Algebraization in Descartes' *Géométrie*

by

H.J.M. Bos

1. It is a pleasure for me to speak at this symposium on a topic from one of the fields in which Professor Monna has been, and still is very active: the history of mathematics. To me, one of the most inspiring aspects of Professor Monna's work in history of mathematics is that he tackles the major issues and does not avoid the basic problems by concentrating on detail. This becomes clear if we consider the range of his interest in the history of mathematics: fundamental concepts such as the concept of function; extensive subfields of mathematics, I am thinking here of his book on the history of functional analysis; major problems, for instance the book on the Dirichlet principle; and basic changes of style in the development of mathematics. Professor Monna deals with the latter topic in his study on algebraization.

At the moment I am engaged in studying [1] the concept of curve in 17th century mathematics, with particular reference to Descartes' *Géométrie*, a book which caused a complete revolution in the mathematical study of curves. The *Géométrie* marks the beginning of analytic geometry, that is, the study of curves by means of their equations. Hence in Descartes' book we have an example of the algebraization of a part of mathematics. It therefore seemed appropriate on this occasion that I should follow up one of Professor Monna's interests and say something about the process of algebraization as it is found in Descartes' *Géométrie*.

2. Strictly speaking the *Géométrie* is not a book but an essay serving as an appendix to another treatise, namely the *Discourse on Method* [2], which was first published in Leiden in 1637. The method was a philosophical method and the *Géométrie* was one of three "essays" (try-outs) of that method, showing its effectiveness in three different branches of science.

In many respects the *Géométrie* was revolutionary. Descartes introduced the elementary algebraic notation which we still use, namely x, y, and z for variables or unknowns, a, b, c etc. for constants and the usual notations for sums, products and powers

of these quantities. But the most important aspect of the *Géométrie* was that Descartes worked out the relation between algebra and geometry much more fully than had been done before. And so the *Géométrie* became a captivating mixture of geometry and algebra. What did Descartes achieve? He achieved two things:

He showed that the algebraic operations of addition, subtraction, multiplication, division and root taking ($\sqrt[k]{\ }$ with k positive, integer) can be interpreted geometrically as operations on line-segments. He also showed that the curves that one meets in geometry can be characterized by their equations in rectilinear coordinates x and y. Because he worked with algebraic operations, the equations were also algebric, in fact they were usually polynomial equations in x and y. The only types of equations which Descartes considered, were algebraic ones.

Thus it would seem that Descartes was doing analytic geometry of algebraic curves. But there is a complication. Descartes said that all curves in geometry have such equations. He did not say that all algebraic equations describe curves that occur in geometry - or at any rate he did not say so explicitly. In other words, the fundamental principle of analytic geometry, namely the correspondence between curves and equations, is not explicitly stated in Descartes' *Géométrie*.

What were the geometrical curves which Descartes had in mind? On this point he was explicit:

> To trace all the curved lines which I want to introduce here nothing else needs to be supposed than that two or several lines can be moved one by the other, and that their intersections describe other lines. ([3], *G*, p. 316)

Such curves may be very complicated but they are acceptable in geometry provided

> one can imagine them as described by one continuous motion, or by several motions which follow each other, the last ones of which are completely regulated by those which precede them. For in this way one can always have an exact knowledge of their measure. (*G*, p. 316)

Note that there is no trace of algebra in this definition or characterization of the curves that are acceptable in geometry. Descartes set great store on this characterization and repeatedly returned to it.

Fortunately, he gave examples to illustrate what he had in mind. His first example [4] is the famous "compass" of figure 1.

