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On the representation of curves in Descartes' *Géométrie*

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ON THE REPRESENTATION OF CURVES

IN DESCARTES' *GÉOMÉTRIE*

BY

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1. *Introduction. Curves and their representations in 17th century mathematics.*

1.1 From antiquity till the beginning of the seventeenth century the collection of plane curves known to mathematicians did not change. It consisted of the conic sections, some higher algebraic curves such as the conchoid of Nicomedes and the cissoid of Diocles, and a few transcendental ones of which the most important were the Archimedean spiral and the quadratrix of Deinostratos. This situation changed drastically in the seventeenth century. In a short period mathematicians enormously expanded the realm of curves open to mathematical treatment. Through the new analytic geometry of Fermat and Descartes the collection of mathematical curves came to include all algebraic curves, that is, all curves whose equation in rectilinear coordinates involves only the algebraic operations +, -, x, ÷ and $\sqrt[k]{\ }$, ($k > 1$, integer). Also the collection of transcendental curves, that is curves which do not admit an equation as above,was expanded. The cycloid appeared on the mathematical scene around 1630; the logarithmic curve in the 1660's. After that mathematicians encountered many more curves that depended algebraically on these two fundamental transcendentals. These curves occurred especially as solutions of inverse tangent problems.

The new curves, like the earlier known ones, occurred in mathematics in three different roles. They could be object of study, they could serve as means for the solution of a problem and they could themselves be the solution of a problem. Thus Pascal, in his famous challenge to mathematicians of 1658, proposed the cycloid as object of study and required the determination of its tangents, its area, the areas and centres of gravity of segments etc. An example of a new curve introduced as a means to solve problems is the third degree curve which Descartes discussed in his Géométrie, and which became known as the Cartesian parabola. The curve was used in the geometrical construction of the roots of 5th and 6th degree equations (cf. 3.4). Finally, the curve with modern equation

$$ae^{-y/a} = a - y + x \qquad (1;1)$$

forms and example of a (transcendental) curve which originated as the solution of a problem, namely the famous problem of Debeaune of 1638 (cf. note 26).

1.2 The tremendous increase in the number of curves confronted 17th century mathematicians with the problem how to introduce, describe or define new curves. In each of the three roles which the curves could have, object of study, means of solution and solution itself, the curves had to be or to become known. In the previous period this was no problem; all curves were already known to mathematicians and one could refer to any of them by its name (ellipse, conchoid, spiral etc.) and its basic parameters.

But when is a new curve sufficiently known? Seventeenth century mathematicians did not have a uniform definition of the concept of curve (nor does it seem that they felt the need for such a definition) and therefore there was no standard form of specifying the curves one had in mind. Indeed there were many ways to specify curves. One could, for instance, indicate how points on the curve could be constructed, one could describe a machine by which the curve could be traced, and, after the introduction of analytic geometry, one could give the equation of the curve. Some of these ways of describing curves were considered satisfactory, others less so, some not at all.

I shall use the term "representations of curves" for ways of specifying curves which were considered to make the curve in question sufficiently known. This term was not used in the 17th century in that meaning ; there was no general such term in that period. Mathematicians did use the terminology "construction of curves" which comes near to it but has a more restricted meaning.

The different ways in which curves were specified in 17th century mathematics, the preferences which mathematicians expressed for certain among these and the reasons given for these preferences form an important and interesting theme of historical study. It is important because these ways and preferences influenced the direction in which mathematics developped in that period. Historians of mathematics have up till now been little aware of this theme, mainly because a too rapid translation of 17th century mathematical argument in modern analytical symbolism has obscured these aspects of the treatment of curves. The subject has also a more general interest because it touches on a wider mathematical, or perhaps metamathematical, question, namely when is a mathematical entity known or when is a problem solved.

1.3 In the present study I shall in particular deal with the representation of curves in Descartes' Géométrie[1]. I intend to follow up the theme in one or two subsequent articles on the representation of curves in the works of later 17th century mathematicians.

For several reasons Descartes' Géométrie is the obvious starting point of a study on the representation of curves. It was this book that brought in one stroke all the algebraic curves into focus. But, as has been remarked [2] (with wonder) by historians of mathematics, Descartes did not consider the equation of a curve a sufficient representation of it. Hence it is of interest to study which representations he did find acceptable.

Moreover, Descartes introduced a sharp distinction between admissable and non admissable curves. The first he called "geometrical" the others "mechanical". The geometrical curves are what we now call the algebraical curves (although Descartes does not quite clearly state that in the Géométrie, it can be inferred from what he says), the mechanical curves are those which now are termed transcendental curves. But because Descartes did not consider the equation a sufficient representation of the curve he could not argue a sharp division between geometrical and non-geometrical curves on the basis of their equations; he had to argue this on the basis of such representations of curves as he did find acceptable. It is therefore important to study which these representations were.

Also Descartes' distinction between "geometrical" and "mechanical" curves was a serious issue in seventeenth century mathematics. The increasing interest in transcendental curves (curves therefore that to Descartes were not acceptable in geometry) forced mathematicians to take position with respect to the question in how far these curves could be considered geometrical or acceptable in general. Again this question could only be dealt with in terms of the representations of these curves and several of the representations used in these debates occur already in Descartes' Géométrie.

Finally, I have found that by considering the representation of curves in Descartes' Géométrie we can gain a better understanding of the structure of that book and of its underlying programme. This structure, and in particular the different roles of curves in the Géométrie and Descartes' different criteria for geometrical acceptability of curves, have, I think, not yet been satisfactorily untangled [3]. The consideration of the representation of curves offers a fruitful way to understand these aspects of Descartes' great contribution to geometry and algebra.

2. *The problem of Pappus*

2.1 Descartes expounded in his Géométrie a new programme for dealing with geometrical problems. In explaining his programme he used one problem as key example : the problem of Pappus. I shall explain Descartes' programme in section 3, but before doing so I shall discuss the Pappus problem and Descartes' solution of it. That discussion may serve as an explanation of the sort of geometrical problems for which Descartes presented a new programme, and of the roles of geometrical constructions, curves and algebraic calculations in that programme.

In explaining the Pappus problem [4)] I shall use symbols for the elements (lines, distances, numbers) in the problem. Descartes presents the problem in prose and with reference to drawings. Let (see figure 1)

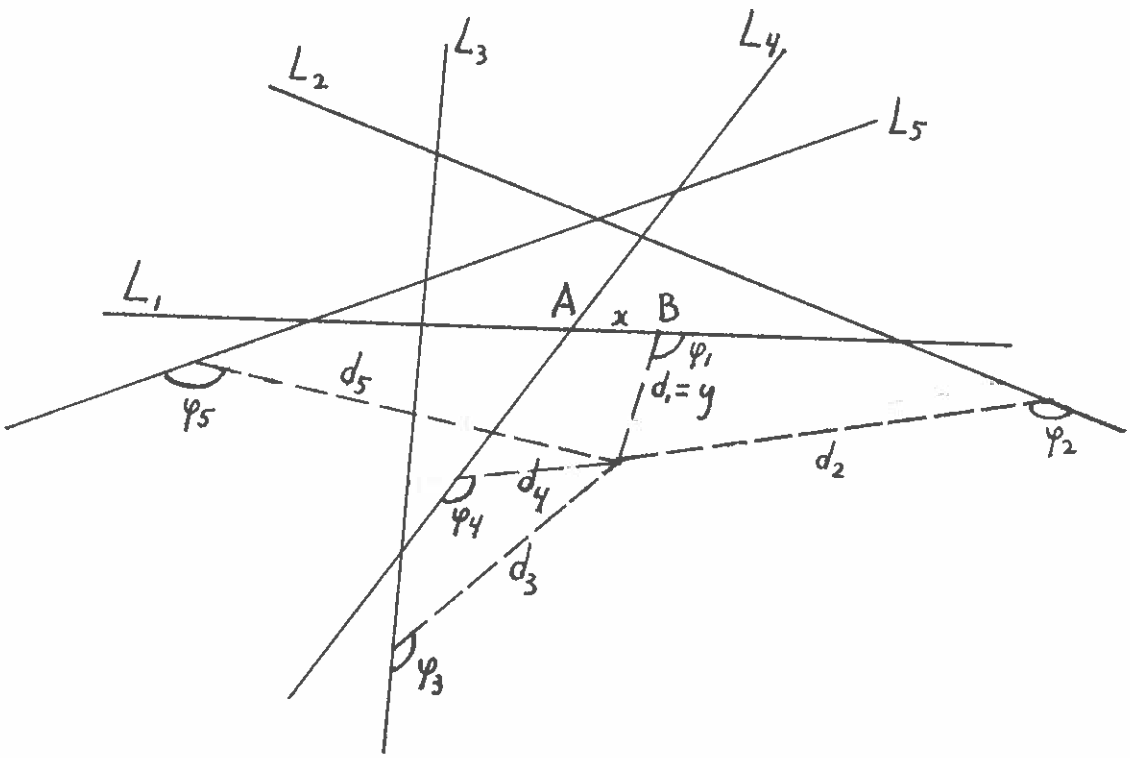


figure 1

a number of lines L_i be given in position in the plane. To each line

L_i corresponds a fixed angle φ_i. For a point P in the plane let $d_{P,i}$, or d_i for short, denote the length of the linesegment from P to L_i which makes an angle of φ_i with L_i. (In the case that φ_i is 90^o, d_i is the distance to L_i). Let $\alpha : \beta$ be a given ratio and a a given line segment. It is required to find points with the following property :

in the case of three lines :

$$(d_1.d_2):(d_3^2) = \alpha : \beta \qquad (2;1)$$

in the case of four lines :

$$(d_1.d_2):(d_3.d_4) = \alpha : \beta \qquad (2;2)$$

in the case of an uneven number (2n-1) of lines, $n > 2$:

$$(d_1...d_n):(d_{n+1}...d_{2n-1}.a) = \alpha : \beta \qquad (2;3)$$

and in the case of an even number (2n) of lines, $n > 2$:

$$(d_1...d_n):(d_{n+1}...d_{2n}) = \alpha : \beta \qquad (2;4)$$

Pappus gives the problem for three and four lines as well as its generalization to more lines[4]. The problem is a so-called locus problem: in each case there are infinitely many point which satisfy the condition, these points form a locus in the plane; this locus is, in general, a curve. Pappus states that in the cases of three and four lines the locus is a conic section and that in the cases of more than four lines nothing is known about form of the locus.

2.2 Descartes sketches the general solution of the problem at the end of the first book of the <u>Géométrie</u> (pp.309-314). His method is as follows. He sets

$$d_1 = y \qquad (2;5)$$

and he takes x to be the distance along L_1 from a fixed point A to the intersection of d_1 with L_1. He then shows by simple geometrical arguments that all d_i can be expressed linearly in x and y :

$$d_i = a_i x + b_i y + c_i \qquad (2;7)$$

He notes that in the exceptional case when all lines are parallel, the x does not occur in the expressions for the d_i.

He then remarks that the products $d_1 \ldots d_n$, $d_{n+1} \ldots d_{2n}$ and $d_{n+1} \ldots d_{2n-1} . a$ become expressions in x and y of degree at most n. The conditions (2;1) - (2;4) can therefore be rewritten as equations. In the case of n lines the equation will be of degree at most n. In the case of n-1 lines the choice of $d_1 = y$ and the occurrence of a in the second product of lines implies that x occurs at most to the power n-1, so that in that case the equation is of degree at most n, but the highest power of x is at most n-1. This does not apply to the case of three lines, because there the problem has an exceptional form. Finally the cases of 2n and 2n-1 parallel lines lead to equations in one unknown, namely y, of degree at most n; the locus in that case consists of a number of lines parallel to the given lines.

