), and

Nouveaux elemens d'algèbre (1702), Harris' Algebra (1702) and Lexicon (1704), Guisnée's Application de l'algèbre à la géométrie (1705), l'Hôpital's Traité (1707), Newton's Arithmetica universalis (1707) and Reyneau's Analyse dimontrée (1708) - to name only works from the two decades around 1700, and the list in by no means exhaustive[44a].

In general, these writings did not bring much new to the technical mathematical side of the subject. By 1710, the theory of construction of equations up to degree 4 had more or less reached its definitive form. It offered algebraic techniques, namely insertion and undetermined coefficients (geometrical techniques were mentioned less often), to find constructing curves for any second-, third- or fourth-degree equation. There was some preference for construction by a parabola and a circle, but most writers discussed variant constructions by any combination of conic sections. There were a number of side-issues, as for instance construction with one prescribed conic section, construction by means of the cubical parabola or by means of the conchoid. I shall return to the motivation of these side-issues in section V. Although the techniques to find the constructing curves were entirely algebraical, the subject kept something of its geometrical setting, witnessed by an interest for the position of the curves in the plane and the preference for circle and parabola constructions. It is noteworthy that the idea of using the graph of the equation (cf V-10) did not occur in these writings on the construction of lower-degree equations.

IV-4

Most writers mentioned in the previous section did not discuss the construction of equations of degree higher than four. This topic came to constitute a separate part of the construction of equations. As we have seen, Fermat and de la Hire had attacked the problem of constructing equations of arbitrary degree. l'Hôpital took up the subject in his Traité analytique des sections coniques (1707); his treatment was followed by most later writers.

In the long (70 pages) ninth book of the Traité, entitled De la construction des egalitez[45], l'Hôpital first gave a by that time customary account of the construction of lower-degree equations, including Descartes' con- construction of fifth and sixth-degree ones. For higher-degree equations he first explained the insertion method, and derived the constructions, as de la Hire had done earlier, by taking $y = x^{\kappa}$ as first constructing curve. He showed that the constructing curves thus found have a too high degree. For instance for a sixteenth-degree equation de la Hire's method gives curves of degree four and five, whereas one would think that two fourth-degree curves could be found. l'Hôpital then stated the "main result" and gave the proof by a dimension argument which I have skteched in section III-8. l'Hôpital also mentioned the possibility of constructing an equation H(x) = a by intersecting the graph y = H(x) with the horizontal line y = a (I shall return in V-10 to this use of the graph in constructions of equations).

Books on algebra and analytic geometry that appeared after l'Hôpital's Traité occasionally mentioned the construction of higher-order equations. Wolff devoted some space to it in his Elementa matheseos (1743) for instance. There is also an interesting example of a university-based study of the construction of equations. This is a dissertation written by J. Kraft while studying at Kopenhagen university, and published in 1742 (Kraft 1742). It concerned mainly lower-degree equations, but it did containt a statement of the "main result", though without proof. The book brought nothing new and it treated the subject without elegance. But, as a student's work, it did show competence in handling an advanced mathematical subject.

In the 1740's, Euler still considered the subject important enough to devote a full chapter of his Introductio in analysin infinitorum (1748) to the construction of equations[46]. He gave the subject a new place with respect to other topics of algebra and geometry, namely as an application of the theory of intersections of algebraic curves. He had treated this theory in the previous chapter, which also contained methods of elimination. Thus Euler formulated the main problem of the construction of equations explicitly as an inverse elimination problem. He treated the construction of lower-degree equations by an undetermined-coefficient method; for higher-degree equations he explained insertion, and gave, without proof, the "main result".

The last major algebra book in which a substantial section was devoted to the construction of equations was Cramer's Introduction a l'analyse des lignes courbes algebriques (1750). Like Euler, Cramer dealt with the subject[47] after having treated elimination theory. He explained the insertion method for finding constructing curves; he quoted l'Hôpital's formulation of the "main result" but he did not offer a proof. The last part of his chapter on the construction of equations was devoted to the graphs of polynomials H(x) and their use in estimating the positions of the roots of H(x) = 0 on the X-axis. Cramer also added comments on the various procedures for constructing equations. The gist of the comments (to which I shall come back in V-8) was that mathematicians had different opinions on what the best constructions are, but that the whole enterprise, apart from studying the graphs of polynomials, had little use.

IV-5

Cramer's treatment of the construction of higher-degree equations epitomizes the final state of the subject. The construction of equations was recognized as an inverse elimination problem, thereby it had become a purely algebraic technique. As method insertion was used. There was the conviction that construction according to the requirements of the "main result" was possible, but the requirement that the constructing curves should be of lowest possible degree was questioned. Indeed the motivation of the whole subject was in doubt.

After 1750 the construction of equations quickly fell into oblivion. Apart from reprints of earlier works, no books appeared that devoted considerable space to it. For some time the subject was still mentioned in lexica and encyclopedia[48)], but even there the suggestion was given that it was no longer of contemporary interest.

V The arguments on motivation and method

V-1

I now come to the arguments on motivation and method that were put forth by mathematicians with respect to the construction of equations. From these arguments will appear the causes for the flourishing of the subject and for its later decline and death.

For Descartes, the construction of equations was the necessary final step in his programme for dealing with construction problems; it was the general method for finding the constructing curves. Its importance was thereby evident. In later textbooks on algebra and analytic geometry we often find Descartes' programme formulated explicitly[49)]: give names to the known and unknown quantities, derive equations, eliminate to get one equation in one unknown, construct that equation by the intersection of curves. Some mathematicians even considered these constructions as the main raison d'être of curves. Newton, for instance, wrote in his treatise on the classification of third-degree curves: "the use of curves in geometry is that by their intersection problems can be solved"; that opinion was expressed more often in the contemporary literature[50)].

Nevertheless, later presentations of the construction of equations showed a decreasing attention for the original geometrical motivation of the subject. For instance: Descartes had stressed the necessity first to check whether the equation to be constructed was irreducible, because otherwise one would not find the simplest possible geometrical construction. Later writers usually did not mention this requirement[51)]. Descartes had given explicit attention to the way the constructing curves were actually traced. Most later writers were satisfied when the equations of the constructing curves were known and did not consider how the curves could be traced in actually performing the construction. That is, they considered the constructing curves merely as loci, to be defined by an equation[52)].

This conception of the constructing curves as loci could be misleading, as is shown by a curious argument of de la Hire. In an article (de la Hire 1712) on the construction of loci and of equations he claimed that quadratic equations can be constructed by straight lines only, without using a circle. Hir arguments was as follows: A quadratic equation can be considered as a special fourth-degree equation, it can therefore be constructed by the intersection of two conics. In this case one has much freedom in choosing the conics and one can choose them as degenerated hyperbolas, that is, pairs of straight lines. Thus the equation

$$ax^2 + bx + c = 0$$

can be rewritten as

$$(x - \frac{b}{2a})^2 = \tfrac{1}{4}(b^2 - 4ac),$$

and as constructing curves one can choose

$$F : (x - \frac{b}{2a})^2 = y^2,$$

that is, a pair of straight lines, and

$$G : y^2 = \tfrac{1}{4}(b^2 - 4ac),$$

another pair of straight lines. Each of these lines is a locus of the first degree, so, if we chose one line out of each pair, we have constructed the quadratic equation by two loci of degree one. De la Hire presented this as a paradoxical result: apparently a compass was not necessary to construct quadratic equations.

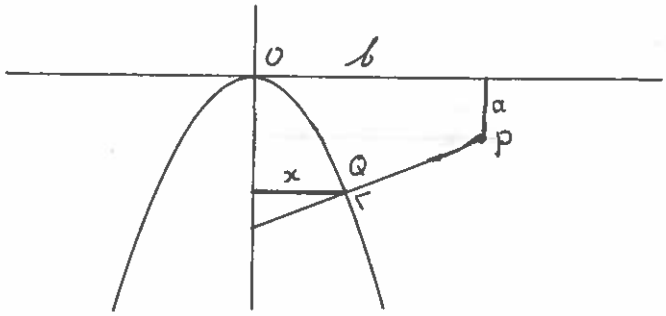
Now de la Hire knew that the compass is necessary for actually drawing these straight lines in the plane: one has to draw perpendicular axes, one has to mark off lengths along them and one has to determine the length $\frac{1}{2}\sqrt{b^2 - 4ac}$; all these operations require a compass. De la Hire admitted this, but did not accept it as objection to his argument. For him this use of the compass somehow did not belong to the construction of the equation proper. He wrote "one does not use it (i.e. the circle) as a locus in the construction"[52a] and, in his view, Descartes' method only required that the loci are of lowest possible degree.

The argument shows that de la Hire did not fully understand Descartes' original geometrical motivation of the construction of equations, and that he was prepared to consider it as a purely formal manipulation of formulas.

V-2

Those mathematicians who did understand Descartes' programme encountered other difficulties. One of these was that the conviction of a strict correspondence between algebra and geometry, which underlay Descartes' approach, proved doubtful. One example of how mathematicians came to question this correspondence occurred in van Schooten's commentaries in his latin editions (Descartes <u>1649</u>, <u>1659</u>) of the <u>Géométrie</u>. It concerned Descartes' statement that if a problem leads to an irreducible third-degree equation, then that problem is "solid", that is, it cannot be constructed by ruler and compass. Van Schooten saw a difficulty here which he illustrated by means of an example provided by Christiaan Huygens[53]. The example concerned the construction of a normal to a parabola through a given point outside the parabola (see figure 8):

figure 8



Let a parabola be given with vertical axis and top in O. Let P be a given point outside the parabola with coorinates a and b as indicated in the figure. It is required to find the line through P which cuts the parabola perpendicularly. Let PQ be this line, with Q on the parabola. Let Q have coordinates x and y and let the unit be chosen such that the parabola has equation $y = x^2$. Huygens calculated, using the known tangent-properties of the parabola, that x must satisfy the third-degree equation

$$x^3 + (\tfrac{1}{2} - a)x - \frac{b}{2} = 0$$

This equation is irreducible, and therefore according to Descartes' recipe for the construction of third- or fourth- degree equations (in fact this is precisely the case discussed in II-4) one needed in the construction a circle and the parabola $y = x^2$. But in this case that parabola was already given, so one actually only needed a circle, and therefore the problem might with equal right be called "plane". Van Schooten asserted that indeed every construction problem in which a conic section is given and which leads to a third- or fourth-degree equation, can be constructed by using the given conic section and a circle, and consequently could be called "plane". (The assertion is true, it can be proved by algebraic manipulations.) Van Schooten ultimately decided to keep to Descartes' classification on the basis of the equation, and decided to call these problems "solid"; Huygens, however, kept his doubts.