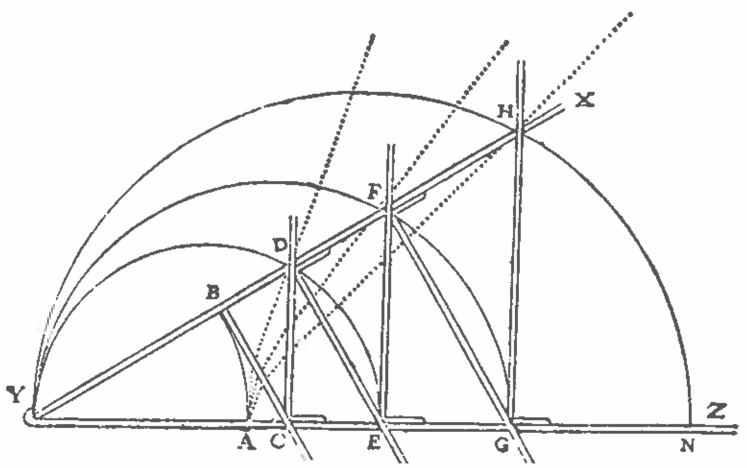

figure 1

It is discussed in the *Géométrie* and since then has often been reproduced in books on the history of mathematics, sometimes purely for aesthetic reasons. The rulers YZ and YX pivot in Y. Ruler BC is fixed to YX at B. The rulers CD, EF, GH can slide along YZ in such a way that they remain perpendicular to YZ. Similarly the rulers DE, FG can slide along YX and remain perpendicular to it. When one now opens the compass, C moves to the right and pushes CD, so that D also moves and describes the curve AD. At the same time D pushes the ruler DE, which in its turn pushes EF and the intersection F describes another curve AF. The next curve thus produced is AH and the process can be continued to yield further curves [5]. According to Descartes' criterion these curves are acceptable as geometrical curves: they are described by a combination of continuous movements that are regulated directly or indirectly by the first movement of the opening of the compass.

Descartes also gave another example [6] of a curve acceptable in geometry (see figure 2):

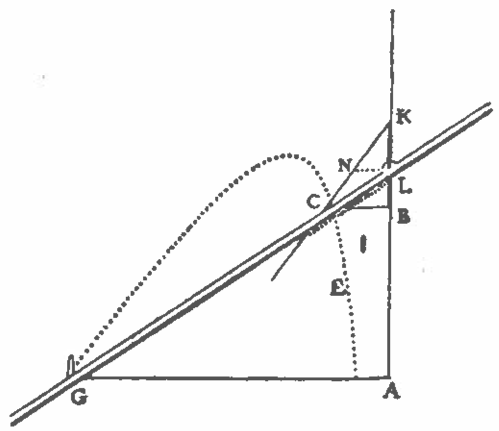

figure 2

GL is a ruler which can turn about G. At L the ruler is connected to a triangle KNL that can move up and down along the vertical axis AL. If the triangle is so moved the intersection C of KN and the ruler describes a curve. This curve (in fact a hyperbola) is one which, according to Descartes' criterion, is acceptable in geometry.

Finally, a variant [7] of the above construction is given, from which it becomes clear that the moving lines in Descartes' statement may themselves be curves. The example is as follows (see figure 3):