Descartes then turns to the question of how the points satisfying the requirements of the problem (the points on the locus) can be constructed. His idea is to choose arbitrary values for y and then to construct geometrically the corresponding values for x, In this way arbitrarily many points on the locus can be constructed. In section 6 I shall discuss this type of pointwise constructions in more detail. Descartes remarks that for any chosen value of y, the corresponding x's are the roots of an equation of which the degree is, in the case of 2n lines, at most n, and in the case of 2n-1 lines, at most n-1. The case of three lines leads in general to an equation of degree 2. The exceptional case of 2n-1 parallel lines leads directly to an equation in y of degree n.

So the problem is reduced to the geometrical construction of roots of equations. Now Descartes anticipates on results which he is to explain in the third book of the Géométrie. These results are: The roots of second degree equation can be constructed by ruler and compass. The roots of third and fourth degree equations can be constructed by the intersection of conics, in particular the intersection of a parabola and a circle. The roots of fifth and sixth degree equation can in general not be constructed by the intersection of conics; more complex curves have to be used in that case. It is possible to construct these roots by the intersection of a circle with a certain third degree curve, namely the "Cartesian parabola".

Based on these results Descartes gives, at the end of the first book, the following classification of the cases of the Pappus problems (G pp.313-14) :

a) 3,4 or 5 lines, but not 5 parallel lines :
the equation in x is of degree $\leqslant 2$ and therefore points on the locus can always be constructed with ruler and compass.

b) 5 parallel lines, 6,7,8 or 9 lines, but not 9 parallel lines :
the equation in x (or for 5 parallel lines, in y) is of degree $\leqslant 4$ and therefore points on the locus can always be constructed by means of intersections of conics; in some cases, construction by ruler and compass only may be possible (namely if the equations happens to be of degree $\leqslant 2$ or if they are reducible to such equations).

c) 9 parallel lines, 10,11,12,13 lines but not 13 parallel lines :
the equation in x (or in y in the case of 9 parallel lines) is of degree $\leqslant 6$, the construction by means of intersection of conic sections will in general not be possible and a more complicated curve has to be used.

d) etc.

2.3 This classification concerns the constructability of the locus. Descartes returns to the Pappus problem in the second book. He there gives another classification, now according to what he calls the "genre" of the locus; I shall translate "genre" with "class".This related to a classification of curves according to the degree of their equations, which Descartes explains in the second book (pp.319-323). The first class contains the curves with equations of the second degree: the circle, the parabola, the hyperbola and the ellipse. The second class contains the curves with equations of degree 3 and 4; the third class those with equations of degree 5 and 6 and so forth. I shall return to this classification in section 3. It leads to the following classification of the cases of the Pappus problem (G pp.323-324) :

a') 3 or 4 lines :
The equations of degree at most 2; the locus is of the first class.

b') 5,6,7 or 8 lines :
The equation is of degree at most 4; the locus is of the second class or, in exceptional cases, of the first.

c') 9,10,11 or 12 lines :

The equation is of degree at most 6; the locus is of the third class or of a lower class in exceptional cases.

d') etc.

In this connection Descartes states that all equations can occur as equations for the locus of some Pappus problem :

> And because the position of the given lines can vary in all sorts of ways, and thereby change the given quantities and the signs + and - of the equations in all imaginable ways, it is evident that there is no curved line of the first class which would not be of use in this problem if it is proposed in four straight lines, nor one from the second which would not be of use if it is proposed in eight, nor from the third when it is proposed in twelve, and likewise with the others. So that there is no curved line which is subject to calculation and which can be accepted in geometry, which is not of use for some number of lines.(G p. 324)

The statement is incorrect[5]. But it is important in Descartes' further classification of curves; I shall return to it in section 9.

Descartes then gives a complete solution (G pp.324-334) of the Pappus problem in three and four lines, calculating the equations explicitly and discussing the positions of the resulting conics in the plane. This section is well known[6] and it is not important for my present subject so I shall not discuss it here.

Finally he treats special cases of the five line locus problem, namely when $L_1,\ldots,L_4$ are parallel and L_5 perpendicular to them. Descartes considers first the problem (G pp.335-339)

$$d_1 \cdot d_2 \cdot d_3 = d_4 \cdot d_5 \cdot a \qquad (2;6)$$

and finds that the locus is the "Cartesian parabola" (see section 5.2). Then he considers (G p.339)

$$d_1 \cdot d_2 \cdot a = d_3 \cdot d_4 \cdot d_5 \qquad (2;7)$$

and he gives a rather complicated prose description of the locus in that case (see also section 8.1).

2.4 Descartes' solution of the Pappus problem illustrates well the different roles of curves in solving locus problems : curves can occur as locus; they can also occur as the means to construct points on the locus. Descartes treats the curves in quite different ways according to the role they have. Consider for instance the conic sections. They occur as locus of the solutions of the 3 and 4 line locus problem. Points on them can be constructed by ruler and compass. Descartes considers this an adequate solution,for he states that this case the problem is "plane" which means that it can be solved by ruler and compass. So if a conic occurs as a locus it can be constructed, pointwise, by ruler and compass.

Now consider the case that a conic is used as a means of construction. This occurs in the 6,7,8 and 9 line locus problem, where points on the locus are constructed by means of intersections if conics with circles and straight lines. Apparently the conics involved here cannot be constructed by ruler and compass because of that were acceptable the whole construction could be performed by ruler and compass, and that is what Descartes denies. Hence pointwise constructions for conics by ruler and compass are acceptable if the conics serve as locus but they are not acceptable if the conic serves as means of construction; in that case their construction has apparently to satisfy stronger criteria. The same applies to other curves than conics.

That this is so is understandable because if the conic is used as a means of construction, it is supposed that its intersection with circles, straight lines or other conics can be found. But if the conic is only given through a pointwise construction as in the locus case, the intersections with other lines cannot be determined. To illustrate this let (see figure 2) C_1 and C_2 be two conics whose intersections I and J we want to construct. Let AB be the axis of the x's and Y the direction of the y's. Points on C_1 and C_2 can be constructed by ruler and compass by taking arbitrary values for x and constructing the corresponding y's. But in that way we cannot precisely construct I and J; we can approximate them but their exact position

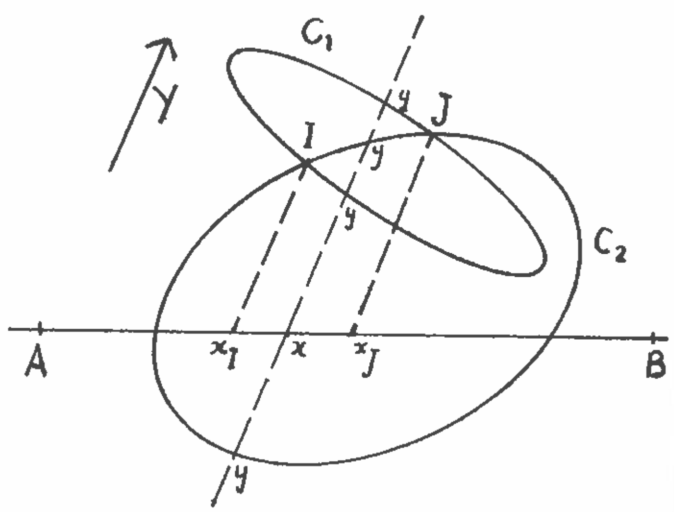


figure 2

would only occur if by accident we started our construction with x_I or x_J.

If a curve is used as means for construction it must be possible to find its intersection with other curves. A pointwise construction is not sufficient for that purpose. In stead it is natural to require a method to trace the curve by a continuous motion, so that the intersections with other lines are actually marked. We will see (sections 4 and 5) that the requirement that curves be traceable by continuous motion, a requirement induced by the use of curves as means for construction, plays a crucial role in Descartes' Géométrie.

3. *Descartes' Programme for Geometry.*

3.1 In his Géométrie Descartes presented an approach to the solution of geometrical problems which contrasted with earlier usages in so strong a way that one may speak here of a new paradigm. Before giving his own opinion on the programme of geometry Descartes explained how mathematicians before him, especially the mathematicians from antiquity, had thought about this matter. He says (G pp. 315-317) that, traditionally, geometrical problems had to be solved by ruler and compass. However, classical mathematicians had already encountered problems which could not be solved in this way. They had solved them by means of intersections of conics or even more complicated curves such as the conchoid. But they called these curves mechanical, thereby expressing that they did not consider them genuinely geometrical. Descartes then speculates about the reasons which the ancients may have had for this, and rejects these reasons. His rendering of the classical arguments is oversimplified, if not inaccurate[7)], but as an introduction and contrast to Descartes' own view it serves very well.

Descartes' view can be summarized as follows: Construction of problems by ruler and compass is certainly simpler than, and therefore preferable over, construction by means of the intersection of more complex curves such as conics or higher curves. In the construction of problems one should always use the simplest possible curves

But this does not imply that more complex curves are less geometrical than the straight line and the circle, or that constructions by means of these curves are less geometrical than constructions by ruler and compass. If a problem can be constructed by the intersection of conics (or of more complex curves) and it cannot be constructed by simpler curves, then that construction is the appropriate one and it is not less geometrical than a construction by ruler and compass.

This vision of the geometrical procedure of construction of problems defines a programme, consisting of three parts.

First it has to be determined which curves are acceptable as genuinely geometrical means for the construction of problems. These curves should include the straight line, the circle, the conic sections and higher curves of ever increasing complexity. Secondly, it must be made clear on which criteria some curves will be considered simpler than others; this will induce a classification according to simplicity within the collection of geometrically acceptable curves. Finally a method has to be found by which, for each problem, the simplest possible curves can be found by which the problem can be constructed. This is essentially the programme which Descartes works out in his <u>Géométrie</u>.

3.2 The first point of the programme, differentiating between the curves which are acceptable in geometry and those that are not, has caused Descartes (and his successors) the greatest number of conceptual problems. As the question concerns curves, and in particular new, hitherto unknown curves, it was discussed in terms of the representations of these curves. Basically Descartes takes as geometrical curves those "which can be described by some regular motion" (<u>G</u>. p.369). But this is not a very clear criterion and Descartes also wants to include in the collection of geometrically acceptable curves all curves that may occur as locus solutions of problems like the Pappus problem. This means that in fact - although he never explicitly says so - Descartes wants to take all algebraic curves as geometrical. But to do so he would have to prove that all algebraic curves can be traced by a continous and geometrically acceptable motions, or that they can be traced by other means which are as geometrical as the tracing by continuous motion. In sections 4 - 9 I shall discuss how Descartes dealt with this very complex part of his programme.

Thus algebra, the algebraic equation of the curve, was the essential criterion in the first part of the programme, but it had, so to speak, to remain under the surface. Descartes could not just take as geometrical all curves that admit an algebraic equation, because then he would have no argument to defend that these curves were truly the only curves acceptable in geometry, in other words he would not be doing geometry.

3.3 In the second and third part of the programme algebra could be quite openly used and it formed the crucial tool. Descartes classified curves with regard to simplicity through a division in classes ("genres") according to the degrees of the algebraic equation of the curve. The first class consists of the curves with equations of degree 2. These are the conic sections; Descartes does not incorporate the straight lines in his classification. Curves with equations of degree 3 and 4 are of the second class; those of degree 5 and 6 of the third etc. (G p.319).

Descartes stresses elsewhere that in constructions we should always use curves of lowest possible class (G p. 371). He notes that within one class some curves may be simpler than the others in the sense that one cannot construct with them as complicated problems as with the others. He mentions the circle as example; it is of the first class but there are constructions that can be performed with the other curves of that class (the conic sections) but not with the circle. Descartes also mentions the conchoid as such an exceptional curve within the second class (G p.323).