Indeed doubts were justified because, contrary to the belief underlying Descartes' programme, here the geometrical classification of problems as to their constructibility does not coincide with the classification of the corresponding equations as to their degree. Geometrically, the problem is plane, algebraically it is solid. The reason of this discrepancy is that essential information can be lost when a construction problem is translated, in the manner prescribed by Descartes, into an equation. In the case of the perpendicular to the parabola, the lost information is the fact that the parabola is given.

Van Schooten's and Huygens' arguments did not give rise to much discussion among mathematicians. However, throughout the seventeenth and early eighteenth centuries one finds echoes of these arguments in the treatment of construction problems in which certain curves are given. Several mathematicians explicitly tried to use the given curve as constructing curve, aiming in that way to choose the other constructing curve as simple as possible. Thus van Heuraet[54)] and Newton[55)] put much algebraical effort into finding a construction of the points of inflexion of a conchoid, using only a circle and the conchoid itself. And, in another field of research, l'Hôpital[56)] found a construction of the arc-length of segments of the logarithmic curve by using only circles, and the logarithmic curve itself; and his construction was praised precisely because of this constructional simplicity and economy. In both cases the calculations can only be understood in the context of geometrical constructions; if one only sees the algebraical side of the problem, the efforts do not make sense.

V-3

Jakob Bernoulli also expressed doubts about the direct connection which Descartes had supposed between geometrical construction and algebraic calculation. He criticised Descartes' procedures in the notes which he wrote for the 1695-edition of the latin text of the Géométrie (Bernoulli 1695). He formulated his critique with respect to a special construction problem, namely: given a triangle, to find two perpendicular straight lines which divide the triangle in four equal parts.

Bernoulli's critique[57)] can be understood without going into the details of the mathematics. He introduced two unknowns x and y and derived two equations which x and y had to satisfy: I shall denote these equations as

$$P(x,y) = 0 \quad \text{and} \quad Q(x,y) = 0$$

respectively. At this stage Descartes' method prescribed that one of the unknowns, say y, should be eliminated from these equations, and that the resulting equation

$H(x) = 0$

should be constructed by finding, in the manner also prescribed be Descartes, two curves F and G whose intersections have x-coordinates equal to the roots of $H(x) = 0$. Jakob Bernoulli found this procedure annoyingly cumbersome and most unnatural. Why introduce new curves which have nothing to do with the original problem? Why not skip the whole procedure of deriving $H(x) = 0$, and use the curves P and Q, corresponding to the equations $P(x,y) = 0$ and $Q(x,y) = 0$? Their intersections yield x as well as y immediately, they have a natural interpretation in connection with the problem and they serve better for getting insight in the different cases of existence or non-existence of roots that may occur. Such constructions, Bernoulli wrote

> "present to our eyes the whole nature of the problem in a much better way than those which, according to the method of the author (i.e. Descartes) should be chosen, on the basis of a third equation (i.e. $H(x) = 0$) with long detours and often insuperable work, and which therefore have to be considered rather as forced and unnatural". (Bernoulli 1695 p. 671).

Of course it may happen - as indeed it happens in Bernoulli's example - that the curves P and Q have higher degree that F and G, and in that sense are not the simplest possible constructing curves for the problem. Bernoulli implied that that is a price one should be willing to pay for the naturalness of the choice of P and Q.

Here again (as in the case discussed in the previous section) the algebra did not fit the geometry. Descartes' algebraical requirement of lowest possible degree would lead to a construction which, according to Bernoulli was geometrically unacceptable because it had no natural link with the problem it was meant to solve.

V-4

Even more than the mathematicians previously discussed, Newton was aware that the algebraic and the geometrical approach to the solution of problems are not analogous and may even be incompatible. In a series of studies he tried to sort out the fundamental questions about using algebra in geometry. The prime theme of these studies can be summarized thus: contrary to what Descartes had thought, algebra is not the means which should bring order into geometry. Algebra is useful as a tool (indeed Newton used it brilliantly) but in all questions concerning the aim of geometry, its proper methods and the criteria for correctness of geometrical constructions, algebra is a bad guide. Descartes had introduced algebraic criteria into geometry: constructing curves should be algebraical; they are simple in as much as their degree is low. Newton came to criticize both these standpoints and he tried to work out alternatives. His work, thereby, was the most consistent and acute critique of Descartes' programme. I shall discuss it here in some detail.

Newton's arguments are to be found in three manuscript studies (ms 1665, ms 1670 and ms 1705) that have remained unpublished until recently, and in the Arithmetica Universalis (1707) which was published in 1707 but whose text dates from 1683-1684.

The 1665-manuscript, entitled "the theory and construction of equations" is a perspicacious but uncritical reaction on Descartes. The work is remarkable because in it Newton worked out most of the results on the construction of higher-order equations that were later found by de la Hire and l'Hôpital. Newton first dealt with constructions of low-degree equations. He then treated the construction of equations of arbitrary degree, taking as the first constructing curve, F, the cubical parabola $y = x^3$. He conceived of this curve as being cut out as a brass template and thus serving as an intrument. To construct equations $H(x) = 0$ with this F, he removed the second term in $H(x)$, increased its degree, if necessary, to a multiple of three and then used insertion to find G. For degrees 1-3, 4-6, 7-9, 10-12 etc. of $H(x)$ he found G of degree 1, 2, 3, 4 etc. After this he worked through a similar scheme taking $y = x^4$ as first constructing curve. He then boldly generalized what he had found, stating, though without proof, both Bezout's theorem and what in III-8 I have called the "main result". Finally he expressed doubt about the practicality of these constructions for higher-order equations: the curves G found in this way, although they have lowest possible degree, may turn out to be very complicated.

The study shows that by 1665 Newton clearly saw the programmatic aspects of Descartes' Géométrie, and that he had mastered the algebraic techniques and formulated the algebraic insights that were first to appear in print in l'Hôpital's Traité (1707). Although he expressed some doubt about the requirement that the constructing curves should have lowest possible degree, he presented no alternative.

V-5

It appears that by 1670 Newton's doubts had become stronger, so much so that he sketched an alternative approach to the construction of geometrical problems and of equations. This is the manuscript "Problems for construing equations" (Newton ms 1670), which is a truly remarkable piece of work because of the seriousness with which Newton pursued the consequences of rejecting Descartes' approach. Euclid had laid down the axioms and postulates for a geometry allowing ruler and compass constructions. Descartes had extended this geometry by allowing in principle all algebraic curves as means of construction. He had done this by postulating that the intersections of moving geometrical curves trace new curves that are also geometrical (cf. III-4). For Newton, this went too far; allowing all algebraic constructions was to him, one might say, giving the geometrical game away. He found himself therefore confronted with the question, which postulates should be added to Euclid's in order to arrive at a geometry in which the constructional possibilities were extended (allowing, for instance, the construction of two mean proportionals and the trisection of the angle) but still suitably restricted.

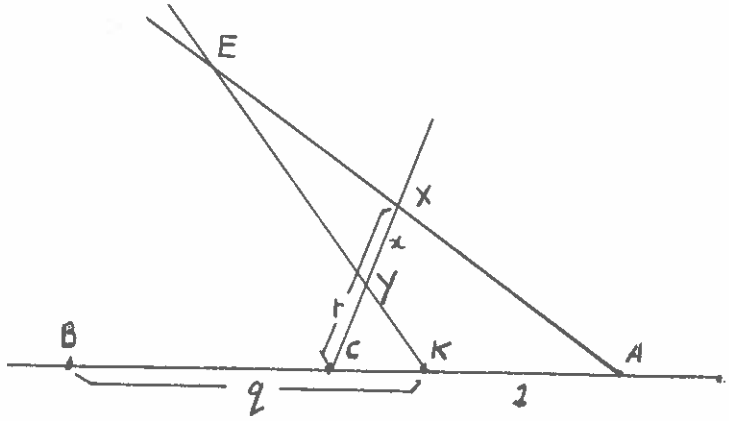
The 1670-manuscript is, in fact, a sketch of such an extended Euclidean geometry. It consists of three parts; the first contains definitions and postulates; the second constructions of problems based on the postulates (in particular a whole series of constructions for finding two mean proportionals); and the third, construction of equations.

It is not possible to treat here Newton's postulates in detail. For my present purpose it is enough to state that they postulate the possibility of certain motions, which in turn are the foundation of two constructions in addition to ruler and compass. There two constructions are the classical neusis construction (cf II-1) and the tracing of ellipses. As the classical neusis construction is equivalent to the tracing of conchoids, one may summarize Newton's approach by saying that he added postulates through which, in addition to the straight line and the circle, the conchoid and the ellipse become acceptable means of construction.