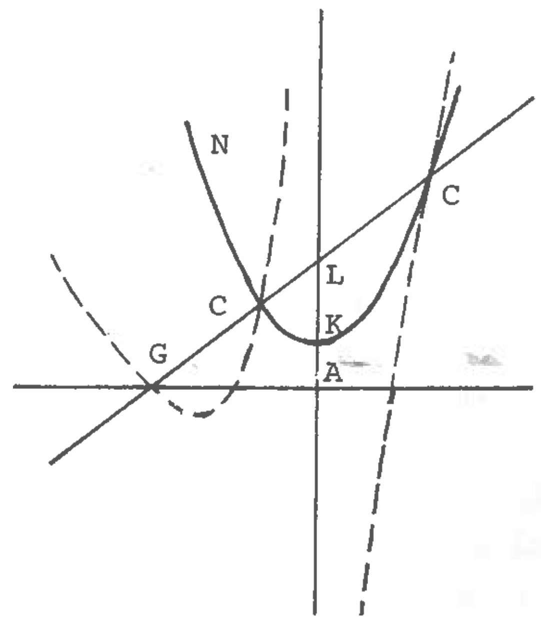

figure 3

The straight line KN in the previous example is now replaced by a parabola KN. Again the parabola moves vertically along KA; the point L moves with it, in such a way that the distance KL remains constant; the ruler GL turns around G and during the motion the intersection C of ruler and parabola describes a geometrically acceptable curve. This curve played a central role in Descartes' *Géométrie*. It later became known as the "Cartesian parabola". The equation of the curve is of the third degree. Curves that can be described in the way explained in the examples are geometrical, according to Descartes, and curves that cannot be so described are to be excluded from geometry. Descartes called the non-acceptable curves "mechanical" and he gave two examples of such curves: the spiral and the quadratrix.

This kinematic approach and the restriction of the class of acceptable curves are a curious starting point for someone who is about to create analytic geometry. This starting point raises a number of questions. Descartes claimed that geometrical curves (in the sense explained above) have algebraic equations. But it is possible to imagine instruments similar to the ones described in the examples which trace non-algebraic curves, such as spirals or cycloids. How did Descartes, then, regard these curves? Descartes indeed knew about the possibility of tracing such curves in this way and he stipulated further conditions for the process of curve tracing which served to exclude these non-algebraic curves. These requirements are very interesting but they fall outside my present theme [8]. Another, more complicated question is this: Descartes said that geometrical curves, in the sense explained above, have algebraic equations. But how did he regard the converse: namely, does every algebraic equation correspond to a curve which is geometrical in Descartes' sense? Descartes did indeed think so but he did not say it explicitly, and he could not prove it. It is of interest to find out precisely what Descartes said about this question. But to do so I first have to say something about the so-called problem of Pappus.

3. In the *Géométrie* Descartes gives particular attention to one special geometrical problem, namely the problem of Pappus [9]. It is a classical problem explained in the *Mathematical collections* of Pappus (ca. 300 A.D.) but is certainly earlier. To explain it I shall use the figure which Descartes gave himself (see figure 4).

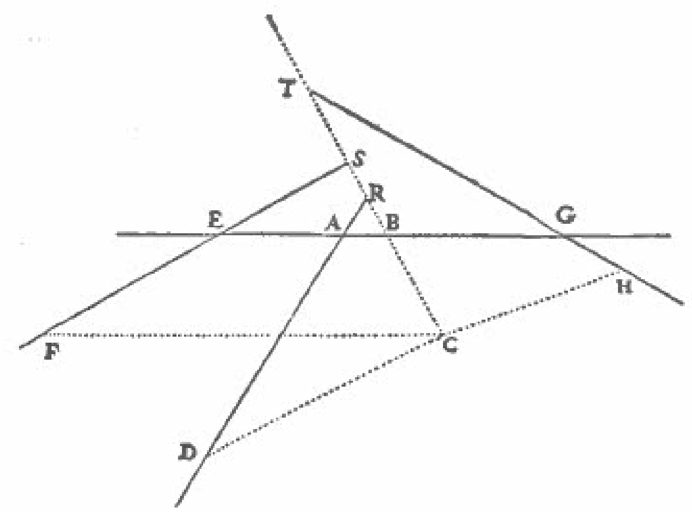

figure 4

Four lines DA, FE, EG and TG are given. From point C lines CD, CF, CB and CH are drawn towards these lines at certain angles. I shall call the length of these lines d_1, d_2, d_3, and d_4, respectively. It is required to find the locus of the points C for which

$$(d_1 . d_2) : (d_3 . d_4) = a : b, \qquad (1)$$

where a : b is a given fixed ratio. The d_i should always form the same angle with the given lines. Pappus himself also formulated a generalization of this problem for more than four lines; in that case the requirement was

$$(d_1 . d_2 \ldots d_n) : (d_{n+1} . d_{n+2} \ldots d_{2n}) = a : b, \qquad (2)$$

and a variant if the number of lines is uneven.