Descartes gives as reason for taking two degrees together in one class that there is a general rule to reduce fourth degree problems to third degree ones, and sixth degree problems to fifth degree ones, etc. (G. p.323). It seems likely that in the case of fourth and third degree problems he had in mid Ferrari's rule for reducing fourth degree equations (in one unknown) to third degree ones. But there is no such rule for sixth and fifth degree equations, nor for higher degree ones, so in this case Descartes has made a rather rash extrapolation.[8)]

The classification may also have been induced by the methods for constructing roots of equations by the intersection of curves.

For the third and fourth degree equations the roots can be constructed by circle and parabola, those of fifth and sixth degree equations by circle and Cartesian parabola, and so, by introducing higher curves as means of construction one can construct roots of equations of two successive higher degrees. But there is a puzzling aspect here. Descartes presents his classification as a classification for curves serving as means of construction, whereas the argument about construction by intersection of curves would classify problems (in this case equations in one unknown) rather than constructing devices; indeed, for the constructing devices involved, parabola (degree 2), Cartesian parabola (degree 3) etc., the degree rises by single steps.

The classification, and especially Descartes' arguments about the subdivision within one class, involve a contradiction between algebraic criteria of simplicity (the form of the equation, in particular its degree) and geometrical criteria of simplicity (the use of the curve as a constructing device). The special role of the circle within the first class shows that the classification is not completely adequate to distinguish means of construction. I shall return to this contradiction in connection with pointwise constructions in section 10.

3.4 For the third part of the programme, to find the simplest geometrical construction of the solution of a given geometrical problem, the crucial tool again was algebra. A problem should be reduced to an equation in one unknown (G pp. 300-302). Then the roots of this equation should be constructed geometrically through the intersection of certain curves, which should be as "simple" as possible that is, of lowest possible class . The simplicity of the curves by which a problem can be solved determines to which class the problem belongs. Here Descartes conformed to classical usage and called problems plane if they can be solved by circles and straight lines, and solid if they also require a conic section. Descartes devoted most of the third book of the Géométrie to this point of the programme. He proved there that every third and fourth degree equation can be constructed by the intersection of a circle and a parabola, and every fifth and sixth degree equation by the intersection of a circle and a Cartesian parabola.

3.5 At the beginning of the third book of the Géométrie Descartes gave a succinct formulation of the programme which I have been describing. He writes there

> Although all curved lines which can be described by some regular movement must be admitted in geometry this is no to say that for the construction of any problem we may use indifferently the first one that occurs. We must always take care to choose the simplest through which the solution is possible. And it should be noted that by simplest curves one should not only understand those which can most easily be described, nor those which the construction or the proof of the proposed problem easier, but primarily those which are of the simplest class which can be used to determine the required quantity. (G p. 369'-370).

There follows an example of the construction of two mean proportionals between two given linesegments, by means of curves traced by a certain machine (which I shall discuss in section 5.1) Descartes remarks that this construction may well be the easiest possible construction and proved with great clarity, but it uses curves of a higher class than would be necessary and therefore

> ... it would be a mistake in Geometry not to use them (namely the curves of a simpler genre, HB). On the other hand it is also a mistake to try vainly to construct a problem by a simpler class of lines than the nature of the problem allows. (G p.371).

4. *The representation of curves in Descartes' Géométrie .*

4.1 Descartes deals with the fundamental question of his programme at the beginning of the second book of the Géométrie. He frames that question in the margin title as : "which are the curved lines that can be accepted in geometry" (G p.315). He criticises the classical mathematicians for having called certain curves used in geometrical constructions "mechanical" rather than "geometrical". Descartes says that the fact that such curves are described by certain machines does not make them less geometrical than the straight line and the circle, which, after all, are also traced by machines, namely by the ruler and the compass. Descartes does not want to impose such very strict requirements for geometrical curves; he accepts many more curves as geometrical :

> To trace all the curved lines which I want to introduce here, nothing else needs to be supposed than that two or several lines can be moved one by the other, and that their intersections mark other lines ... " (G p.316).

Such curves may be very complicated, but that needs not make them less geometrical :

> It seems very clear to me that if we consider, (as is customary), geometrical that which is precise and exact, and mechanical that which is not, and if we consider geometry as the science which furnishes a general knowledge of the measures of all bodies, we have no more right to exclude the more composite lines than the simpler ones, provided that one can imagine them as described by a continuous motion, or by several motions which follow each other, and of which the last ones are completely regulated by those which precede. For in this way one can always have an exact knowledge of their measure.(G p.316).

Descartes criterion, then, to accept curves as geometrical is that they can be traced by continuous motion. The tracing of the curve is basic for understanding its nature; Descartes significantly combines the word "tracing" with understanding and conceiving; he speaks about "ways to trace and conceive curved lines" (G p.319) and "to know and trace the line" (G p.307).

In view of Descartes' programme, which I have outlined in section 3, the fact that he considers curves primarily as traced by continuous motions generated by certain machines, means that he was confronted with a number of deep conceptual problems. These problems are the following :

a) There are certain curves, such as the spiral and the quadratrix which Descartes does not accept as geometrical but considers mechanical in the sense of inprecise and unexact. These curves, however, can be traced by a continous motion. Descartes had therefore to specify which motions he accepts and which he rejects.

b) In the course of his studies Descartes came accross several curves which he could not, or would not, present as traced by some continuous motion. In stead he presented them as constructed pointwise or as traced by machinery involving strings. He had therefore to argue that such constructions or ways of tracing are equally acceptabel in geometry as tracing by continuous motion.
c) Moreover, pointwise constructions and tracing machinery involving strings can also be given for curves which Descartes did not accept in geometry. Therefore he had to specify which pointwise constructions and which tracing methods with strings were acceptable.
d) Finally, algebra was the crucial tool in Descartes' new programme for geometry, and the new curves he wanted to introduce had to be amenable to algebraic treatment, that is, they had to have an algebraic equation. Thus Descartes had to consider the question whether his new curves had such equations and conversely whether equations resulting from the use of the algebraical methods would always correspond to geometrically acceptable curves.

The representation of curves is central in these questions. The arguments about the acceptability of curves can only be formulated in terms of the representation of the curves, and the discussion is so complex because three different methods of representation play a role in it : representation by specifying the continuous motion which traces the curve, representation by the method to construct points on the curve, and representation by specification of a tracing machinery involving strings. To these three one may add the fourth method of representation of a curve by its equation. However, the whole complex of questions occurs precisely because Descartes did not consider that last representation as sufficiently geometrical.

In the following sections I shall record Descartes' arguments with respect to the four problems mentioned above.

5. *Curves described by continuous motion*

5.1 To trace the curves that are acceptable in geometry, Descartes stated, "nothing else needs to be supposed than that two or several lines can be moved one by the other and that their intersections mark other lines". (G p.316). In the second book of the Géométrie he illustrates the kinds of motions he has in mind here by two examples.

The first example concerns the famous instrument of figure 3 (G p.318 and p. 370).

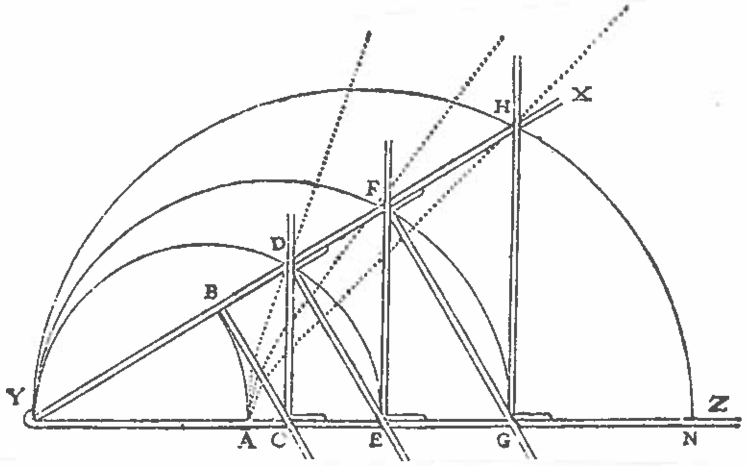


figure 3

It is a system of linked rulers. The rulers YX and YZ are connected in Y by a pivot. Ruler BC is fixed to YX in B. The rulers CD, EF and GH are made in such a way that they can slide along YZ while keeping perpendicular to it. Similarly the rulers DE and FG slide along YX while keeping perpendicular to it. At the beginning of the motion of the instrument angle XYZ is supposed to be zero and all the rulers coincide in point A. Now the angle XYZ is opended by keeping YZ fixed and rotating YX. Ruler BC pushes CD outwards, CD pushes DE, DE pushes EF etc. The point B (fixed on YX) describes a circle; the points D, F and H, sliding along YX, describe other curves dotted in the figure. Descartes argues that these curves, although described by ever more complicated combinations of motions, should all be accepted in geometry :

> "The latter are subsequently more composite than the first, and this more composite than the circle. But I do not see what could prevent us to conceive the description of the first (i.e. the curve described by D) as clearly and distinctly as that of the circle, or at least as that of the conic sections, nor what could prevent us to conceive the second one and the third one and all the others, which one can describe equally well as the first one; nor therefore what could prevent us to accept all these curves in the same manner to serve the speculations of geometry." (G p.318-19) .

The instrument of figure 3 occurs already in very early studies of Descartes; I shall return to its use and origin in section 10. Here it should be noted that Descartes' discussion of the instrument in this passage does not serve primarily to explain which motions he had in mind for the tracing of acceptable curves; he goes on to add another example which explains that better. Rather, the instrument serves to show that, however composite a motion is, the resulting curve can be conceived in a clear and distinct way, and is therefore acceptable in geometry. The instrument is a precise illustration of the description, quoted above, of acceptable tracing motions :

> " a continuous motion, or(-) several motions which follow each other and of which the last ones are completely regulated by those which precede." (G p.316).

Here the first motion is the rotating motion of the rulers YX and BC, the subsequent motions are those of the rulers CD, DE, EF etc., BC regulates the motion of CD, CD that of DE and so forth.

The text, especially the use of the key words clear and distinct ("nettement", "distinctement" G p.318) shows that Descartes saw a parallel between the series of interdependent motions in the machine, all regulated by the first one, and the "long chains of reasoning" in mathematics, discussed in the Discours de la Methode, which, as long each step in the argument is clear, yield results as clear and certain as their starting point.[9)]

5.2 But the example of the instrument of figure 3 does not cover all the combinations of motions which Descartes had in mind, because it involves only straight lines as moving parts. When Descartes wrote "nothing else needs to be supposed than that two or several lines can be moved the one by the other, and that their intersections mark other lines", he had also moving curved lines in mind. This becomes clear in the second example of a tracing instrument which Descartes gives. He uses this example to explain that every curved line traced by such a continuous motion has an equation, and that in general the curve traced by the instrument is more complicated than the curve which is used in the instrument. The example is as follows (G. pp.319 sqq., see figure 4):

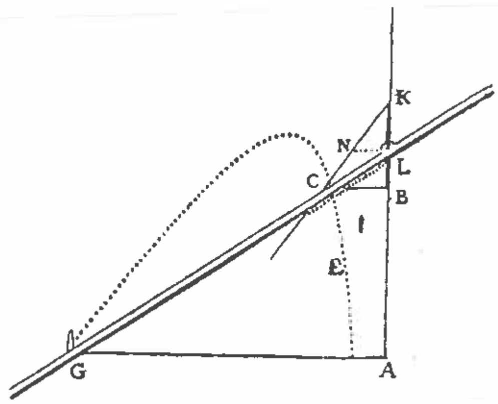


figure 4

A ruler GL is pivoted in G. It is linked in L with a device NKL which is movable along vertical axis while keeping the direction of the line KN constant. When L is moved along the vertical axis the ruler turns around G and the line KN is moved downward, staying parallel to itself. The intersection C of these two moving straight lines described the curve GCE. Descartes derives the equation of this curve,

$$y^2 = cy - \frac{c}{b}xy + ay - ac \qquad (5;1)$$

(where GA = a, KL = b, NL = c, CB = y and AB = x) and concludes that it is a curve of the first class; he adds that it is indeed a hyperbola. Thus the straight line KN in the machine produces a curve of the first class.