Newton used these means of construction in the solution of a series of problems, among which the trisection of the angle and the finding of two mean proportionals. After that he turned to the construction of equations. I shall illustrate his approach by the neusis construction he gave for the cubic equation

$$x^3 = qx + r.$$

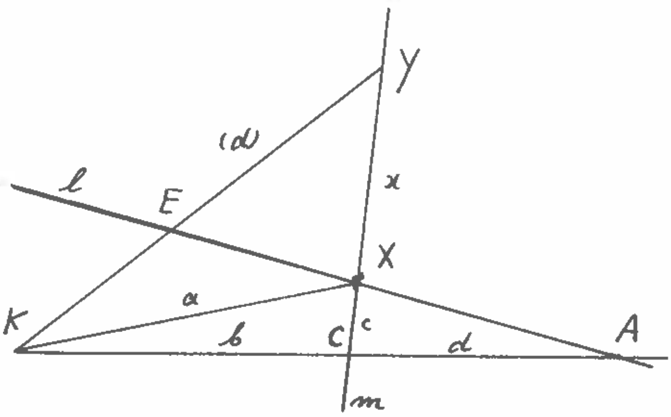
figure 9



It is as follows[58] (see figure 9). Draw a straight line with points BKA (in that order) such that KA = 1 and BK = q. Take C such that BC = CA. Draw a triangle CXA such that AX = AC and CX = r. Prolong AX and CX. (Until now all constructions can be performed by ruler and compass). Now by a neusis construction, insert a line segment EY = AC = $\frac{1}{2}(q + 1)$ between the lines AX and CX and "verging" towards K. Then XY is a root of the equation. -

Here, as with most of the other problems and equations he constructed, Newton added a synthetic geometrical proof. These synthetic proofs do not make clear how the constructions were found. But Newton added near the end of the manuscript[59] some examples of algebraic analysis, and from these it appears that he had found the construction discussed above by an undetermined-coefficients method, as follows (see figure 10):

figure 10



Let ℓ and m be two given lines, intersecting in X, K a point outside ℓ and m, KCA a line through K intersecting ℓ in A and m in C. Call KX = a, KC = b, CX = c, CA = d. Suppose that, by means of a neusis, line KEY is constructed such that EY = d. Call YX = x. Using the theorem of Menelaus and the cosine theorem, Newton now derived an equation for x, which turned out to be of the third degree. Its coefficients involve the undetermined lengths a, b, c and d. To find the neusis construction for, say,

$$x^3 = qx + r,$$

Newton adjusted the values of a, b, c and d such that the coefficients of the two equations coincide. Newton used variants of this approach to find other constructions.

The example well illustrates how strongly Newton's use of algebra in geometry here differs from Descartes'. There are no axes involved, and no equations of curves. The algebraical analysis serves exclusively to find a neusis construction. Descartes' constructions strongly bear the mark of the algebra he used - one needs only to fill in the equations of the curves and to eliminate to have the algebraic translation of the geometrical procedure. In contrast, Newton's neusis construction does not show much trace of algebra; the algebra here is truly subservient to the geometry.

Newton also worked out constructions of third- and fourth-degree equations with the ellipse and stated a preference for these over the neusis constructions because the ellipse is simpler than the conchoid. He preferred the ellipse over other conic sections as well, because tracing the ellipse was one of the motions he had postulated.

In the last part of the manuscrips Newton also dealt with the construction of higher-order equations. As in the 1665 study he used $y = x^3$ as universal first constructing curve. He explained how this curve can be traced by a combination of motions, which should make it acceptable as a constructing curve. Here, however, Newton was no longer able to actually reduce these tracing movements to the ones he had introduced in his additional postulates. Indeed, near the end of the manuscript (which is unfinished) Newton seems to give up the strictly geometrical approach of the first parts.

Thus the 1670 manuscript shows that Newton's wish to work out a truly geometrical approach to the construction of problems and equations, led him to prefer, for third- and fourth-degree equations, constructions that are strongly different from the parabola and circle constructions of Descartes' Géométrie. It also shows that Newton could not pursue that line for higher-order equations in a natural and convincingly geometrical way.

V-6

Newton returned to the subject again when in the years 1683-84 he wrote down his Lucasian lectures on algebra (Newton Luc. Lect.). These lectures were not written for publication, but later Whiston proposed to publish them, and Newton, though reluctantly, assented. They came out under the title Arithmetica universalis (Newton 1707). The work became very popular and was reedited and translated often. The last part of the lectures concerned the construction of equations. This part was printed in the published version as an appendix with the title aequationum constructio linearis, the linear construction of equations[60)].

The aequationum constructio linearis is a curiously unbalanced piece. On te one hand Newton defended it with very strong words (much stronger than in his 1670 study) that the construction of problems and equations should be part of pure geometry in which algebra should not determine the aims. Consequently he presented constructions of equations by neusis, ellipse or conchoid and claimed these to be better than constructions with other conics. On the other hand he did not include, or even refer to, the postulates about acceptable motions in geometry, on which these preferences rested. Moreover, Newton wrote that he gave these constructions as an auxiliary technique for finding the numerical values of the roots of equations. He explained that the geometrical construction of the equation provides approximations of the roots which can be used as starting values for further arithmetical approximation procedures. This was a practical motivation and rather at odds with the strong defence of pure geometry. Indeed why should strongly worded methodological arguments for protecting the purity of synthetic geometry against the influence of algebra be expounded in the context of a mere auxiliary method to find the first digits of an approximation of a root? - that question Newton left to the reader.

Some quotations may illustrate the style and the arguments in the Appendix. Recent mathematicians, Newton wrote, have "wellcomed in geometry all lines that can be expressed by equations" and stipulated that constructions should be performed with curves of lowest possible degree. The degree offers a good classification for studying the curves themselves, but not for their use as constructing curves:

> "Yet it is not its equation but its description which produces a geometrical curve. A circle is a geometrical line not because it is expressible by means of an equation but because its description (as such) is postulated. It is not the simplicity of its equation but the ease of its description which primarily indicates that a line is to

be admitted into the construction of problems. To be sure, the equation to a parabola is simpler than that to a circle, and yet because of its simpler construction the circle is given prior admission. A circle and the conics are, if regard be paid to the dimensions of their equations, of the same order, and yet in the construction of problems a circle is not numbered with these latter curves but, because of its simpler description, is reduced to the lower order of the straight line; as a result it is not impermissible to construct by means of a circle what can be constructed by straight lines, but to construct by means of conics what can be constructed by a circle is to be reckoned a fault". (Luc. lect. p. 425).

Geometrical simplicity, namely the simplicity of tracing should be the criterion, not algebraic simplicity. Newton spelled out the programmatic choice that had to be made:

"Either, then, we are, with the Ancients, to exclude from geometry all lines except the straight line and circle and maybe the conics, or we are to admit them all according to the simplisity of their description". (Luc. lect. p. 427)

Choosing for the latter option, Newton was prepared even to admit non-algebraic curves, such as the easily traceable cycloid, in preference to high-degree algebraic ones.

The final statement on the proper place of algebra with respect to geometry:

"Multiplications, divisions and computations of that sort have recently been introduced into geometry, but the step is ill-considered and contrary to the original intentions of this science: for anyone who examines the constructions of problems by the straight line and circle devised by the first geometers will readily perceive that geometry was contrived as a means of escaping the tediousness of calculation by the ready drawing of lines. Consequently these two sciences ought not to be confused. The Ancients so assiduously distinguished them one from the other that they never introduced arithmetical terms into geometry; while recent people, by confusing both, have lost the simplicity in which all elegance in geometry consists. Accordingly, the arithmetically simpler is indeed that which is determined by simpler equations, while the geometry simpler is that which is gathered by a simpler drawing of lines - and in geometry what is simpler on geometrical grounds ought to be first and foremost. It will not therefore be interpreted as a fault in me if with the prince of mathematicians, Archimedes, and others of the Ancients I should employ a conchoid in the construction of solid problems."

But immediately there follows the disclaimer:

"Nonetheless, if anyone does feel differently, I want him to know that my immediate concern is not for a construction which is geometrical, but for one of any sort whereby I may attain a numerical approximation to the roots of equations." (Luc. lect. p. 429)

V-7

By the time that the Arithmetica universalis was being printed, about 1705, Newton once more considered the geometrical construction of problems and equations. He wrote drafts (Newton ms 1705) for what appears to be a revision of the appendix Aequationum constructio linearis in the Arithmetica universalis. In these drafts he incorporated the postulates from the 1670-manuscript and he left out the argument that the constructions could provide starting values for numerical approximations.

Newton's attempted revision was left unfinished; the drafts were published only recently. They repeat the statements on the purity of geometry. This shows that by 1705 Newton was still convinced, as strongly as he had been when he wrote the Lucasian lectures, of the necessity to keep the geometry of the ancients "pure and uncontaminated" (ms 1705, 211).

Newton's strong words in the Arithmetica universalis made an impression; they were often referred to in eighteenth-century mathematical literature[61). But nobody took up Newton's views and developed them further. In fact, Newton himself, when writing the drafts for a revision, must have experienced that it was very problematical to consistently work out the approach to pure constructional geometry which he so strongly advocated. So we find the same outcome here as in the cases studied in sections V-2 and V-3: considerable and well-founded doubt and critique of Descartes' programme of merging algebra and geometry, but no workable alternative.

V-8

As we have seen, both Jakob Bernoulli and Newton rejected Descartes' requirement that the degrees of the constructing curves should always be lowest possible. Several other mathematicians also expressed opinions on the right choice of the constructing curves. In this section I shall survey these opinions.

Descartes had not provided arguments why the degree of the constructing curves should be lowest possible, nor had he given much guidance about the question which curves to choose among many possible curves of the same degree. In his notes to the 1695 edition of the Géométrie Jakob Bernoulli uttered clear annoyance about Descartes' silence on these points; he wrote: "but when we ask for reasons of the assertion, complete silence"[61a].