How did Descartes approach this problem? He wrote out the requirement (which Pappus gives entirely in prose) as an equation in x and y. This is easy because after a suitable choice of coordinates every d_i can be expressed in the form $lx + my + n$. From the equation so formed Descartes derived a geometrical construction for points on the locus. He also deduced from the equation further data about the locus as curve (e.g. vertex, axis etc.).

4. In connection with the Pappus problem, Descartes made a statement which, together with two further arguments in the *Géométrie* help us to find out whether or not he thought that all algebraic equations correspond to geometrical curves.

Descartes said [10] that if one takes a sufficient number of lines in the Pappus problem (formula (2)) it is possible to get all possible equations (he says: all possible coefficients and signs in the equations) by varying the positions of the lines. In other words: every equation can occur as an equation of the locus in a Pappus problem. Descartes did not prove this and in fact the statement is false [11]. But for low degrees of the equations the statement is plausible enough, and Descartes had found that in the Pappus problem for four lines all quadratic equations did indeed occur.

If the equation in x and y of a curve is given, Descartes argued that points on it can be constructed by arbitrarily assuming a value x_o for x and constructing geometrically the root (or roots) y_o of the resulting equation in y. I shall return to this procedure, called "the construction of equations", later; here it is important to note that it yields a pointwise construction for the curve whose equation is given; that is, one can construct arbitrarily many points on it but one cannot by this means trace the curve by continuous motion. However, after giving some examples of such pointwise constructions Descartes stated [12] that these were equivalent to tracing by continuous motion. Descartes gave no proof at all for this statement, which is crucial in the structure of the Géométrie.
Combining these three statements: the occurrence of all equations for loci, the pointwise constructability of each curve whose equation is given, and the equivalence of pointwise constructions to tracing by continuous motion, we may conclude that Descartes effectively considered every algebraic equation to correspond to a "geometrical" curve. In other words, his "geometrical curves" are what we now call algebraic curves. But in the course of his argument Descartes had to give up the strict criterion of traceability by continuous motion. And the crucial step, the equivalence of pointwise construction to continuous motion-tracing, could not be proved.

5. This leads to a further question, namely: Why make matters so complicated? Descartes could have made things much easier for himself by simply stating: Geometry concerns precise and exact operations; algebraic operations, when interpreted geometrically

are precise and exact; therefore all curves which admit an algebraic equation are acceptable in geometry and geometry will henceforth be simply the analytic geometry of algebraic curves. He would then have obtained the same class of geometrical curves without the complications of a definition requiring continuous motion-tracing by complicated instruments as those in the examples explained above. He could have done this, but he did not. The question then is: Why didn't he? Why did he keep to his criterion of continuous motion-tracing for geometrical curves?

In my opinion, this question has never been answered satisfactorily. In the many studies on Descartes' *Géometrie* the question is usually not mentioned at all. Most writers are already convinced that Descartes invented analytic geometry and so they do not seriously consider his actual definitions and opinions. I have come up against this problem through my interest in the concept of the construction of curves in the 17th century. In fact, what Descartes is struggling with here is the problem of which constructional procedures are acceptable for the definition of a curve. In the 17th century many mathematicians faced this problem and so it is an interesting theme in the history of mathematics.

But also from the point of view of algebraization the question of why Descartes kept to his kinematic criteria for the acceptability of curves is of interest, because these criteria introduced a tension and a restriction in a book which otherwise constituted an important step towards the algebraization of geometry. I shall now go on to say why, in my opinion, Descartes kept to these kinematic criteria.

6. We must first ask what Descartes' aim was in writing the *Géometrie*. Descartes had a very clear programme for geometry. According to him the aim of geometry was the solution of geometrical problems. Geometrical problems at that time were mostly of the following type. A certain geometrical figure is given and it is required to find one or several points which satisfy a certain given property.

Finding mean proportionals is a characteristic example of such a geometrical problem (see figure 5).

figure 5

Two line segments AB and AC are given and it is required to find a point D such that the segment AD satisfies

AB : AD = AD : AC, (3)

i.e. AD is the mean proportional between AB and AC. If two mean proportionals are required, points E and F have to be found such that AE and AF satisfy

AB : AE = AE : AF = AF : AC (4)

AE and AF are then the two mean proportionals between AB and AC.