Next Descartes asserts (G p. 322) that if the straight line in the machine is replaced by a curve of the first class, the resulting curve will be of the second class. He mentions the case that KN is a circle with centre L; the resulting curve will then be the conchoid of Nicomedes. (Indeed that curve (see figure 5) has the property that on all the lines through a fixed point (G) the intercepts between he curve and an axis (KA) are equal.) The conchoid is of a higher class than the conics.

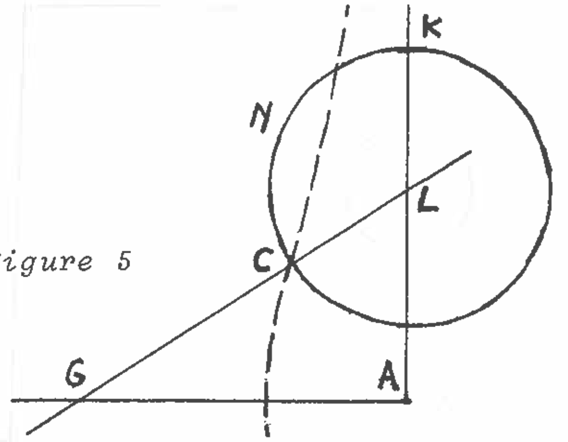


figure 5

Then Descartes replaces the circle by a parabola (see figure 6), and states that the resulting curve will be the "first and simplest curve for the problem of Pappus if there are only five lines given in

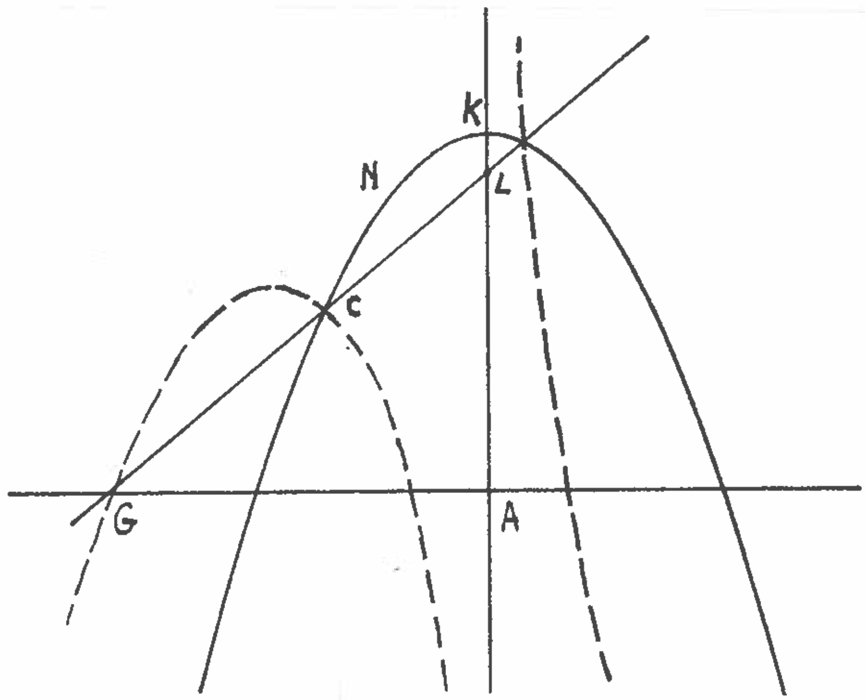


figure 6

position" (G p.322). This curve plays a central role in Descartes Géométrie, it became later known as the "Cartesian parabola".[10]) Later on in the second book (G 335-337) Descartes showed that it is the solution of the five line locus (cf.2.1)

$$d_1.d_2.d_3 = d_4.d_5.a, \qquad (5;2)$$

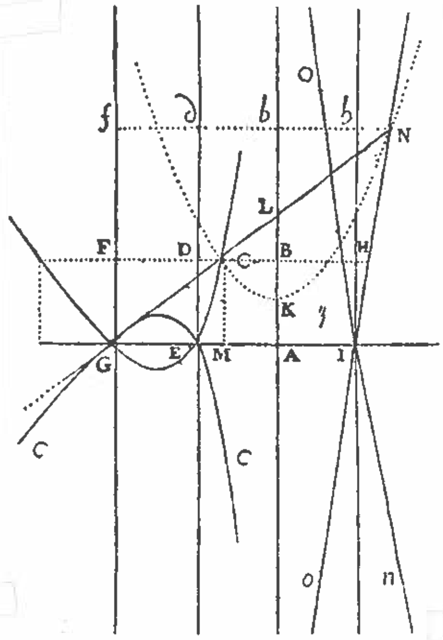
if L_1, L_2, L_3 and L_4 are equidistant and parallel and L_5 is perpendicular to the other lines; a is the distance between L_1 and L_2 and all the distances d_i are taken perpendicular to L_i. He gives the equation of the curve as

$$y^3-2ay^2-a^2y+2a^3 = axy \qquad (5;3)$$

(See figure 7, G p.336. The lines L_i are GF, ED, IH, AB and GA respectively. The distances are taken perpendicular to the lines, d_1 = CF, d_2 = CD, d_3 = CH, d_4 = CB = y, d_5 = CM = x. GL in the ruler moving around G. CKN is the parabola moving vertically along its axis AB. GEC is the branch of the "Cartesian parabola" described by C, NIO is the branch described by the other intersection N, cGc and oIn are the branches of the other "Cartesian parabola" which occurs if the distance to L_5 is taken positive in the other direction.)

In the third book Descartes explains how this curve can be used in finding the roots of a sixth degree equation (G p.403); he repeats there in more detail the way to trace the curve through the motion of a ruler and a parabola. In the passage from the second book where he introduces this curve Descartes says that it is of the second class. Furthermore he claims that if a curve of the second class is used in the tracing, the resulting curve will be one of the third class etc. (G p.332). He does not prove this.

figure 7



5.3 The linkage machines and the device of a moving curve whose intersection with a ruler traces new curves, are the examples which Descartes gives to illustrate his concept of tracing curves through combinations of motions. It is a fundamental concept because Descartes states that he will only introduce new curves traceable in this way. That means that there are other curves which he will not accept as geometrical because they cannot be traced in this way. Descartes mentiones the spiral and the quadratrix as examples of such curves. However, both the spiral and the quadratrix can be traced by a combination of continuous motions, indeed they were defined in such a way. Descartes had therefore to precisize a further requirement for the motions in order to rule out these curves. Before discussing this requirement I shall indicate the ways how the two mentioned curves can be traced by continuous motions.

The Archimedean spiral (see figure 8) is described by two motions, one rotatory motion of a ruler OR which turns uniformly

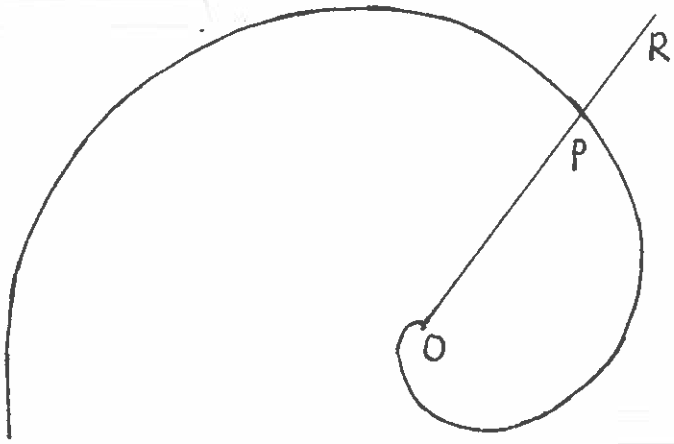


figure 8

around O, and one rectilinear motion of a point P which moves uniformly along the ruler OR. The point P traces the spiral.

The quadratrix[12] (see figure 9) can be described by a combination of separate motions, namely again one rotatory motion of a ruler OA which turns uniformly around A from position OA to position OC, and one rectilinear motion of a ruler PQ which moves uniformly downwards from position AB to position OC, and this in the same time that OR turns from OA to OC. The intersection S of both rulers traces the quadratrix ASD. The construction implies that during the motion always

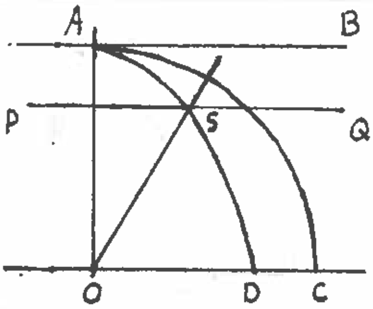


figure 9

$$AP : \overset{\frown}{AR} = AO : \overset{\frown}{AC}.$$

Descartes says about the spiral, the quadratrix and similar curves, that "they are conceived as described by two separate movements, between which there is no relation (" raport") that can be measured exactly" , and that for that reason they "really only belong to Mechanics".(G p.317) The absence of a measurable "raport" is the essential point here for in both cases the two movements could in principle be linked in such a way that the one determines the other, namely by a string mechanism which I shall discuss in section 7. Considering the methods to trace the quadratrix and the spiral we may conclude that Descartes, in speaking about the measures of the motions, means their velocities and that these measures have no exactly measurable "raport" because the comparison of the velocities involves the comparison of the lenghts of straight and curved lines, in particular the ratio AO : $\overset{\frown}{AC}$ of the radius of a circle to the quarter arc of that circle. The argument returns some pages later in the Géométrie in connection with the tracing of curves by machines involving strings (see 7). There Descartes writes:

> "... the proportion between straight lines and curves is not known and I even believe that it can never be known by man."

(G p.340). Thus the separation between geometrical and non-geometrical curves, which is fundamental in Descartes' vision of geometry, rests ultimately on his conviction that proportions between curved and straight length cannot be found exactly. This, in fact, was an old and in Descartes' time generally accepted doctrine, going back to Aristotle.[13)] The central role of the incomparability of straight and curved in Descartes' geometry explains why the first rectifications of algebraic (i.e. for Descartes geometrical) curves[14)] in the late 1650's were so revolutionnary: they undermined a cornerstone of the building of Descartes' geometry.

5.4 These, then, are Descartes' arguments about the tracing of curves by continuous motion. The possibility of an acceptable way of tracing the curve constitutes the fundamental criterion for accepting the curve as geometrical. Obviously this criterion is connected with the use of the curve as means of construction (see 2.4 and 10); the intersections of the curve with other lines can be considered constructable only in the case that the curve is actually traced. But in the Géométrie Descartes also accepts other ways to represent curves. I shall discuss these, and their relation to tracing by continuous motion, in the next sections.

6. Pointwise constructions for curves

6.1 As we have seen (section 2.2), Descartes' solution of the Pappus problem consisted of a method to construct arbitrarily many solutions, that is, arbitrarily many points on the locus. The method was: first to derive the equation of the locus in indeterminates x and y; then to chose an arbitrary value η for y and to form the equation in one unknown for the corresponding value or values of x; then to solve this equation geometrically, that is to construct the root or roots ξ; and finally to construct the point or points with coordinates ξ,η on the locus. By repeating this process with other values for y arbitrarily many solutions or points on the locus can be found. However, it is not at all obvious that this construction may be considered as a satisfactory construction for the whole curve which forms the locus. It is not a construction by continuous motion. The process yields only a finite number of points on the curve. And it is, in general, not possible to determine with this construction the intersection of the locus with some given curve.