Bernoulli's annoyance is understandable because, especially in the early phase of the development of the construction of equations, the right choice of the constructing curves was still felt as the central problem. In connection with it we often find quoted the almost moralistic terms of Pappus, who spoke about guilt of mathematicians proposing wrong theorems and the "considerable error" of those who solve plane problems with solid means[62]. Descartes himself had used these words in the Géométrie: it would be "an error in geometry" to construct by means of curves with a too high degree as well as it would be an error to try to construct with curves of too low degree[63]. Fermat used the words against Descartes in his Dissertatio, criticizing him for using curves of too high degree:

> "Certainly it is an offence against the more pure Geometry if one assumes too complicated curves of higher degrees for the solution of some problem, not taking the simpler and more proper ones; for it has been often declared already, both by Pappus and by more recent mathematicians, that it is a considerable error in geometry to solve a problem by means that are not proper to it". (Fermat Dissertatio p. 121)

In an article on Cartesian geometry (1688), Jakob Bernoulli called Descartes' construction of four mean proportionals (i.e. of the fifth degree equation $x^5 = a^4b$) by circle and cartesian parabola "most prolix", gave an alternative and commented

> "... I can see nothing that could in this case acquit Descartes from the vice of acting ungeometrically (αγεομετρησιασ) which he mentions so often". (Bernoulli 1688 p.349)

De la Hire also quoted the expression "considerable error" in connection with constructions by means of curves with too high degree[64]. The frequent allusion to Pappus' phrase shows the importance attributed to choosing the simplest possible constructing curves.

Fermat, de la Hire and (though less strictly) l'Hôpital accepted Descartes' requirement that the constructing curves should have lowest degree. They interpreted this in the sense that the two curves should not differ much in degree; these requirements guided the elaboration of the "main result" (cf. section III-8).

Descartes had left open the problem how to choose the simplest curve among curves of the same degree. Some mathematicians valued the freedom of choice here positively; van Schooten, Sluse and other writers took pleasure in working out constructions of third- and fourth-degree equations by all sorts of combinations of conic sections. Probably inspired by Descartes, the parabola was often considered the simplest among the conic sections (apart from the circle, which, as means of construction, was considered a "plane" curve). But some mathematicians (Newton for instance, cf. V-6), considered the ellipse as the simplest conic, because its mechanical description (by a trammel construction or by the "gardener's" construction) was almost as simple as tracing a circle with a compass.

Wallis even proposed an alternative classification of curves to incorporate gradation of simplicity within the class of conics. In his Algebra (1685) he suggested to call straight lines of degree 1, circles of degree 2, the other conic sections of degree 3 and the cubical parabola of degree 4. With that classification his construction of a third-degree equation by the cubical parabola and a straight line (given in his 1657, see IV-2) would be as good as Descartes' construction by parabola and circle; in both cases the sum of the "degrees" of the constructing curves is five[66]. Wallis must have realized that his argument was rather ad hoc, and, more important, could not in an obvious way be generalized for higher-order curves. Still, the fact that he put forth the argument shows that he considered the matter important.

As Jakob Bernoulli had done earlier (cf. V-4), Guisnée noticed in his textbook on analytic geometry (1733) that the search for lowest possible degree could lead to inappropriately complicated constructing curves. He wrote

> "... in a way it is embarrassing to geometry if one introduces, often with much difficulty, certain curves preferable to others which present themselves in a natural way and whose description is often very simple; therefore I wished that curves would be preferred without reference to their degree, in the way they are ordinarily determined". (Guisnée 1733 pp. 27-28)

This statement was quoted later, with approval, by Rabuel[67].

Some writers (Rolle, Cramer)[68] stated explicitly that the requirement of low degree was purely a matter of "elegance". Several others (Kraft, Euler, Cramer)[69] wrote that simplicity of tracing was a better criterion for the choice of constructing curves than low degree. Cramer for instance wrote:

> "It seems that, in choosing the proper curves to construct an equation, one has to aim at the easiness of description rather than the simplicity of the equation. One can say that for each problem there is some curve by which it is solved more naturally than by all other curves even of lower degree. Indeed a curve which has high degree but whose equation has only few terms will mostly be easier to describe, be it by points or by some intrument, than a curve of lower degree but whose equation, even if simplified as far as possible, has a high number of terms. There are even various examples of curves that are easy to describe although their nature can be expressed only by most complicated equations. Well, should one not prefer the simplest constructions in geometry? And are not the simplest constructions those that are performed with the easiest traceable curves? The equation is really only a symbol which guides us in calculations; fundamentally it is the description of the curve which resolves the problem. Whether one arrives at the construction by a long or short, an easy or a difficult calculation does not have any influence on the operation itself that really constitutes the solution". (Cramer 1750 p. 91)

The words are strong enough, but there is no attempt to establish an alternative criterion of simplicity which could really be operative in finding constructing curves. Hence by 1750 a definite, workable criterion for the geometrical simplicity of constructing curves had failed to turn up, the arguments on the crucial point of method - how to choose constructing curves - had proved inconclusive.

V-9

The arguments discussed in the previous sections illustrate that mathematicians were confronted with a fundamental problem in working out Descartes' programme: The geometrical criteria of adequacy for solutions (simplicity of construction, acceptability of the means of construction) could not be translated in a natural way into algebraical language and procedures. On the other had (as Newton's work shows) a purely geometrical approach to the construction of higher order problems and equations was not feasible either.

Thus after 1700, the construction of equations, having started as a sensible, indeed necessary, part of Descartes' programme for geometry, was losing much of its original motivation. Still, there remained interest in the subject and we even see new motivations for it being put forth. One of these was Newton's argument that the construction of equations can supply a first approximation of the root, which can serve as a starting value for a further numerical approximation. We find the same argument in Halley's lectures (1725).

Several other mathematicians alluded to practical use of the construction of equations by suggesting that a special first constructing curve F should be cut out from metal and used as a template in constructions. As we have seen (cf. V-5) Newton had suggested this in his 1665 manuscript. Baker considered the rules he published in his Geometrical Key as practical rules:

> "Sit down therefore at thy study-table (reader) seek the aequation whose construction thou designest, in the central table, or synopsis, which will guide thee, to its rule for its construction, its demonstration, figure, or (at least) to one suitable to it. Take thy compass and the Scale of inches (for that scale only have I used through the whole) and having described according to art (which in Chap. 1 is taught) a Parabole, let all things be applied accordingly, as we have prescribed; and thou shalt find all things forthwith exactly to answer thy expectation". (Baker 1684 en of preface)

It is very difficult to assess in how far the construction of equations was ever put to actual practical use. I have not found examples of such use, but it may be that some mathematicians actually solved equations by geometrical construction, with or without subsequent numerical approximation. Many mathematicians, on the other hand, were sceptical about the practicality of these constructions. Halley wrote that no geometer will actually use ruler and compass to find exact solutions of geometrical problems "because of the defect of instruments and of our senses, whereby the intersections of lines imperfectly drawn, are yet more imperfect"[70]. He considered constructions useful for giving a global insight in the problem:

> "a geometrical construction, rightly manag'd lays open the whole mystery in a short view, and at once shews directly as well the number and quantities of the roots, as their signs...". (Halley 1725 p. 2)

Halley had worked out this idea in a separate article (1687). In that article he investigated the values of the coefficients in third- and fourth-degree equations for which there are 1, 2, 3 or 4 roots. He did this by studying geometrically the positions of the centres and the values of the radii of circles that intersect a parabola in 1, 2, 3 or 4 points. The results are convincing in the third-degree case; for the fourth degree they become very involved.

Wolff denied, in his Elementa Matheseos that the construction of equations has any practical use; its value lay in training the force of the mind.[71].

In fact, one quite outspoken attack on the methods of constructing equations came from a mathematician involved in the practice of mathematical instruments, namely Stone. He added to the second edition (1758) of his translation of Bion's great treatise on mathematical instruments, an appendix in which he commented, among other things, on the construction of equations. He repeated Newton's arguments of the Arithmetica universalis (cf. V-6): strong critique on the use of algebra in geometry; preference for neusis constructions for practical reasons. He then presented a series of construction problems, in solving which he deliberately broke all the rules of established theories of construction, using parallel rulers, sliding rulers, sliding squares, cords and several other impromptu means of construction. He commented:

> "I have given the instrumental constructions of the few problems above, as a specimen of the most easy, natural, and obvious way possible of performing the business, in order to invite others to proceed in this way in the resolution of difficult geometrical problems, rather than by that usual one so long in vogue, of first obtaining an algebraic equation by means of the given conditions of the problem; and then finding the linear roots of that equation, which in almost all cases in troublesome, unelegant and unnatural, and in many other cases is intolerable, and almost impossible". (Bion/Stone 1758, p. 324)

The arguments reviewed in this section show that the attempt to provide new, and in particular practical, motivations for the techniques of the construction of equations, failed. Indeed we find little conviction left in the argumetns, practical or fundamental, which mathematicians put forth in the first half of the eighteenth century as motivation for the construction of equations.

V-10

Although ultimately no new convincing motivations emerged, there was a side issue within the theory of constructing equations which attracted attention when mathematicians started searching for other than purely geometrical motivations. This was the interest in graphs of polynomials[71a]. The requirement of lowest possible degree rules out construction of an equation by intersecting its graph with the X-axis (cf. III-2). However, Jakob Bernoulli, having criticized the requirement, suggested in his 1695 notes to Descartes' Géométrie[72], to construct the equation

$$x^5 = ax^4 + b^2x^3 + c^3x^2 + d^4x + e^5$$

by rewriting it as

$$x = a + \frac{b^2}{x} + \frac{c^3}{x^2} + \frac{d^4}{x^3} + \frac{e^5}{x^4},$$

tracing the graph of the right hand side,

$$y = a + \frac{b^2}{x} + \frac{c^3}{x^2} + \frac{d^4}{x^3} + \frac{e^5}{x^4},$$

and intersecting it with the straight line

$$y = x.$$

He noted as advantage that the graph can easily be constructed pointwise, that is, for each x-value it is easy to construct (by elementary constructions for multiplication and division) the terms $\frac{b^2}{x}$, $\frac{c^3}{x}$ etc., and their sum. Hence points on the graph could be easily found. He suggested that this advantage might compensate the disadvantage of the high degree of the graph.