Solving such a geometrical problem means constructing the required point or points. But which constructions are acceptable? The ancient mathematicians, according to Descartes, required that the construction be performed with ruler and compass. But they also encountered problems which could not be constructed with these means. In those cases they constructed the required points by means of the intersection of curves other than circles and straight lines. That is, they allowed other curves to be used as means of construction. These curves could be conic sections or certain curves called conchoids, or other curves. They also devised instruments, kinds of generalized compasses, with which such curves could be traced, so that the points of intersection could actually be determined.

For instance it is generally not possible to construct two mean porportionals with ruler and compass. But they can be constructed as follows by two conic sections (see figure 6):

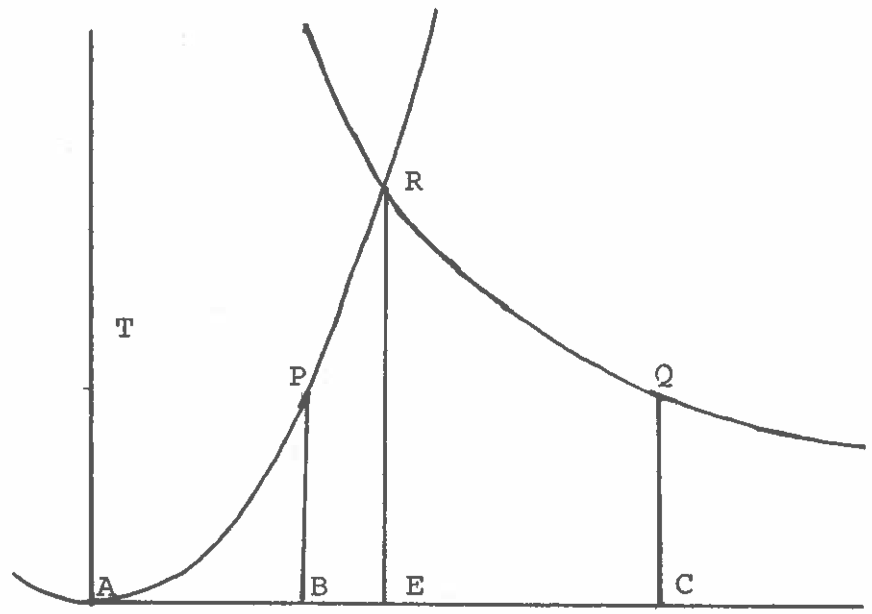

figure 6

Let AB and AC be the given line segments. Draw an axis AT perpendicular to AC and mark points P and Q such that PB = QC = AB and PB ⊥ AC and QC ⊥ AC. Now draw through P a parabola with axis AT and vertex A, and draw through Q a hyperbola with the axes AC and AT as asymptotes. Mark the intersection R of the parabola and the hyperbola and draw RE perpendicular to AC. Then AE is one of the required mean proportionals. It is easy to check this. Such constructions of the problem were known in antiquity.

The essential point of the construction is that it is supposed to be possible to draw the parabola and the hyperbola and to actually determine their intersection R. It is not evident that this is possible and hence that the construction is an acceptable solution of the problem. The ancient mathematicians, Descartes said, had doubts about this kind of construction; they were not sure if such constructions were geometrical, perhaps they were rather mechanical, depending on certain mechanisms for tracing the constructing curves.

Thus there was uncertainty among the ancient mathematicians about when a geometrical problem was adequately solved. Descartes' programme was to eliminate this uncertainty. He stated that conic sections, conchoids and certain other curves which the ancient mathematicians used for constructions were no less geometrical than the circle and the straight line. He based this argument on the fact that one could imagine instruments for drawing these curves, and the operations of these instruments could be conceived as clearly and distinctly as the operation of an ordinary compass. These curves are indeed more complicated but for that reason no less geometrical than the circle. Descartes stated also that one should always use the simplest possible means of construction, that is, the simplest possible curves.