Descartes' discussion of the Pappus problem in the first book of the Géométrie leaves the question open whether this pointwise construction can be considered as a construction of the locus as curve. In the case of the three and four line locus, where the locus is a conic, Descartes does more than giving the pointwise construction, he indicates how in each case the position of the vertices,axes, latus rectum and latus transversum[15] can be found, thus giving a representation of the locus curve by naming it (ellipse, hyperbola etc.) and giving its basic parameters. (G pp. 327-332). However, later on in the second book Descartes returns to pointwise constructions of curves and states that in certain cases curves constructed pointwise should be accepted in geometry.

The occasion for him to do so is the five line locus. Descartes solves it for two special cases. He considers (G pp.335-339) four equidistant parallel lines $L_1,\ldots,L_4$ and one perpendicular L_5. In the case

$$d_1 . d_2 . d_3 . = d_4 . d_5 . a \qquad (6;1)$$

(where a is the distance between L_1 and L_2) Descartes finds, as I have discussed in section 5.2 that the locus is the "Cartesian parabola" which he had introduced earlier as the curve described by a combined motion of a ruler and a parabola. For the case

$$a.d_1.d_2 = d_3.d_4.d_5 \qquad (6;2)$$

Descartes also gives an explicit solution, be it in a very complicated formulation which does not give a description of the locus by continuous motion, but rather a property of it,[16] from which, at most, a pointwise construction can be derived. (G p.339) He then decides not to give more details because he has already indicated in the first book how, in general, points on the locus can be constructed:

> As to the lines serving in the other cases, I shall not bother to distinguish them into different kinds, for I have not undertaken to say everything. And now that I have explained the way to find an infinity of points through which they pass, I think that I have sufficiently given the way to describe them. (G p.339)

Thus Descartes states that pointwise constructions are sufficient to describe the curves.

6.2 Descartes says more about the acceptability of these constructions in the next section, of which the margin title is :

> Which are the curved lines that one describes by finding many of their points and that can be accepted in geometry. (G p.340)

As that title indicates, Descartes accepts pointwise constructions under certain conditions as sufficient constructions for curves. As in the case of tracing curves by continuous motion, these conditions must be exclude the curves such as the spiral and the quadratrix. Indeed there are, as Descartes says, pointwise constructions for these curves as well. Descartes probably had in mind here the following pointwise construction for the quadratrix[17] (see figure 10). Divide arc AC in 2, 4, 8, 16 etc. parts (this can be done by ruler and compass) and do the same with the radius OA. Then draw radii as OR to the divisionpoints on ARC and draw horizontals as TS through the divisionpoints of OA. The intersections as S of corresponding radii and horizontals are on the quadratrix. In this way

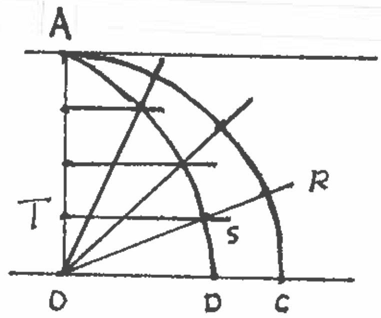


figure 10

arbitrarily many points on the quadratrix, lying arbitrarily close to each other can be geometrically constructed. For the spiral there is a similar construction.

These pointwise constructions then, Descartes has to exclude and so he has to explain the difference between these, unacceptable constructions and the acceptabel pointwise constructions of, for instance, the loci for the Pappus problem. According to Descartes the difference lies in the fact that for curves such as the quadratrix the constructable points are special points (for the quadratrix those with ordinate $\frac{k}{2^n}.OA$). In the case of the acceptable pointwise constructions every point is in principle constructable because the construction may start from every given value of one of the coordinates. Descartes explains this as follows:

> It is worthy of note that there is a great difference between this method of finding several points to trace a curved line, and that used for the spiral and similar curves. For by the latter one does not find indifferently all points of the required curve, but only those points which can be determined by a simpler measure than that required for the composition of the curve. Therefore, strictly speaking, one does not find any one of its points, that is, not one of those which are so properly points of the curve that they cannot be found except by means of it. On the other hand there is no point on the curves which serve for the proposed problem [the Pappus problem H.B.] that could not occur among those which are determined by the method explained above. And because this method of tracing a curved line by finding a number of its points taken at random is only applicable to curves that can also be described by a regular and continuous motion, one may not exclude it entirely from geometry. (G pp.339-340).

Thus Descartes states firmly, be it without any attempt at proof, that curves admitting a pointwise construction in which every point on it can, in principle, be constructed, can also be traced by continuous motion and are therefore geometrical. The passage suggests that Descartes saw a correspondence between the complete arbitrariness of the constructed points on the curve and the continuity of the motion.

6.3 After this passage Descartes repeatedly uses pointwise construction in the same way as tracing by continuous motion, namely to represent a curve. For instance he introduces the famous ovals[18], which are curves with certain optical properties, through a pointwise construction. As an illustration of such a representation by pointwise constructions I summarize Descartes' introduction of the first oval (G p.352, "this is the way how I describe them"): Let two lines (see figure 11) be given, intersecting in A under a given angle. A lies between the points F and G on the one line; the ratio of AF to AG is given. R lies on the other line, AG = AR. To construct points on the oval, take an arbitrary point K on AG. Draw a circle with centre F and radius FK. Draw KL perpendicular to AR. Draw a circle with centre G and radius RL. The two intersections of the two circles lie on the oval. By repeating this construction starting from other points K on AG, arbitrarily many points on the oval can be found. - The construction yields a geometrical curve, because the choice of K is completely arbitrary.

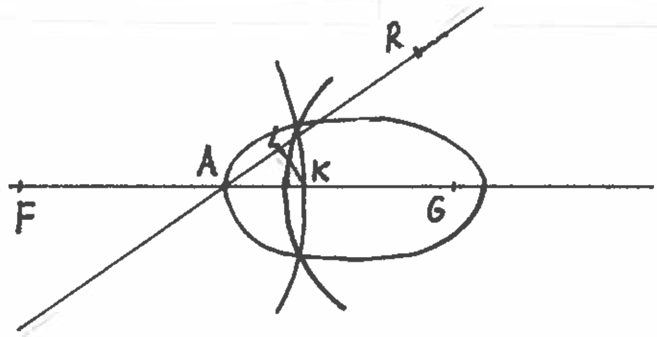


figure 11

7. String constructions for curves

7.1 The passage in the second book on the geometrical acceptability of curves given by pointwise constructions is followed by a passage on a third kind of representation of curves, namely tracing by machines which involve strings. The margin title of that section is:

> And which curves that one describes by means of a string can be accepted. (G p.340).

Descartes then refers to his Dioptrique[19], in which he has given such string-constructions for the ellipse and the hyperbola. For the ellipse it is the well known "gardeners construction" (see figure 12):

A cord is fixed in the points A and B. It is stretched by a tracing pin T which is moved around A and B, always keeping the cords straight. It then traces an ellipse with foci A and B (A.T[20] 6 p.166).

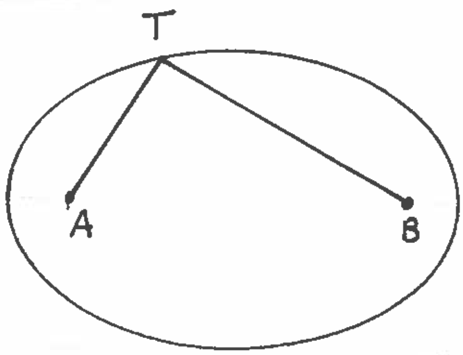


figure 12

For the hyperbola (see figure 13) a ruler AR is pivoted in A; a string is fixed at B and at point R on the ruler. The string is stretched by a tracing pin T which is kept against the ruler. By turning the ruler around A and keeping T to the ruler and AT stretched, T describes one arm of a hyperbola with foci A and B. (A.T. 6,p.176)

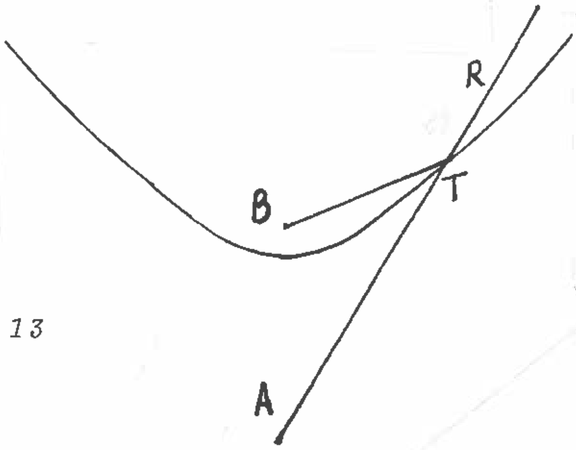


figure 13

It is noteworthy that in the Dioptrique Descartes calls this construction of the ellipse "rather rough and not very exact" (A.T. 6 p. 166) but thinks that it serves better than the section of a cone or a cylinder to understand the nature of an ellipse. In the case of the hyperbola Descartes pictures "a gardener who uses it to measure off the border of some flowerbed" (A.T. 6 p.176). Nevertheless, in the Géométrie Descartes accepts these constructions as genuinely geometrical representations of curves. This shows that he is not primarily concerned with the exactness of his constructions in practice but with their clarity and understandability in principle.

7.2 On the other hand, string constructions can also be used to trace curves which Descartes does not accept as geometrical. Descartes mentions this but does not give examples. He may have had in mind methods as the following to trace the Archimedean spiral (see figure 14). Consider a ruler AR pivoted in A. Around A there is a circular disk BC. In B a string is fixed to the disk; it is kept slung along the disk and the ruler (for instance by letting it pass through an eye fixed on the ruler in D.)At the end of the cord a tracing pin T is fixed. When now the ruler AR is moved clockwise around A, the pin T traces an Archimedean spiral. (such a construction is mentioned by Huygens[21]; it was certainly easy for Descartes to think out similar machinery to trace the quadratrix e.g.)

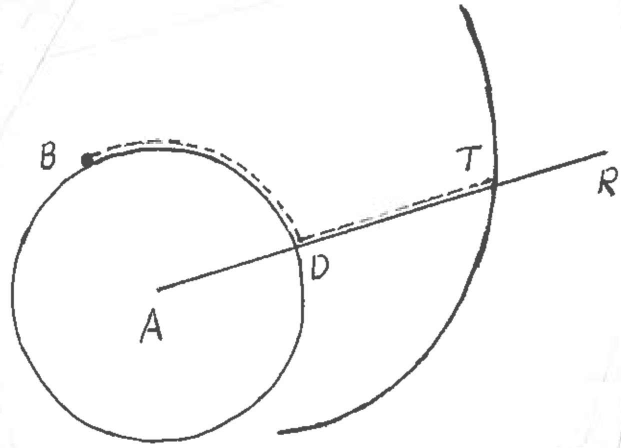


figure 14

Such use of strings in tracing curves Descartes had to exclude. He did so by excluding the cases in which the string is partly curved and partly straight and during the motion curved parts change into straight ones or vice versa. His reason for excluding this case is, as I have already discussed above (section 5.3), his conviction that ratios between straight and curved lines cannot be given exactly. Descartes argued as follows:

> Nor should we reject the method in which a string or a loop of thread is used to determine the equality or the difference of two or more straight lines which can be drawn from each point of the required curve to certain other points or toward certain other lines under certain angles. We have used this method in the Dioptrique to explain the ellipse and the hyperbola. It is true, though, that one cannot accept in geometry any lines which are like strings, that is, which are sometimes straight and sometimes curved, because the proportion between straight lines and curved lines is not known and I even believe that it can never be known by man, so that one cannot conclude anything exact and certain from it. Nevertheless, because in these constructions one uses cords only to determine straight lines whose lengths are perfectly known, this should not be a reason to reject them. (G pp. 340-341).