In l'Hôpital's Traité we find a similar argument[73], now actually concerning the graph of the equation itself. He suggested constructing the equation

$$x^n + \ldots + a_1x + a_o = 0$$

by intersecting the graph

$$y = x^n + \ldots + a_1x$$

with the horizontal line

$$y = -a_0.$$

He mentioned easy pointwise constructibility of the graph as advantage and he stressed that this procedure was useful for reading off the limits between which the roots of the equation lie, as well as their global position on the X-axis.

Cramer treated in his Introduction[74] the graph of an equation in the same way as l'Hôpital, he also saw it as a special kind of construction of an equation, with advantages for gaining global insight in the location of the roots.

In later mathematics, the interest in the graph of polynomials remained[75]; but soon it was no longer seen as a special kind of construction of the equation.

V-11

An inconclusiveness similar to that of the general arguments can be discerned in reactions of mathematicians to a more technical critique on the usual procedure of the construction of equations. This critique was voiced by Rolle in two articles, 1708 and 1709. Rolle was interested in determining the limits between which the real roots of equation lie, and in the real and imaginary intersections of algebraic curves. In studying these questions he found many instances in which the insertion method for the construction of equations, as explained by de la Hire, produced anomalous results.

In his articles he presented these cases as a critique on the usual method for constructing equations. He did this in a quite effective manner. He gave nine different equations ranging from degree 6 to degree 20, and 15 possible choices for the first constructing curve. Out of these he took 20 different combinations, calculated the second constructing curve by insertion, and showed that in each case something anomalous happened. The number of intersections of the constructing curves turned out to be larger or smaller than the number of roots; the intersections yielded roots that did not satisfy the equation; the second constructing curve turned out to have no real points etc.

In particular Rolle showed that an intersection corresponding to a real root may become imaginary, in which case it does not appear at all when one actually constructs the equation. His example[76] was the equation

$$H : x^6 + 21a^5x - 22a^6 = 0,$$

hwose real roots are $-2a$ and a. When one chooses

$$F : ay^2 = x^3$$

the second curve becomes

$$G : 21a^3x + y^4 = 22a^4.$$

F and G have two real points of intersection, namely (a,a) and $(a,-a)$, which correspond to the root a of $H(x) = 0$. But there is no real point of intersection corresponding to the root $-2a$ (the corresponding points of intersection are $(-2a, 2ai\sqrt{2})$ and $(-2a, -2ai\sqrt{2})$).

The example implied quite serious a critique on the usual method for the construction of equations. It showed that by simply performing the rules one might arrive at results that were geometrically meaningless, whereas nothing in the algebraic manipulations warned that something anomalous was happening.

It is noteworthy that Rolle, in these and other articles, in fact undertook an exploratory study of the intersections of algebraic curves, with particular interest in the occurrence of imaginary points of intersection. Still, apparently the only form in which he could present his results was that of a critique of existing method of constructing equations. This shows that at that time intersection theory of curves was still so strongly bound up with the construction problem that it could not be treated as a separate theory[77].

Several writers reacted on Rolle's critique. De la Hire himself, whose method was explicitly criticized, answered

> "it seems to me that one could not say that this is a defect of the method, but only of the application of it, as is not uncommon in geometrical and analytical operations". (De la Hire 1710 p.29)

He suggested that if anomalies as the ones indicated by Rolle occurred one had to start again somewhat differently. Experience learned that after a few trials one always found a good solution. More specifically he suggested that the first constructing curve should be chosen such that its real x-values cover the segment on the X-axis in which the roots of the equation lie[78].

Hermann suggested in his 1727 that Rolle's critique could be avoided by always taking the first constructing curve in the form

$$\mathcal{F} : yf_1(x) + f_2(x) = 0$$

which has no points with real x-coordinate and imaginary y-coordinate. Euler and Cramer also restricted the choice of the first constructing curve in this way to avoid the occurrence of imaginary points of intersection[79].

At first sight it seems indeed an effective answer to the critique raised by Rolle. But there is a complication. Cramer as well as Euler stated the "main result". Cramer referred to l'Hôpital, Euler just stated it. L'Hôpital's argument for the main theorem (and it seems likely that both Cramer and Euler would use the same argument) was that the number of coefficients free to choose in polynomials F(x,y) and G(x,y) of degree p and q, is larger than the number of coefficients in a polynomial H(x) of degree pq. But if one restricts the choice of $\mathcal{F}$ by stimpulating that F should be of the form

$$yf_1(x) + f_2(x),$$

this argument is no longer valid. The number of coefficients in F is then

$$2p + 1,$$

that in G remains

$$\tfrac{1}{2}(q + 1)(q + 2),$$

and that in H is

$$pq + 1.$$

For the main theorem still to be valid, the requirement would be

$$pq + 1 \leqslant 2p + 1 + \tfrac{1}{2}(q + 1)(q + 2),$$

which, for higher values of p and q (e.g. $p,q > 7$) does not apply.

It is significant that neither Euler nor Cramer (not to speak of Hermann) did see this. The argument is certainly not difficult; once one has seen the problem, the result is immediate. We can conclude, therefore, that neither Euler nor Cramer took the problem sufficiently seriously to become aware of an obvious problem. I take this as a sign that their treatment of the subject was determined by tradition rather than by active interest. The tradition was not strong enough to keep the attention of mathematicians after Euler and Cramer.

VI Conclusion

VI-1

In this concluding chapter I shall analyse the causes that led to the decline of the construction of equations, and I shall briefly discuss the applicability of concepts and models of Lakatos' "methodology of research programmes" to this case of a "degenerating" mathematical theory. Before that, however, something has to be said about the relation of the theory to other parts of mathematics. The construction of equations was not an isolated theory; there were two other theories to which, in subject and methods, it was near. These were: the algebraic theory of equations, whose aim was to solve equations by radicals, and the theory of algebraic curves.

To solve an equation by radicals means to express its roots in terms of its coefficients by means of a formula involving only the arithmetical operations and the extraction of n-th roots. In the sixteenth century solutions by radicals were found for third- and fourth-degree equations (the formulas of Cardano and related formulas). During the seventeenth and the eighteenth centuries mathematicians tried to understand the mathematics behind these solutions and to find such solutions for higher-degree equations.

The algebraic solution of equations by radicals may at first sight seem the theory which was most akin to the construction of equations. Both theories concerned equations and aimed at the exact (not approximate) determination of roots. However, each theory defined exactness in its own way: the geometrical theory by generalizing the exact ruler-and-compass constructions to constructions by algebraic curves; the algebraic theory by accepting solutions involving n-th roots as exact. These approaches were indeed so different that the two theories had in fact little in common. Nor were they seen as strongly related or comparable. One might argue that for some time the construction of equations was more successful than the algebraic approach to equations - it provided solutions for equations up to degree four which were easy and more elegant than the Cardano or related formulas; and, contrary to the algebraical approach, it was not blocked at the fourth degree. Still I have found this comparison only once made in a side remark by Roberval[80]. Significantly, most mathematicians did not compare the two theories in this way, they saw them as entirely distinct in aims and in methods.

While, after 1750, the construction of equations fell into oblivion, the algebraic theory of equations gained new impetus under the influence of a new approach advocated especially by Lagrange. This approach led to most important results: the proofs (by Ruffini and Abel) that the solution by radicals of the general fifth-degree equation is impossible, Galois theory and, later, modern algebra. Despite the chronological coincidence of the decline of the geometric theory and the renewed impetus of the algebraic one, this is not a case of replacement of an unsuccessful approach by a successful rival. The theories had been separate already for a long time, their aims were different and none of the techniques worked out in the geometrical theory were taken over by the algebraical one.

However, not all mathematical insights gained by the theory of constructing equations were lost when the subject fell into oblivion. There was indeed a natural inheritor, namely algebraic geometry, the study of algebraic curves. Construction of geometrical problems and of equations provided the early motivation for the general study of algebraic curves and their intersections. This was expressed by Newton when he wrote, in connection with his classification of third-degree algebraic curves, that "the use of curves in geometry is that by their intersections problems can be solved"[81]. The construction of equations also promoted the interest in techniques of elimination which later became central in the study of intersections of algebraic curves. During the first half of the eighteenth century the relation between the two theories gradually changed. The theory of algebraic curves acquired an independent status; the construction of equations was seen as one of its applications. Several insights concerning intersections and elimination had been taken over and when the construction of equations finally disappeared, these insights remained alive within algebraic geometry.

VI-2

Although its techniques and insights were partly taken over in algebraic geometry, the construction of equations did die; its central aim, the exact geometrical construction of roots of equations, was no longer acknowledged as important, and subsequenty forgotten; the construction of equations lost its place as standard part of algebra, the newer textbooks omitted the subject. The question remains, why did the construction of equations die?

In the previous chapter I have reviewed the various processes that combined to invalidate the subject: its original motivation was misunderstood by later mathematicians; contradictions appeared in the efforts to translate the original geometrical criteria into algebraic ones; new motivations for the subject were not satisfactory; convincing criteria of adequacy for constructions of higher-order equations could not be formulated; fundamental critique was answered without sufficient care by ad hoc arguments. As I mentioned in the introduction, these processes form part of a general development of de-geometrization of mathematics which took place in the seventeenth and eighteenth centuries. In other fields of mathematics (in particular in differential and integral calculus) the de-geometrization was very beneficial; in the case of the construction of equations it led to the disappearance of the subject after 1750.

It is important to note that the construction of equations did not disappear because the problems in the field were unsolvable (it was, for instance, quite feasible to study in detail constructions for equations of degrees 5 and 6, or 7, 8 and 9). Nor did the theory loose interest because all problems were already solved. Nor, again, can the loss of interest be attributed to a rival theory which produced better constructions, because there was no such theory.