Descartes worked out these two statements in a programme. It is obvious that his programme would deal with three points:

a. The programme should distinguish the collection of curves that are acceptable in geometry as means of construction from the collection of curves that are not acceptable.

b. The collection of curves acceptable as geometrical means of construction should be ordered as to simplicity; then it will be clear what is meant when the simplest possible curves are required.

c. The programme should provide a method for finding for a given problem, the construction which uses the simplest possible curves.

This was the programme of the *Géométrie*, and Descartes was able to carry it out to a large extent. His treatment of the three points of the programme can be summarized as follows.

a. Curves that are acceptable in geometry as means of construction are, according to Descartes, those that can be described by one continuous motion or by a combination of such motions in the way explained above (section 2). Within the programme, the requirement of traceability is obvious: the curves serve to determine their intersections with other curves or lines, and these intersections can be found if the curves can actually be traced.

b. In order to classify the acceptable curves Descartes used the degree of their equation. His classification is somewhat complicated and not completely clear, but roughly he argued that a curve is simpler when the degree of its equation is lower [13].

c. Algebra is fundamental to the third part of Descartes' programme. He explained that first of all the geometrical problem to be solved should be reduced to an algebraic equation. The way to do this is to call the known quantities in the figure a, b, c etc., and the unknown quantities x, y, z etc., after which the given and the required relations between these quantities are translated into equations. Eventually there should be as many equations as there are unknowns. If there are not, then the problem is indeterminate (a locus problem) and one should arbitrarily fix some of the unknowns and thus reduce the problem to a system of as many equations as there are unknowns. Then the unknowns should be eliminated one after the other from the system till one equation in one unknown is left. If we can find that one unknown we can find the others. But for Descartes "finding" meant constructing geometrically. Therefore he added a method for constructing, by the simplest possible and acceptable geometrical means, the roots of an equation in one unknown. That is, for every equation in one unknown, he indicated how one should find two curves, of the lowest possible degree, such that the x-coordinates of the intersection points are precisely the roots of the equation This procedure is called the "construction of an equation". Descartes showed [14] that:

-quadratic equations can be constructed by the intersection of a circle and a straight line (that was a classical result);

-cubic and quartic equations can be constructed by the intersectio of conics (that was known also) and in particular by the intersection of a parabola and a circle (that was new);

-5th and 6th degree equations can be constructed by the intersecti of a circle with the "Cartesian parabola" mentioned above (cf. figure 3) (that was quite new);

-and, said Descartes,

> One only has to go on in the same way to construct all equation of higher degree, to infinity. For in mathematical progressions once one has the first two or three terms, it is not difficult to find the others. (*G*, p. 413).

This overconfident statement is characteristic for Descartes' approach. Nevertheless, in working out the third point of his programme, he did present a large number of techniques that have proved very important for algebra. These include the sign rule for the number of roots of an equation, techniques for shifting roots for letting them change signs, techniques for removing middle terms in an equation and techniques for finding whether an equation is reducible. He needed the last technique, of course, for finding the simplest constructing curves for a given problem. All these techniques, therefore, were devised and presented with a geometrical aim. This aim was to construct geometrical problems, by the intersection of the simplest possible, geometrically acceptable constructing curves, i.e. curves that can be traced by a continuous motion.

7. Before returning to the question of why Descartes kept to his kinematic criterion for geometrical acceptability I want to point out a characteristic difficulty in the second part of Descartes' programme. There the simplicity of a curve is measured by the degree of its equation. But within Descartes' programme, simplicity of curves has to do with simplicity as means of construction and it is not at all evident that simplicity of construction corresponds to low degree. On the contrary; the circle for instance is a simpler means of construction than a conic, but the degree of its equation is the same as that of a conic. Descartes has to make an exception for this case.

Furthermore, the curves traced by the instrument discussed in section 2 (cf. figure 1) have equations with rather high degrees. But their kinematic origin in fairly straightforward and hence as means of construction these curves could be regarded as rather simple. Descartes remarks this himself [15]; nevertheless in this case he opted for the algebraical classification of simplicity by the degree of the equation.