7.3 Further on in the Géométrie Descartes uses string constructions as an alternative to pointwise constructions in the representation of his ovals. To illustrate this I summarise the stringconstruction for the first oval (see figure 15) : FE is a ruler pivoted in F. A string is fixed in E on the ruler and in G on the axis FAG. It is slung around a pin K on the axis and it is kept straight by a tracing pin in C against the ruler. Thus the string is kept as E-C-K-C-G. Now the ruler is turned around F and in that motion the tracing pin C traces the oval. The points F, A, K and G on the axis can be chosen such that the oval has the required optical properties.

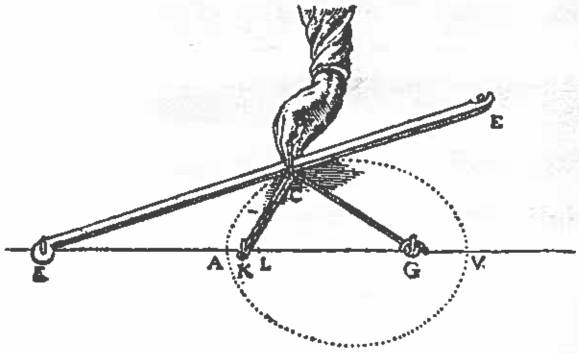


figure 15

8. *Curve equations in the Géométrie.*

8.1 We have seen how Descartes makes use of three different kinds of representation of curves: by tracing machines, by pointwise constructions and by tracing machines involving strings. In each case certain further conditions (which exclude the transcendental curves) should be satisfied in order that the resulting curves should be acceptible in geometry. In the first and the third kind of representations these conditions have to do with the axiom of the incommensurability of straight and curved, in the second with the randomness of the constructible points on the curve. It is noteworthy that Descartes does not try to connect these two types of condition. In fact they relate to different apsects of curve tracing by continuous motion. The incommensurability of straight and curved relates to the condition that the combined motions which trace the curve regulate each other in a measurable way (cf. section 4.1). The randomness of the constructable points relates to the continuity of the tracing motions (cf. section 6.2).

There remains the question of the role of equations as representations of curves: in how far did Descartes consider the equation a sufficient representation of a curve? Descartes is convinced that the equation of a curve incorporates all information on its properties; He writes :

> "Now if one knows the relation that all points of a curved line bear to all points of a straight line in the way I have explained (i.e. as soon as the equation is known H.B.) it is easy also to find the relation they have to all the other given points and lines; and subsequently to find the diameters, axes, centers and other lines or points to which each curve has some special or more simple relation than to others, and in that way to conceive various ways of describing the curves, and to choose the easiest."(G p.341).

The passage suggests that it needs still an effort to find the description of the curve from its equation, so that the equation itself is not an appropriate representation of the curve.

This is in agreement with the fact that nowhere in the Géométrie Descartes uses an equation to introduce or represent a curve. In several cases he treats curves without giving their equations, in other cases he gives the equation almost casually in the course of his arguments.The solution of the Pappus proplem in five lines of which four are parallel, equidistant and perpendicular to the fifth, (cf. sections 2.3 and 6.1)

$$a.d_1.d_2 = d_3.d_4.d_5 \qquad (8;1)$$

is given in book II in the form of a very complicated prose description of a defining property of the locus. The description could have been translated into an equation, and would certainly have been more informative in that way. Equations of the curves traced by the machine discussed in section 5.1 are not given in the Géométrie, nor does Descartes present the equations for the ovals. The Cartesian parabola, so fundamental in the Géometrie, is introduced in book II as the curve traced by the intersection of a parabola and a ruler. Its equation is given afterwards, and clearly not as a representation of the curve but as a means to prove that it solves the five line locus problem (G p.337), or to determine its tangents (G p.344). For readers to whom the description of the curve by ruler and parabola "seems difficult" Descartes adds a alternative representation not the equation but a pointwise construction (G. p.407).

The conclusion from these facts must be that for Descartes the equation of a curve was primarily a tool and not a means of definition or representation. It was part of a whole collection of algebraic tools which in the Géométrie he showed to be useful for the study of geometrical problems. The most important use of the equation was in classifying curves into classes and in determining normals to curves. Here the equation must actually be written out. In many other cases Descartes could get through his calculations about problems without explicitly writing down the equation of the curve.

9. *Geometrical curves*

9.1 Within his programme for geometry, Descartes did not, and could not simply state that geometrical curves are those which admit algebraic equations. Hence the question arises how Descartes saw the class of geometrical curves; did he really consider this class to be the same as the class of curves admitting algebraic equations and did he think that every such equation could occur as the equation for a geometrical curve? And was he aware of the extension of the class of curves which he decided to banish from geometry? In this section I shall deal with these questions.

Descartes states firmly that all geometrical curves have equations After explaining the curve tracing machine discussed in section 5.1 he wrote :

> "I could give here several other ways of tracing and conceiving curved lines, which would be ever more complicated by degrees to infinity. But to understand the totality of all curves that are in nature and to distinguish them orderedly in certain classes, I do not know a better way than to say that all points of those that can be called geometrical, that is those which admit some precise and exact measure, necessarily have some relation to all points of a straight line, which can be expressed by some equation, the same equation for all points." (G p.319)

He goes on to explain how, in the case of the tracing machines discussed in section 5.2 these equations can be found.

The converse question, namely whether all algebraic equations describe geometrical curves, is a much more difficult one and Descartes does not answer it explicitly. Taken in its strict sense the question is whether for every algebraic equation a tracing machine, or a combination of continuous motions in the sense explained in section 5, can be found that describe the curve having that equation. Descartes nowhere explicitly deals with that question. Still it is so fundamental a question in the whole Cartesian programme of geometry that it seems very unlikely that Descartes had not seen the question. Rather his silence on this question must be explained by his inability to answer it. That Descartes could not answer it is not surprising; it has been found only in the 19th century that the answer to the question is positive [21a].

Implicitly, Descartes' answer to the question is positive. An equation of a curve represents a pointwise construction; one takes successiveley fixed values for one of the variables, say for y, and constructs geometrically the corresponding values for x as the roots of the equation in x that results. That this can always be done follows from the third book of the Géométrie in which Descartes shows that the roots of equations (in one unknown) up to the sixth degree can be found by the intersection of geometrically acceptable curves, and in which he claims that the same can be done for higher degree equations. (G p.413, cf. section 3.4). This is how Descartes solves the Pappus problem and he even claimed that every equation can occur as the equation for the locus in a problem of Pappus in some number of lines. Hence algebraic equations yield pointwise constructions for the curves they describe and these constructions are acceptable in geometry because the choice of the starting point of the construction of the points (namely, the choice of the y) is completely free (see section 6.2). Moreover, Descartes claims that such pointwise constructions of curves are equivalent to tracing by continuous motion, and hence, implicitly, he claims that all algebraic curves are geometrical in the sense of being traceable by continuous motion.

It is clear that the crucial step in this argument is the equivalence of pointwise and continuous motion constructions. Through this equivalence curves described by equations, in particular curves occurring in locus problems, acquire a status in geometry equal to that of curves traced by continuous motion in particular curves used as means of construction. But we have seen that Descartes' arguments for the equivalence are weak (section 6.2). He must therefore have had strong reasons to incorporate it in his geometry. In section 10 I shall say something more on the reasons Descartes may have had for this and on some conclusions which may be drawn from these concerning the formation of his ideas in geometry in the years before the publication of the Géométrie.

9.2 As to the transcendental curves, it is noteworthy that Descartes' basic argument in rejecting them, the incommensurability of straight and curved lines, serves only in the case of transcendental curves depending on the quadrature of the circle, such as the quadratrix and the spiral, which are the only ones Descartes mentions explicitly. The question then arises how many transcendental curves did he know and, more importantly, did he know curves depending on logarithmic relations and by which arguments did he exclude these from geometry?

The idea that curves generated by motions not mutually subordinated are to be rejected from geometry occurs already in Descartes' letter to Beeckman of 26 march 1619[22] (see section 10.2), he mentions there the quadratrix as example. By that time he had hit upon the logarithmic relation also, namely in connection with the problem "de reditu redituum" (income on income, i.e. compound interest). In one study[23] he does not yet use a curve to represent the relationship between an axis divided in equal parts and one divided in proportional parts, but the idea of such a curve does seem to underlie his argument. In another study[24] he actually draws that curve, calls it the linea proportionum and recognizes it as one of the same class as the quadratrix :

> The line of proportions is to be put in the same class as the quadratrix, for it is generated by two motions not subjected to each other, one circular and one straight (A.T. 10 pp.222/223).

It is not clear how Descartes got the (wrong) idea that the line of proportions is generated by a combination of a straight and a circular motion ; perhaps he only referred to the quadratrix in mentioning these motions. The figure in the published text suggests that he had no clear idea about the form of the curve.

There is no evidence that Descartes actively studied other transcendental curves than the quadratrix and the spiral before 1637. But shortly after the publication of the Géométrie we find Descartes discussing the logarithmic spiral in a letter to Mersenne[25] and another logarithmic curve in connection with one of the problems set by Debeaune[26]. Around this time he also studied the cycloid.[27] In the case of Debeaune's problem Descartes does not explicitly recognize the connection of the curve with logarithms, although he may well have seen it. He works out two motions which together describe that curve and he finds that these two motions

> are so incommensurable that they cannot be regulated by each other in an exact way; and therefore that this line belongs to those which I have rejected from my Geometry as being only mechanical. (A.T. 2 p.517).

From the little information we have, then, it seems that before the publication of the Géométrie Descartes may have had the idea that by rejecting the quadratrix the spiral "and the like" (G p. 317) he did not reject many interesting curves and only those originating from motions which involve the relation between curved and straight lines. Shorly after 1637 he was confronted with several other "non geometrical" curves, some of them not obviously depending on the relation between curved and straight, and some of them indeed quite interesting.

10. *Once more: Descartes' programme*

10.1 We have seen that in Descartes' programme for geometry as expounded in the Géométrie there is a contradiction with respect to criteria for geometrical acceptability of curves. On the one hand Descartes claims that he only accepts curves as geometrical which can be traced by certain continuous motions. This requirement relates to curve tracing and the possibility to determine intersections of the curve with other curves; it is induced by the use of the curve as means of construction in geometry. On the other hand Descartes states that, under certain conditions, curves represented by pointwise constructions are truly geometrical. Pointwise construction are related to curve equations in the sense that an equation for a curve directly implies its pointwise construction. Pointwise construction occur primarily for curves that occur as solutions to locus problems.

The link between the two criteria, namely Descartes' argument why curves that are pointwise constructable can also be traced by continuous motion, is very weak. This makes the rôles of these two criteria all the more interesting. Descartes could not strictly keep to the continuous motion criterion because in that case certain curves (such as the locus solution of the general Pappus problem) which he wanted to accept in geometry, could not be proved to be geometrical. And of course he could not take pointwise constructability itself as criterion because pointwise construction of points on the curve is a construction by geometrical means, that is by the intersection of curves that are accepted as means of construction.

The question then arises: why did Descartes not cut this Gordian knot in the way which seems so obvious to us, namely by defining geometrical curves as those which admit algebraic equations ? Why did he not simply state that all such curves are acceptable means of construction, that their intersections can be found (which would then be an axiom), and that they are ordered as to simplicity by the degrees of their equations ? Descartes could have done so, and it would have removed the contradictions mentioned above. But he did not. In order to understand why we have to look at the development of Descartes' ideas on geometry.

10.2 In 1619 already Descartes had a programme for his geometrical research. We know it from his letter to Beeckman of 26 March 1619. He wrote there :

> I hope to prove (-) that certain problems can be solved with straight and circular lines only; that others can only be solved with other curved lines, but such that originate in one single motion, and that therefore can be traced by the new compasses, which I do not think are less certain and geometrical than the ordinary ones by which circles are drawn; and that finally other (problems) can be solved only by curved lines originating from different motions that are not subordinated to each other and that certainly are only imaginary (imaginariae); such is the well known quadratrix. I think that one cannot imagine problems that cannot be solved by at least these lines; but I hope that I shall come to demonstrate which questions can be solved by the first or the second method and not by the third; so that in geometry nothing would remain to be found. (A.T.10 pp. 157).