Rather than in the sphere of the mathematical problems and techniques, the causes for the disappearance of the construction of equations lay in the sphere of motivation and method. They were connected with the reasons why mathematicians considered certain problems as meaningful, and with the criteria of adequacy which mathematicians set for the solution of these problems. Such reasons and criteria are very essential in the development of mathematics. They guide the research in a field, and as such they are necessary for its development. The essence of their role becomes particularly clear in the case of the construction of equations, because the original motivation of the subject lost its meaning and adequate criteria to choose between the many possible constructions failed to turn up. Hence the arguments on motivation and method remained inconclusive, they could not be translated into convincing criteria or guidelines for research.

Why, then, did the arguments on motivation and method remain inconclusive? They did so because of a contradiction, built in from the very beginning in the Cartesian mixture of algebra and geometry. Purely algebraically, the procedure of the construction of equations does not make sense, its sense must come from geometry. Therefore the criteria of adequacy must come from geometry. In its later development the whole procedure became algebraical, but the geometrical meaning - exact construction -, and the geometrical criteria of adequacy - simplicity of instruments or easiness of tracing the curves - appeared not to have a natural translation into algebra. This is the basic contradiction, and in it we can sum up the cause for the death of the construction of equations: the subject had a natural tendency to become algebraic, but it could not bring over its original geometrical criteria to algebra, hence it lost its motivation and it died.

VI-3

The construction of equations thus provides an example of a mathematical theory which, having started off with considerable vitality, entered in a degenerating phase from which it did not recover. The process of decline of scientific theories has recently attracted much interest from philosophers and historians of science, especially because "degenerating" and "progressive" research programmes are key concepts in the "methodology of researchprogrammes" proposed by Lakatos as a way to study and understand the development of science[82]. Lakatos himself, and several other writers have studied historical examples of degenerating (and progressive) scientific theories. These examples concern natural science rather than mathematics. Recently Hallet[83] has studied the special case of mathematical theories from a Lakatosian viewpoint.

In preparing my analysis of the development of the construction of equations I have found these studies[84] illuminating and inspiring, but I have come to the conclusion that the Lakatosian approach is not applicable in this case. This conclusion may be of some methodological interest and I shall therefore briefly discuss the reasons why I have found the concepts and models of the "methodology of research programmes" insufficient for understanding the case of the construction of equations.

The central aim of Lakatos' methodology of research programmes is to distinguish between "progressive" and "degenerating" researchprogrammes. Lakatos has worked out the distinction primarily for programmes within natural science. He has suggested that the historical development of science can be fruitfully understood - in his terms "rationally reconstructed" - in terms of competing researchprogrammes, whereby scientists finding themselves in a degenerating programme tend to switch over to a rival programme which is in a more progressive phase. Lakatos has found the criterion for researchprogrammes to be progressive or degenerating in their success or failure to produce significant and successful predictions. Predictions are considered significant especially if they concern natural phenomena outside the realm for which the theories on which they are based were originally created. The basis for assessing the significance of predictions, and thereby the state of progress or degeneration of the programme, in the informed intuitive opinion of the scientists working in the fields in question.

It is difficult to apply this criterion directly in the case of mathematics. Hallet has sketched how a related criterion can be worked out for mathematical research programmes. His proposed criterion is the success or failure of programmes to solve mathematical problems for whose solution they were not specifically created. He has illustrated his proposal by applying the criterion in a number of case-histories in nineteenth- and early twentieth-century mathematics. In these studies he bases his assessment of the success or failure of programmes primarily on the opinions expressed by leading mathematicians involved in the fields in question.

The construction of equations was not a large scale endeavour within mathematics, so the term "research programme" may be somewhat too broad to use in this case. Still, the subject had strong programmatic aspects, which played an important role in its development, and, after 1700, it was clearly in a degenerating stage. So it is tempting to consider it as a degenerating research programme in the Lakatosian sense. Hallet's criterion indeed confirms that the construction of equations was degenerating after ca. 1700; the theory certainly did not solve problems for whose solution it was not created, in fact, in its later phase it did not solve any problems at all. But it should be noted that this failure to solve problems is not metioned in the arguments of the leading mathematicians at the time on the state of the subject. As we have seen, the main theme in the critical opinions expressed about 1750 on the subject was not that it did not solve problems, but that it did not make sense. Hallet's criterion, then, although formally applicable in this case, is unsatisfactory because it is unrelated to the actual causes of the degeneration of the construction of equations, which were not in the sphere of results but in the sphere of motivation.

It is significant that another central process in the Lakatosian model for the "rational reconstruction" of developments in science does not apply either in this case: there was no rival programme. In losing interest in the construction of of equations mathematicians did not switch over to a rival, more succesful approach to the same questions; they just dismissed the questions.

I find, then, that an analysis of the development of the construction of equations along the lines suggested by the "methodology of research programmes" is not satisfactory. The reason for the inapplicability of Lakatosian concepts and models in this case is that they only concern success and failure on the level of <u>results</u>. Here, however, is a case in which the significant factors in the development were not on the level of results but on that of meaning and motivation.

The case of the construction of equations shows that factors concerning meaning and motivation, and the related issue of criteria of adequacy of methods and solutions, can be crucial in the development of mathematics. The inapplicability of Lakatosian concepts and models is, to me, one of the signs that the importance of these factors in the development of mathematics (and indeed of science in general) has been underestimated. The present article is meant as a contribution to an approach to the development of scientific theories which takes these factors seriously.

Acknowledgement

I have presented parts of this material during lectures at Leeds, Aarhus, Utrecht, Odense, Heidelberg and Amsterdam. The discussions on the occasion of these lectures have been very helpful for me. Lektor K. Andersen (Aarhus) and dr J.R. Ravetz (Leeds) have read preliminary versions of the present text. I am very thankful to them for their active interest and their stimulating comments.

Notes

1) Zeuthen 1903 p. 199.

2) Wieleitner 1911 pp. 54-58.

3) Boyer 1943 and 1956, index s.v. "graphical representation".

4) The term "construction" in the sense of solution occurs in the titles of five of the seventeen articles in Euler's Opera (1) 22 (early papers on differential equations).

5) I have dealt with the programmatic aspects of the Géométrie in my 1981, see esp. pp. 304-307.

6) Cf. Steele 1936 and Niebel 1959.

7) The construction was mentioned by Eutocius. For text and translation see Thomas 1967 1 pp. 278-283.

8) Pappus Collectio 1 pp. 54-57 and 270-273 (i.e. lib. III 20-22 and lib. IV 57-59).

9) On neusis constructions see Heath 1921 (index s.v. "neusis") and Archimedes Works pp. c-cxxii.

10) Heath 1921 1 pp. 235-236.

11) Heath 1921 1 pp. 238-240.

12) Heath 1921 1 pp. 236-237.

12a) Newton, in manuscripts written c. 1693, recognized this problem: "...a problem is not something which is postulated to be done. We must therefore take care lest, in our zeal to augment geometry, we at the same time pollute it with postulates of this sort." (Math. papers 7 pp. 388-389, tr. Whiteside).

12b) The term is Pappus', cf. V-8 and note 62.

13) Cf. ref. in note 5.

13a) Descartes knew (see e.g. 1637 p. 335) that in some problems there would be more unknowns than equations, so the elimination would result in one equation in two or more unknowns. In that case an infinity of points satisfy the problem; these points form a locus. Descartes explained that to construct points on the locus one assumed values for all but one of the unknowns in the final equation and then proceeded as in the case of the ordinary problems.

14) Descartes 1637 pp. 380-389.

15) Descartes 1637 pp. 302-304.

16) Descartes 1637 pp. 389-395.

16a) Descartes states that he takes the latus rectum of the parabola equal to 1. Latus rectum and latus transversum are the classical terms for certain line-segments occurring in the defining properties of conic sections. If the vertex of the conic section is taken as origin and the Y-axis is along the diameter, then the latus rectum a and the latus transversum b occur in the analytical formulas for the conics in the following way: $x^2 = ay$ (parabola); $x^2 = ay - \frac{a}{b}y^2$ (ellipse); $x^2 = ay + \frac{a}{b}y^2$ (hyperbola).

17) Descartes 1637 pp. 402-412, cf. Bos 1981 p. 306 note 11.

18) Cf note 14.

19) Rolle (cf. V-11 and note 76) even discussed the construction of equations which were obviously reducible without commenting upon the fact.

20) Descartes 1637 pp. 381-386.

20a) De la Hire 1679 p. 429.

21) Descartes 1637 pp. 403-405.

22) Cf. e.g. Descartes 1637 p. 371.

22a) Descartes 1637 p. 413.

23) Kinckhuysen 1660 pp. 63-65; de la Hire 1679 p. 111; Jakob Bernoulli 1687 p. 349.

24) Fermat Dissertatio p. 121.

25) Descartes 1659 1 p. 324.

26) De la Hire 1679 pp. 418-421.

27) Sluse 1668 pp. 51-65.

28) Sluse 1668 pp. 76-78.

29) l'Hôpital 1707 pp. 346-347.

30) Newton ms 1665 p. 498.

31) Bezout 1779 pp. 30-33.

32) Newton (ms 1665 p. 498) and l'Hôpital (1707 p. 346) state it as a matter of course; l'Hôpital claims that it is evident by the "rules of algebra".

33) MacLaurin 1720 p. 136, Euler 1748a, Cramer 1750 pp. 660-676.

34) (Note cancelled.)

35) l'Hôpital 1707 p. 346.

36) Formulated in modern terms the problem is this: Let K be a field (in particular the real or the complex numbers). Are there, for every polynomial $H(x) \in K[x]$ of degree p.q, polynomials $F(x,y)$ and $G(x,y) \in K[x,y]$ of degrees p and q respectively, such that $R_{F,G} = H$? - I have mentioned this problem to several algebraic geometers. Thier reaction was that they did not know whether the

problem has been studied already, that the answer is probably "yes", but that the proof would be difficult. I shall be very grateful to any reader who can give me more definite information about this problem.