8. We see that within the *Géométrie* there is tension between the algebraic and the geometrical criteria for simplicity. It now becomes even more interesting to find out why Descartes insisted that geometrically acceptable curves are those which can be described by continuous motion. Why did he not simply state the equivalence of curves and their equations and thereby avoid all the tensions and contradictions in his programme?

The answer lies in the structure of his programme as I have explained it above. The whole apparatus of algebraic methods which Descartes worked out, especially in the third point of his programme, makes sense only if the aim is the construction of equations with one unknown by the intersection of the simplest possible curves. And that question makes sense only within the geometrical context where curves serve as means of construction, that is, where their intersections are actually found. But that requires tracing of the curves by continuous motion, for only then is it certain that if the curves are given (namely traced) the points of intersection are given also. In other words, if Descartes had given up the criterion of traceability by continuous motion and had simply supposed that once the equations of curves are given their intersections are given too, then the whole programme of constructing roots of equations by the intersection of curves would have become pointless. For in that case, as soon as the equation itself is given, its roots should be regarded as having been given too.

This, then, is the answer to the question: Descartes could not give up the criterion of traceability by continuous motion because his programme would then have made no sense. The tension between the algebraic and the geometrical aspects of the *Géométrie* was inherent in his programme.

9. What fascinates me most in this example of algebraization of a mathematical subject is the dialectic of geometrical and algebraic concepts, methods and problems. On the one hand, the whole enterprise of the *Géométrie*, the elaboration of an analytic geometry for algebraic curves, only made sense for Descartes within the geometrical context of construction by intersection of curves traced by continuous motion. On the other hand, it was in fact the geometrical context and the geometrical starting point which led to tensions, if not contradictions, in the structure of the *Géométrie*; for instance the impossibility of proving that all algebraic curves are traceable by continuous motion, and the fact that the algebraic criterion for the simplicity of curves does not correspond to the geometrical criterion for simplicity.

Thus it seems in this case that the first phase in the emergence of analytic geometry was necessarily full of contradictions. One wonders then whether such contradictory phases have also occurred in other instances of algebraization, and if so, whether these were necessary.

This, I think, is an important general question arising from the study of the process of algebraization in Descartes' *Géométrie*. It also ties up with professor Monna's interest in algebraization, and therefore raising this question seemed to me a good way of ending this lecture at a symposium held in his honour.

Notes

[1] Compare my article "On the representation of curves in Descartes' *Géométrie*", Mathematical Institute Utrecht, Preprint nr. 142, February 1980. This article is submitted for publication in the *Archive for History of Exact Sciences*.

[2] R. Descartes, *Discours de la méthode pour bien conduire sa raison, et chercher la vérité dans les sciences, plus la Dioptrique, les Méteores et la Géométrie qui sont des essais de cete methode*, Leiden, 1637.

[3] The abbreviation *G* refers to the *Géométrie* cited in note 2. The *Géométrie* is easily accessible in a facsimile edition with English translation: *The geometry of René Descartes with a facsimile of the first edition* (tr. and ed. by D.E. Smith and M.L. Latham) New York, (Dover), 1954. The translations in the present article are my own, but I have taken Smith and Latham's translation as a starting point.

[4] *G* p. 318 and p. 370.

[5] The instrument and the curves it traces were meant to be of use in finding mean proportionals; indeed YC, YD, YE and YF are mean proportionals between YA and YG. Compare secion 6.

[6] *G* p. 319 sqq.

[7] *G* p. 322.

[8] Further explanation is given in my article cited in note 1.

[9] In particular *G* pp. 309-314 and pp. 324-334.

[10] *G* p. 324.

[11] A proof that it is false will be given in the published version of my article mentioned in note 1.

[12] *G* p. 339-340.

[13] *G* p. 319.

[14] These results are explained in the third book of the *Géométrie*.

[15] *G* p. 370-371.