The passage shows that by 1619 Descartes had already formed the conception of geometry which he adhered to all his life. He does not consider geometry primarily as an axiomatic, deductively ordered corpus of knowledge about points, lines etc., but at the science of solving geometrical problems. Once all such problems can be solved nothing more needs to be found in geometry.

In referring to compasses Descartes had in mid here two linkage machines for solving problems or tracing curves. One of these we have met already in the Géométrie, it is the machine illustrated in figure 3. Its aim is to find mean proportionals between two given line segments. The similarity of the triangles involved yields immediately

$$YB : YC = YC : YD = YD : YE = YE : YF = YF : YG = YG : YH \quad (10;1)$$

Hence to find, for instance, two mean proportionals between YA and some given line λ , one opens the compass so far that YE = λ ; YC and YD are then the required proportionals. Alternatively, one traces first, by continuously opening the compass, the curve AD. Then when λ is given one intersects AD with the circle with diameter YE = λ. The intersection point is D; YC (the abscissa of D) and YB are the required proportionals. In the same way the curve AF traced by F serves to determine four mean proportionals.

The other compass was not incorporated in the Géométrie. In the same way as the preceeding compass is based on a very simple geometric device to find proportionals, this one employs a very simple device for dividing an angle in any given number of equal parts. For the trisection of the angle the machine is as in figure 16. There are four rulers AB, AC AD and AE, all pivotted in A. On each of them at fixed and equal distances from A, there are adjusted links FG, IH, KG, and LH, all equal in length. FG and KG are joined such that G can move along ruler AC ;

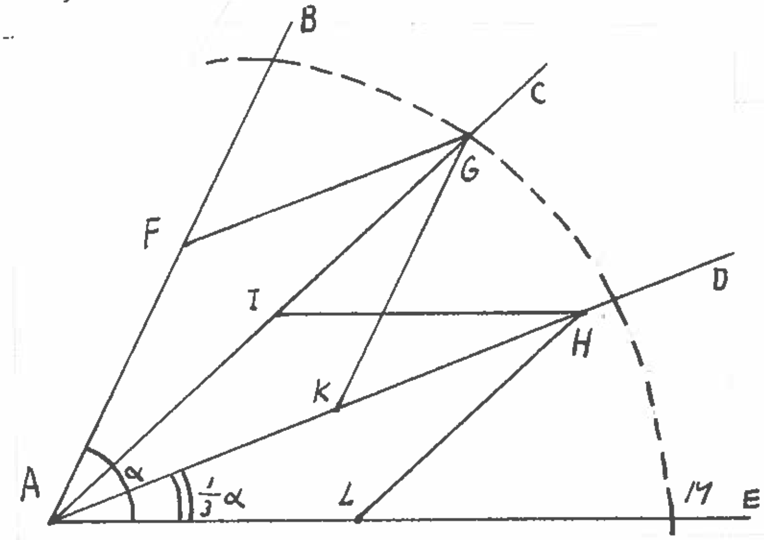


figure 16

IH and LH are joined such that H can move along ruler AD. If now a given angle α has to be trisected the compass is opened till ∠BAE is equal to α, then ∠BAC is equal to $\frac{1}{3}$ α. Alternatively (see figure 17) let the point G trace a curve MG by opening the compass while leaving ruler AE fixed. If then angle α has to be trisected one draws the angle in A with one leg along AE, choses a point F' on the other leg such that AF' = AF and draws a circle around F' with radius FG. Through the intersection G' of the circle and the curve one draws AG' . Then $\angle F'AG' = \frac{1}{3} \angle F'AE = \frac{1}{3}\alpha$.

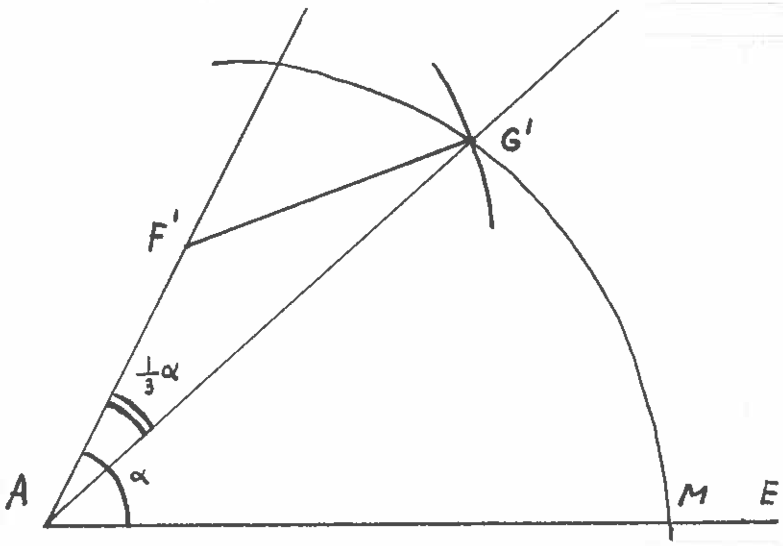


figure 17

Around this time Descartes had experimented with other such compasses, and in the letter to Beeckman he mentions that he has found the construction of all types of cubic equations by means of such compasses. The use of these compasses for various special constructions was by no means new, it occurs in classical mathematics and Pappus' Collections contain several such machinery. Also the idea to consider the curves traced by these machines is not new, the conchoid of Nicomedes, for instance, is a curve traced by a special kind of instrument for certain constructions, the so-called neusis constructions.[28] It is important, however, that in the formulation of his programme Descartes considers the curves themselves, rather than the compasses, as the means for constructions.

10.3 If we compare the programme which Descartes outlined in his 1619 letter to Beckman with the programme of the Géométrie we find significant differences. These differences precisely concern the role of algebra and pointwise constructions. We see that by 1619 Descartes' programme contained already the following ideas: Gêometry is the science of solving, or constructing geometrical problems. Construction by more complex means that the circle and the straight line (the compass and the ruler) need not be less geometrical. Curves that can be traced by one single continuous motion such as provided by the compasses are acceptable in geometry as means of construction. There are also problems that can be constructed only by curves traced by a combination of motions that are not subordinated to each other. Such curves are "imaginary", an example is the quadratrix. With such curves all problems can be solved but Descartes wants to classify the problems that can be solved by acceptable geometrical curves.

All these elements of the 1619 programme also occur in the Géométrie. But several points of the programme in the Géométrie are still lacking in 1619. Most notable is the absence of algebra. It is true that Descartes envisages geometrical solutions, through certain compasses, of algebraic equations, but algebra does not yet play a role in the classification of geometrical means of construction as to their simplicity, nor in a method to find the simplest possible constructions. It seems likely that by 1619 Descartes envisaged to classify the constructing curves as to the simplicity of the compasses involved in tracing them. An echo of this is found in the Géométrie where Descartes, in discussing the compass for mean proportionals, says

> I don't believe that there could be a more easy method to find as many mean proportionals as one wishes, nor one whose proof would be more evident, than to use the curved lines traced by the instrument XYZ... (G p. 370).

But, he goes on to say, the curves traced by that instrument are of a higher class than necessary and therefore they should not be used in a truly geometrical solution of the problem to find mean proportionals (G p. 371). We may conclude that by 1637 Descartes' algebraic criterion of simplicity of curves namely the class, defined through the degree of the equation, had replaced and indeed was in conflict with an earlier criterion of simplicity, namely the simplicity of the compass and the resulting proof of the construction.

The other element lacking in the 1619 programme concerns loci and pointwise constructions. In 1619 Descartes does not want to introduce new curves in geometry for other purposes than as means of construction. The problems to be constructed will have one (or a finite number of) solutions. In 1619 Descartes did not consider the case that the solutions are infinite in number forming as locus a curve which, by the nature of the process of solving the problem, is constructed pointwise. Hence the problem did not occur whether such curves should be accepted in geometry and if so according to which criteria.

10.4 Evidently, then, Descartes' programme of geometry has changed between 1619 and 1637. The stages of this change are in fact fairly well known and dateable.[29)] Probably shortly after the letter to Beeckman of march 1619 and before november 1620 Descartes studied the construction of problems through the intersection of conics and found the solution of all third and fourth degree equations through the intersection of a parabola and a circle. This must have given him the idea that the conics are the class of constructing curves immediately following the circle and the straight line.

This result may have led him to search for a construction of all fifth and sixth degree equation through the intersection of a circle with one special curve more complicate than the conics. He succeeded in finding this construction ; The curve is the Cartesian parabola, the construction is explained at the end of book 3 of the Géométrie (G pp.402-411, cf section 3.4). But we do not know the date of this discovery[30]. These results must have induced Descartes to consider the degree of the equation of the curve, rather than the simplicity of the tracing machine, as criterion for the geometrical simplicity of curves used in constructions.

The other new aspect, loci and pointwise constructions, entered Descartes' programme most probably in 1631 when Golius suggested him to try his hand at the problem of Pappus. We know that Descartes solved the problem in a number of weeks and that the solution appearing in the Géométrie is essentially the one he sent to Golius in January 1632[31]. This study must have turned Descartes' attention the more toward algebra, the equation as embodying all the information about the curve, the necessity to incorporate all curves admitting algebraic equations in geometry and the necessity to admit pointwise construction for curves.

10.5 However, more important than the chronology of these changes in the development of Descartes' geometrical ideas, is the fact that they explain the basic contradiction in Descartes' programme in the Géométrie. The programme of 1619 may have been impracticable, but it was consistent. It claimed to demarcate geometrical from non geometrical procedures and it did so by geometrical means, namely constructing machines that were generalizations of the ruler and the compass.

In the programme of 1637 algebra has become dominant. Descartes now classifies curves according to the degree of their equations and a great part of the Géométrie (especially the third book) is devoted to algebraical techniques concerning the roots and coefficients of equations (reduction of equations, sign rule, removal of terms from the equation, change of negative roots into positive ones, etc.). But despite all this algebra, what has remained is Descartes' conception of geometry as the science of solving geometrical problems by the construction of points through the intersection of curves. And so the main aim of the third book, is the construction of roots of equations through the intersection of curves.

This aim determines the structure of the third book and the nature of the algebraical techniques presented there. The reduction of equations to other equations of lower degree is necessary for finding the construction by the simplest possible constructing curves. The techniques concerning the roots and coefficients of the equation serve to reduce the equations to standard forms for which Descartes then gives standard constructions. For third and fourth degree equations this is the construction by the intersection of circle and parabola; for fifth and sixth degree equations the construction by the intersection of circle and Cartesian parabola.

Thus, although algebra has taken a dominant position in Descartes' programme of 1637, it is still the conception of the geometrical aim of the work which gives it structure and motivation.

10.6 Here lies the answer to the question raised in section 10.1, namely why Descartes kept the criterion of tracing by continuous motion for the geometrical curves, and why he did not simply define geometric curves to be those that have algebraic equations. As we have seen, the whole structure of his Géométrie depends on the conception of construction by the intersection of geometrical curves. For Descartes, these intersections are actually found or constructed only in the case that the curves can be traced by continuous motion. In that case one can conceive clearly and distinctly that the intersections are found. If he were to give up the criterion of tracing by continuous motion and at the same time keep to his programme of construction by the intersection of curves, he would have to state as an axiom that for all curves having an algebraic equation the intersections are given or constructible.

It is evident that Descartes could not do this. An axiom stating that the intersections of curves are constructible is by no means clearly and distinctly evident, so it would not satisfy Descartes' criterion for accepting a statement as basis for further argument.