37) Wallis 1657; the cubical parabola and its use in constructions is treated in the dedicatio of this work (pp. 231-256 of the edition in Opera 1).

38) Descartes 1659 1 pp. 324-328.

39) Descartes 1659 1 pp. 328-330.

40) Kinckhuysen 1660 pp. 56-63.

41) Kinckhuysen 1663 pp. 56-64.

42) Mahoney 1973 p. 130, note.

43) De la Hire 1679 pp. 297-452; this part has a separate title-page.

44) Wallis 1685 pp. 275-277, Sturm 1689 pp. 348-358, Harris 1702 pp. 86-99 and 1704 s.v. "construction of equations", Halley 1687a, Anonymous 1703, Hermann 1727 p. 140, Wolf 1743 1 p. 406, Zedler 1733 6 col. 1098, Euler 1748b 2 p. 277.

44a) Wallis 1685 pp. 273-277 (Opera 2 pp. 295-299), Sturm 1689 pp. 392-474 (= 1700 last part, separately paginated 1-96), Ozanam 1702 pp. 224-233, Harris 1702 pp. 31-99 and 1704 s.v. "construction of equations", Guisnée 1705 pp. 201-211, l'Hôpital 1707 pp. 291-361, Newton 1707 (appendix), Reyneau 1708 pp. 601-621. Note the considerable extent of the treatments by Sturm, Harris and l'Hôpital; the section in Sturm 1689 is in fact a complete treatise (with separate title-page) on Cartesian geometrical constructions.

45) l'Hôpital 1707 pp. 291-361.

46) Euler 1748b 2 ch. 20 "De constructione aequationum" pp. 269-284.

47) Cramer 1750 Ch. 4 "Quelques remarques sur la construction géometrique des egalitez", pp. 80-108.

48) Savérien 1753 1 pp. 220-221; d'Alembert Construction, Encycl. Britt. (3[d] ed. 1797) 1 pp. 441-442. A special case is Crokers Complete dictionary (1765); it has three relevant headings: "Construction", "Construction of equations" and "Geometrical construction of equations". These three form a cross-reference loop, but none of them explains what construction of equations is.

49) Examples are Lamy 1692 pp. 284 sqq and Wolf 1743 1 pp. 302-303.

50) Newton _Enumeratio_ p. 161. Braikenridge states in his book on the description of curves (_1733_) that the central problem in the theory of curves is to find "a general and easy method by which curves can be described in the plane so that by their help problems can be constructed" (p. vii).

51) Cf note 19.

52) (Note cancelled.)

52a) De la Hire _1712_ p. 351.

53) The relevant notes of van Schooten are in Descartes _1649_ p. 278 and in Descartes _1659_ pp. 321-322 (Huygens' example only occurs in the 1659 edition); see also Huygens _Oeuvres_ _12_ pp. 81-82 and _14_ pp. 420-422. De la Hire later also discussed this problem, see his _1679_ pp. 440-452. Compare also Whiteside's note on these and related constructions in Newton _Math. papers_ _7_ pp. 304-305 (note 59).

54) Van Schooten incorporated this study of van Heuraet in his edition Descartes _1659_ _1_ pp. 259-262.

55) Newton _Math. papers_ _1_ pp. 502-505; a study dating from 1665.

56) l'Hôpital's construction can be found in Huygens _Oeuvres_ _10_ pp. 407-408; for further references see note 1 on p. 407.

57) Bernoulli _1695_ pp. 670-675. Related arguments are in his _1695_ passim and in his _1688_.

58) Newton _ms 1670_ pp. 470-475.

59) Newton _ms 1670_ pp. 492-495.

60) Newton _1707_ pp. 279-326; I refer to the edition of the text in _Math. papers_ _5_ pp. 420 sqq.

61) E.g. by Stone, cf V-9.

61a) Bernoulli _1695_ p. 689.

62) Pappus _Collectio_ _1_ p. 271.

63) Descartes _1637_ p. 371.

64) De la Hire _1679_ pp. 306-307.

65) (Note cancelled.)

66) Wallis _1685_ p. 275.

67) Rabuel _1730_ p. 418; Rabuel cited from an earlier edition of Guisnée _1733_.

68) Rolle _1708_ p. 340; Cramer _1750_ p. 88.

69) Kraft (_1742_ p. 8), Euler (_1748b_ _2_ p. 279), Cramer (see below).

70) Halley _1725_ p. 2.

71) Wolf _1743_ _1_ p. 393.

71a) Cf Boyer _1943_.

72) Bernoulli 1695 pp. 689-691.

73) l'Hôpital 1707 pp. 348-349.

74) Cramer 1750 pp. 92-108.

75) For some information on the further history of the use of graphs in calculating roots of equations see Frame 1943.

76) Rolle 1708 pp. 361-363.

77) Rolle presented, in his 1713, a "paradox in geometrical constructions". The paradox was that two graphs, both concave and increasing, may intersect each other in arbitrarily many points. Here again Rolle gave his result, which is purely about intersection theory, in terms of the construction of equations. (The paradox was inspected by members of the Académie, who concluded that "the example is good and the paradox true" (Saurin 1713 p. 262).)

78) Euler later criticised de la Hire's suggestion, noticing that it might still happen that points of intersection corresponding to a real root x were imaginary (1748b 2 p. 283).

79) Euler 1748b 2 p. 280, Cramer 1750 p. 86.

80) Roberval 1693 p. 244.

81) Cf. note 50.

82) Lakatos Papers vol. 1.

83) Hallet 1979.

84) Another example of a study of a declining mathematical theory is the investigation by Fisher of the "death" of invariant theory (Fisher 1966, 1967). Fisher adopts a sociological approach in describing the process of decline in this case. This approach is not applicable in the case of the construction of equations because the number of mathematicians involved is too small and because none of these mathematicians viewed the construction of equations as his specialty or as the main framework of his mathematical activities.

Bibliography

d'ALEMBERT, J. le R.

Encyclopédie — (together with D. DIDEROT, eds) Encyclopédie ou dictionaire raisonné des sciences, des arts et des métiers (28 vols) Paris 1751-1765.

Construction — article "Construction" in Encyclopédie (ed. Bern/Lausanne 1778-1781, vol 9, pp. 203-206.

ANONYMOUS

1684 — (review of BAKER 1684) Philos. Tr. 14 (May 1684, nr 159) pp. 549-550.

1703 — "Problema geometricum de generali via inveniendi omnes regulas centrales Bakeri" Acta Erud. (April 1703) pp. 166-170.

ARCHIMEDES

Works — The works of Archimedes (ed. T.L. HEATH) New York (Dover) 1953 (reprint of ed. Cambridge 1897/1912).

BAKER, Th.

1684 — The geometrical key, or the gate to equations unlock'd London 1684.

BERNOULLI, Jakob

Opera — Opera Omnia (2 vols) Lausanne/Geneva 1744

1688 — "Animadversio in geometriam Cartesianam et constructio qucrundam problematum hypersolidorum" Acta Erud. (June 1688) pp. 323-330; Opera 1 pp. 343-351.

1695 — (Notes to the Frankfurt 1695 edition of DESCARTES 1659) "Notae et animadversiones tumultuariae in geometriam Cartesii" Opera 2 pp. 665-717.

BEZOUT, E.

1779 — Théorie générale des équations algébriques Paris 1779.

BION, N.

1758 — The construction and principal uses of mathematical instruments translated from the french (-) by Edmond Stone (2nd ed. enlarged) London 1758.

BOS, H.J.M.

1981 — "On the representation of curves in Descartes' Géométrie" Arch. Hist. Ex. Sci 24 (1981) pp. 295-338.

BOYER, C.B.

1943 — "Early graphical solutions of polynomial equations" Scripta Math. 11 (1943) pp. 5-19.

1956 — History of analytic geometry New York 1956.

BRAIKENRIDGE, W.

1733 — Exercitatio geometrica de descriptione linearum curvarum London 1733.

CRAMER, G.

1750 — Introduction à l'analyse des lignes courbes algêbriques Geneva 1750.

CROKER, T.H.

1765 — The complete dictionary of arts and sciences (ed. by T.H. Croker e.a., 3 vols) London 1765-1766.

DESCARTES, R.

1637 — La géométrie (one of the essays in his) Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences, plus la Dioptrique, les Meteores et la Geométrie, qui sont des essais de cete méthode Leiden 1637 (pp. 297-413). Original text and translation in: D.E. SMITH and M.L. LATHAM (eds) The geometry of René Descartes with a facsimile of the first edition New York (Dover) 1954.

1649 — Geometria à Renato Descartes anno 1637 Gallicè edita, nunc autem cum notis Florimondi De Beaune (-) in languam Latinam versa et commentariis illustrata opera atque studio Francisci à Schooten (-) Leiden 1649.

1659 — Geometria à Renato Descartes anno 1637 Gallice edita; postea autem una cum notis Florimondi De Beaune (-) Gallicè conscriptis in Latinam linguam versa; et commentariis illustrata, opera atque studio Francisci à Schooten (-). Nunc demum ab eodem diligenter recognita, locupletioribus commentariis instructa (-) (2 vols) Amsterdam 1659-1661. (Later editions: Amsterdam 1683, Frankfurt/M 1695 (with notes by Jakob Bernoulli).)

DIDEROT, D (see d'ALEMBERT Encyclopédie)

ENCYCLOPAEDIA

Enc. Britt — Encyclopaedia Brittanica (3rd ed. 22 vols) Edinburgh 1797-1801.

EULER, L.

Opera — Opera Omnia (4 series, many vols, ed. Soc. Sci. Nat. Helv.) Leipzig etc. 1911- .

1748a — "Démonstration sur le nombre des points ou deux lignes des ordres quelconques peuvent se couper" Mém. Acad. Berlin (1748) pp. 234-248; Opera ser. 1 vol. 26 pp. 46-59.

1748b — Introductio in analysin infinitorum Lausanne 1748; Opera ser. 1 vols 8, 9.

FERMAT, P. de

Opera — Varia opera mathematica Toulouse 1679.

Oeuvres — Oeuvres (eds P. TANNERY and C. HENRY) (4 vols) Paris 1891-1912.

Dissertatio — "De solutione problematum geometricorum per curvas simplissimas et unicuique problematum generi proprie convenantes disseratio tripartita" Opera pp. 110-115; Oeuvres 1 pp. 118-131.

FISHER, C.S.

1966 — "The death of a mathematical theory: a study in the sociology of knowledge" Arch. Hist. Ex. Sci. 3 (1966-1967) pp. 137-159.

1967 — "The last invariant theorists" Arch. Europ. de sociol. 8 (1967) pp. 216-244.

FRAME, J.S.

1943 — "Machines for solving algebraic equations" Math. tables aids comp. 1 (1943-1945) pp. 337-353.

GUISNÉE, ..

1733 — Application de l'algèbre à la géometrie ou méthode de démonter par l'algèbre les theorêmes de géometrie et d'en résoudre et construire tous les problêmes (2nd ed.) Paris 1733 (earlier ed. 1720)

HALLET, M.

1979 — "Towards a theory of mathematical research programmes" Br. J. Phil. Sci. 30 (1979) pp. 1-25, 135-159.

HALLEY, E.

1687a — "De constructione problematum solidorum sive aequationum tertiae vel quartae potestatis, unica data parabola ac circulo efficienda, dissertatiuncula" Phil.Tr. 16 (Jul/Aug 1687, nr 188) pp. 335-343.

1687b — "De numero radicum in aequationibus solidis ac biquadraticis, sive tertiae ac quartae potestatis, earumque limitibus, tractatulus" Phil. Tr. 16 (Nov 1687, nr 190) pp. 387-402.

1725 — Lectures read in the school of geometry in Oxford concerning the geometrical construction of algebraical equations, and the numerical resolution of the same by the compendium of logarithms (printed as first part of) KERSEY 1725

HARRIS, J.

1702 — A new short treatise of algebra, with the construction of *equations as far as the fourth power together with a specimen of the nature and algorithm of fluxions* London 1702.

1704 *Lexicon technicum or an universal english dictionary of arts and sciences (-)* (2 vols) London 1704-1710.

HEATH, Th.

1921 *A history of Greek mathematics* Oxford 1921

HERMANN, J.

1727 "Observationes in schediasma quod Dn Rolle cum hac inscriptione: Eclaircissemens sur la construction des egalitez, in commentariis academiae reg. scient. 1708 edidit" *Misc. Berol.* *3* (1727) pp. 131-146.

HIRE, Ph de la

1679 *Nouveaux elemens des sections coniques, les lieux geometriques, la construction ou effection des equations* Paris 1679

1710 "Remarques sur la construction des lieux géométriques et des equations" *Mém. Acad. Roy. Sci.* (1710) pp. 7-45.

1712 "Remarques sur la géometrie de M. Descartes" *Mém. Acad. Roy. Sci.* (1712) pp. 337-352.

l'HÔPITAL, G.F. de

1707 *Traité analytique des sections coniques et de leur usage pour la resolution des equations dans les problêmes tant déterminez qu'indéterminez* Paris 1707 (sec. ed. Paris 1720) (references to the 1720 edition).

HUDDE, J.

1659 "Epistola prima de reductione aequationum" in DESCARTES *1659* vol. *1* pp. 401-506.

HUYGENS, C.

Oeuvres *Oeuvres complètes* (22 vols) The Hague 1888-1950.

KERSEY, J.

1725 *The elements of that mathematical art commonly called algebra (-) to which is added lectures (-) by dr Edmund Halley (-)* London 1725.

KINCKHUYSEN, G.

1660 *De grondt der meetkonst ofte een korte verklaringe der kegelsneeden* Haarlem 1660.

1661 *Algebra ofte stel-konst, beschreven tot dienst van de leerlinghen* Haarlem 1661.

1663 *Geometria ofte meetkonst, beschreven tot dienst derghene die haer in deze konst zijn oeffenende* Haarlem 1663.

KLÜGEL, G.S.

Wörterbuch — *Mathematisches Wörterbuch* Leipzig 1803- .

KRAFT, J.

1742 — *Theoriam generalem succinctam construendi aequationes analyticas cujuscunque sint ordinis (-)* Kopenhagen 1742.

LAMY, B.

1692 — *Nouveaux elemens de geometrie, ou de la mesure des corps (-)* Paris 1692.

MACLAURIN, C.

1720 — *Geometria organica sive descriptio linearum curvarum universalis* London 1720

1748 — *A treatise of algebra* London 1748.

MAHONEY, M.S.

1973 — *The mathematical career of Pierre de Fermat* Princeton 1973.

MONTUCLA, J.E.

1799 — *Histoire des mathématiques* (4 vols) Paris 1799-1802.

NEWTON, I.

Math. papers — *The mathematical papers of Isaac Newton* (8 vols, ed. D.T. WHITESIDE) Cambridge 1967-1981.

ms 1665 — "The theory and construction of equations (1665-1666)" *Math. papers* *1* pp. 489-540.

ms 1670 — "Problems for construing equations 1670?" *Math. papers* *2* pp. 450-517.

Luc. lect. — "The Lucasian lectures on algebra" *Math. papers* *5*.

Enumeratio — "Enumeratio linearum tertii ordinis" (supplement to) *Opticks or a treatise of the reflexions refractions, inflexions and colours of light* London 1704; also in *Math. papers* *7* pp. 588-645.

ms 1705 — "On the construction of geometrical problems" *Math. papers* *8* pp. 200-231.

1707 — *Arithmetica universalis sive de compositione et resolutione arithmetica liber* London 1707.

NIEBEL, E.

1959 — *Untersuchungen über die Bedeutung der geometrischen Konstruktion in der Antike* Köln 1959.

OZANAM, J.

1702 — *Nouveaux elemens d'algebre ou principes generaux pour resoudre toutes sortes de problêmes de mathematique* Amsterdam 1702.

PAPPUS

1588 Pappi Alexandrini mathematicae collectiones a Federico Commandino Urbinate in latunum conversae et commentariis illustratae Pesaro 1588.

Collectio Pappi Alexandrini collectionis quae supersunt (-) (ed. F. HULTSCH) Berlin 1875.

RABUEL, C.

1730 Commentaires sur la geometrie de M. Descartes Lyon 1730

REYNEAU, C.

1708 Analyse demontrée ou la methode de resoudre les problêmes des mathematiques et d'apprendre facilement ces sciences (-) Paris 1708.

ROBERVAL, G.P. de

1693 "De geometrica planarum et cubicarum aequationum resolutione" Mém. acad. roy. sci. 1666-1699 6 Paris 1730 pp. 136-246. (Earlier published in Divers ouvrages de mathématique et de physique par messieurs de l'academie royale des sciences Paris 1693.)

ROLLE, M.

1708 "Eclaircissemens sur la construction des egalitez" Mém. acad. Roy. Sci. (1708) pp. 339-365.

1709 "Eclaircissemens sur la construction des égalitez, seconde mémoire" Mém. acad. roy. sci. (1709) pp. 320-350.

1713 "Remarques sur un paradoxe des effections geomètriques" Mém. acad. roy. sci. (1713) pp. 243-261.

SAURIN, J.

1713 "Sur une observation de M. Rolle par rapport aux constructions geometriques proposée à l'academie comme un paradoxe" Mém. acad. roy. sci. (1713) pp. 261-264.

SAVERIEN, A.

1753 Dictionnaire universel de mathématique et de physique (2 vols) Paris 1753.

SCHOOTEN, F. van (See DESCARTES 1649 and 1659.)

SEGNER, J.A.

1756 Cursus mathematicus (5 vols) Halle 1756-1768.

SLUSE, R.F. de

1659 Mesolabum seu duae mediae proportionales inter extremas datas per circulum et ellipsim vel hyperbolam infinitis modis exhibitae Liege 1659.

1668 Mesolabum seu duae mediae proportionales inter extremas datas per circulum et per infinitas hyperbolas, vel ellipses et per quamlibet exhibitae ac problematum omnium solidorum effectio per easdem curvas. Accessit part altera de analysi et miscellanea Liege 1668.

STEELE, A.D.

1936 "Über die Rolle von Zirkel und Lineal in der Griechischen Mathematik" Quellen u. Stud. Gesch. Math. Astr. Phys. B. (Stud.) 3 (1936) pp. 287-369.

STONE, E. (See BION)

STURM, J.C.

1689 Mathesis enucleata Nürnberg 1689.

1700 Mathesis enucleata or, the elements of mathematics London 1700.

THOMAS, I.

1967 Selections illustrating the history of Greek mathematics (ed. I. THOMAS) (2 vols) London 1967-1968.

WALLIS, J.

Opera Opera mathematica (3 vols) Oxford 1693-1699.

1657 Adversus Marci Meibomii de proportionibus dialogum tractatus elencticus 1657, in Opera 1 pp. 229-290.

1685 Treatise of algebra both historical and practical Oxford 1685 (Lat. tr. in Opera 2 (1693)).

WIELEITNER, H.

1911 Geschichte der Mathematik II. Teil I. Hälfte, Arithmetik, Algebra, Analysis Leipzig 1911.

WOLF, C. von

1743 Elementa matheseos universae (-) editio novissima (5 vols) Geneva 1743-1752.

ZEDLER, J.H.

1732 Grosses vollständiges universallexicon aller Wissenschaften und Künste (64+4 vols) (ed. J.H. ZEDLER) Leipzig 1732-1754.

ZEUTHEN, H.G.

1903 Geschichte der Mathematik im 16. und 17. Jahrhundert Kopenhagen 1903.