Moreover, by adopting this approach Descartes would lose the claim that he was doing Geometry; he would be doing some kind of algebra. But that would mean giving up the principal aim of his work: to bring order in the science of geometry.

Finally, the whole structure of the Géométrie, determined by the aim to find the simplest constructing curves for a given problem, would lose much of its sense. If the intersections of all algebraic curves are by axiom constructible there is no evident reason for finding the simplest curves for a given problem, and hence there is not much sense in finding constructions for roots of equations. The roots of an equation $x^n+ax^{n-1}+\ldots = 0$ are the intersections of the curve $y = x^n+ax^{n-1}+\ldots$ with the straight line $y = 0$, and so they are given already as intersections of curves with algebraic equations.

So, in conclusion, we see that Descartes could not give up his continuous motion definition of geometrical curves because then he would have lost the claim of doing geometry and hence the rationale of the whole structure of his work.

11. Conclusion

11.1 As I hope to have shown, the representation of curves forms the key to understanding the structure of Descartes' Géométrie and its underlying programme or programmes. The structure and the programme involve contradictions, but there is a unity of vision behind it, which Descartes had already formed as early as 1619 and which found its clear expression in the Géométrie in 1637. The vision was that geometry can and should be structured, that the bewildering jumble of problems, methods and solutions, in which it is not clear where the problems end and the solutions begin, can and should be cleared up. The vision, in short, that geometry concerns a surveyable, orderable collection of well defined problems, well defined also in the sense that there are clear criteria of adequacy for their solutions. Descartes left it to his successors to work out the programme, to encounter its limitations and to come to terms with its contradictions.

Acknowledgement

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Notes

1. R. Descartes, _La Géométrie_, one of the essays in his _Discours de la méthode pour bien conduire sa raison, et chercher la vérité dans les sciences, plus la Dioptrique, les Meteores et la Geometrie qui sont des essais de cete methode_, Leiden, 1637. I shall refer to the _Géométrie_ in footnotes and references by the abbreviation _G_; I shall use the pagenumbers of the original edition (pp. 297-413). The original text is easily accessible in _The geometry of René Descartes with a facsimile of the first edition_ (tr. and ed. by D.E. Smith and M.L. Latham), New York (Dover) 1954. In my translations of texts from the _Géométrie_ I have taken the English of Smith and Latham as starting point. However, their translation is very free and often unreliable, so that in many cases I have had to modify it. The edition of the _Géométrie_ in the _Oeuvres de Descartes_ (eds. C. Adam and P. Tannery, Paris 1897-1913) vol. _6_, pp. 367-485 also indicates the pagenumbers of the original.
2. See, for instance, C.B. Boyer, _History of analytic geometry_ (New York 1956) p. 88 and p. 102; see also M.S. Mahoney, "Descartes: mathematics and physics", _Dictionary of scientific biography_ (ed. C.C. Gillispie, New York 1970 ff) vol. _4_ (1971), pp. 55-61, footnote 7.
3. Many studies have been devoted to Descartes' _Géométrie_. Most of these, however, are unsatisfactory with regard to the questions I discuss here because they are guided by the unfruitful question whether Descartes did or did not invent analytic geometry. The best source on the actual contents of the _Géométrie_ is the _Géométrie_ itself. The best summary of its intention, its development and its place within Descartes' mathematics is still G. Milhaud, _Descartes savant_, Paris

1921. I have found A.G. Molland's "Shifting the foundations, Descartes' transformation of ancient geometry", Hist. Math. 3 (1976), pp. 21-49 very helpful as a discussion of the history of the concepts of construction and classification of curves in antiquity and of Descartes' opinions on these classical ideas. Three other more or less recent studies devote special attention to the concepts of curves and constructions in Descartes' Géométrie. These are: J. Vuillemin, Mathématiques et métaphysique chez Descartes, Paris 1960 (in particular Ch. III "De la classification cartésienne des courbes", pp. 77 ff.), G.-G. Granger, Essai d'une philosophie du style, Paris 1968 (in particular Ch. III "Style Cartésien, style Arguésien" pp. 43-70), and J. Dhombres, Nombre, mesure et continu, épistémologie et histoire, Paris 1978 (in particular the section pp. 134-143). I find that none of these treats these concepts in sufficient detail to explain the structure of the Géométrie, the conflicting elements in Descartes' programme and the genesis of this programme. I must mention here also J. Itards's La géométrie de Descartes (Conférences du Palais de la Découverte, série D, nr. 39) Paris 1956. In the very dense text of this "conférence" there are many remarks that suggest to me that Itard knew many of the things I am dealing with in the present article. However, he does not explain these remarks in this text, so that they remain understandable only to those who know them already. I do not know of other studies of Itard in which he explains his ideas about the Géométrie more fully.

4. Note however that (2;3) is not a generalization of (2;1). In fact (2;1), the problem in three lines, is an exceptional case which arises when in the problem for four lines two lines coincide.
5. A proof will be given in the final version of this article.
6. For an extensive discussion of the section see D.T. Whiteside, "Patterns of mathematical thought in the later seventeenth century", Arch. Hist. Ex. Sci. 1 (1960-1962), pp. 179-388, especially pp. 290-295.
7. See the article by Molland cited in note 1.
8. Descartes tended to underestimate the dangers of extrapolating mathematical results, witness, for instance, the penultimate sentence of the Géométrie: "For in the matter of mathematical progressions, once one has the first two or three terms, it is not difficult to find the others" (G p. 413).

9. Descartes Oeuvres (see note 1) vol. 6 p. 19.
10. See G. Loria, Spezielle algebraische und transzendente ebene Kurven, Leipzig (2d ed.) 1910-1911, vol. 1 pp. 51-52.
11. See Archimedes, On Spirals, definition 1, in The works of Archimedes (ed. T. Heath, New York, Dover reprint) p. 165.
12. Often called the quadratrix of Deinostratos, although the names of Hippias and Nicomedes are also connected with the curve. Pappus discusses the curve in his Mathematical collections. See I. Bulmer Thomas, "Dinostratus", Dictionary of scientific biography (ed. C.C. Gillispie, New York 1970 ff.) vol. 4 pp. 103-105.
13. See T.L. Heath, Mathematics in Aristotle (Oxford 1949) pp. 140-142.
14. Rectifications of algebraic curves were found around 1658, independently, by van Heuraet, Neile and Fermat. See for instance M.E. Baron, The origins of the infinitesimal calculus (Oxford 1969) pp. 223-228.
15. Latus rectum and latus transversum are the classical terms for certain linesegments occurring in the defining properties of conic sections. If the vertex of the conic section is taken as origin and the X-axis is along the diameter, then the latus rectum a and the latus transversum b occur in the analytical formulas for the conics in the following way: $y^2 = ax$ (parabola); $y^2 = ax - \frac{a}{b}x^2$ (ellipse); $y^2 = ax + \frac{a}{b}x^2$ (hyperbola).
16. Descartes' very criptic description of the curve is as follows: The curve is such that

    "if all the straight lines orderedly applied to its diameter (i.e. the ordinates H.B.) are taken equal to those of a conic section, then the segments of the diameter between the vertex and these lines (i.e. the abscissae H.B.) have the same ratio to a given line as that line has to the segments of the diameter of the conic section to which these lines are orderedly applied." (G p. 339)

    Following C. Rabuel, Commentaires sur la Géométrie de M. Descartes (Lyon 1730) p. 271, the passage can be interpreted as follows. If we take the origin in the centre of the figure,

$$a.d_1.d_2 = d_3.d_4.d_5$$

    leads to

$$a(y+\frac{a}{2})(y-\frac{a}{2}) = x(y+\frac{3}{2}a)(y-\frac{3}{2}a)$$

as equation for the required curve. Taking

$$w = \frac{1}{2}a^{-1}(y^2 - \frac{9}{4}a^2)$$

we find

$$w : a = a : (a-x) .$$

If we now take the "vertex" in Descartes' text to be the point V (x=a, y=0), and draw the parabola

$$2aw = y^2 - \frac{9}{4}a^2$$

with w taken along the X-axis from V, then the required curve and the parabola are related in such a way that for points (x,y) and (z,y) on either curve with equal ordinates y, the abscissae x-a and w (taken from V) satisfy

$$w : a = a : (x-a) .$$

This corresponds to what Descartes says, but he does not specify that the conic section is a parabola in this case. If the conic section and the position of the vertex are given, Descartes' description implies a pointwise construction of the curve.

17. This pointwise construction follows immediately from the description of the quadratrix by continuous motion, see note 12. T.L. Heath mentions the construction in his A history of Greek mathematics (2 vols, Oxfors 1921), vol. 1 p. 230, as a means to find points on the curve near D (figure 9) and thus to approximate D. But he gives no reference to classical sources containing this construction.
18. The ovals which Descartes discusses on pp. 352-368 of the Géométrie are curves whose surfaces of revolution provide shapes of lenses with the property that light rays coming from one point converge, after passing through the lense, to another point (and variants of this property). Descartes explains how these ovals can be constructed when the positions of the light source and the converging point, and the refractive index of the lense material, are given.
19. La Dioptrique, one of the three essays of the Discours; in Descartes' Oeuvres (see note 1) vol. 6 pp. 79-228.
20. I use the abbreviation A.T. for Adam and Tannery's edition of Descartes' Oeuvres mentioned in note 1.

21. See C. Huygens, Oeuvres Complètes (22 vols. The Hague 1888-1950) vol. 11 p. 216; a note from 1650.

21a. A.B. Kempe, "On a general method of describing plane curves of the n^{th} degree by linkwork", Proc. London Math. Soc. 7 (1876) pp. 213-216.

22. A.T. 10 pp. 154-158.

23. A.T. 10 pp. 77-78; the study dates from before December 1618.

24. A.T. 10 pp. 222-223, from 1619-1621.

25. Descartes to Mersenne 12-9-1638, A.T. 2 pp. 352-362, in particular p. 360.

26. For Descartes' solution of Debeaune's problem see his letter to Debeaune of 20-2-1639, A.T. 2 pp. 510-523, and C.J. Scriba, "Zur Lösung des 2. Debeauneschen Problems durch Descartes", Arch. Hist. Ex. Sci. 1 (1960-1962) pp. 406-419.

27. Mersenne mentioned the cycloid and Roberval's studies on its quadrature in his letter to Descartes of 28-4-1638 (A.T. 2 pp. 116-122). In his answer, 27-5-1638 (A.T. 2 pp. 134-153) Descartes said that he had never thought of the curve before (p. 135). He discussed the curve and its properties in several subsequent letters to Mersenne.

28. See the chapter "On the problems known as neuseis" in The works of Archimedes (ed. T.L. Heath, Dover edition) pp. c-cxxii, in particular p. cvii. For the conchoid see also figure 5.

29. See G. Milhaud, Descartes savant (Paris 1921) Chs 1, 3 and 6.

30. It seems likely that the discovery occurred later than 1628, for we have a note of Beeckman on his interview with Descartes in October 1628. Descartes had explained to Beeckman the construction of the roots of any fourth degree equation by the intersection of a circle and a parabola. Beeckman noted that "M. Descartes made so much of this invention that he confessed never to have found anything superior himself and even that nobody else had ever found anything better" (A.T. 10 p. 346). It is not likely that Descartes would have commented in this way is by that time he knew already the general construction of the roots of 5th and 6th degree equations.

31. See Descartes' letter to Golius of January 1632, A.T. 1 pp. 232-236. Descartes refers to an "écrit" sent earlier to Golius; this "écrit" is now lost. In the letter Descartes adds a definition of classes ("genres") of curves. The definition is not clear and the terminology is quite different from the one used in the *Géométrie*. Still, from the further indications in the letter it seems likely that by that time Descartes had found the essential elements of the solution of the Pappus problem as it appeared in the *Géométrie*, and that the "écrit" sent to Golius contained a condensed version of this solution.

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