Johann Molther's 'Problema Deliacum', 1619

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Introduction

In 1619 Johann Molther published at Frankfurt a book about the duplication of the cube, entitled *Problema Deliacum*. It attracted little interest at the time and references to the book in historical works are very scarce.[1] Yet for my studies on the concept of construction I have found the book revealing, and the occasion of the Festschrift for Dr Busard provides a welcome occasion to present a short note on its content, purpose and interest.

About Johann Molther himself little seems to be known.[2] He was born on March 28, 1591 in Grünberg in Hessen (Germany) as son of Johann Molther (1561-1618), pastor and professor of theology and Hebrew at Marburg university. About Molther senior's life we know more than about his son's because the latter included in his *Problema Deliacum* a long funereal poem for his father, detailing the qualities and the career of the deceased. Johann junior studied medicine and became professor of medicine at Marburg University in 1621. His date of death is unknown. Besides the *Problema Deliacum* he published an astronomical treatise and three medical disputations.[3]

In the style of his time, Molther did not spare words in phrasing the title of his study:

> The Delian problem of doubling the cube, that is, given any solid, to

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[1]Marin Mersenne expounded Molther's construction of two mean proportionals (see below, p. 43) in his *Harmonie universelle contenant la théorie et la pratique de la musique* Paris, 1636 (also fasimile ed. Paris 1965), p. 68; N.Th. Reimer mentioned Molther in his *Historia problematis de cubi duplicatione sive de inveniendis duabus mediis proportionalibus inter duas datas*, Göttingen, 1798; G.J. Toomer mentioned Molther's book in his article on Nicomedes in the *Dictionary of Scientific Biography* (ed. C.C. Gillispie, New York 1970-1980) vol. 10, pp. 114-116. I have found no other references.

[2]My sources for the biographical information are:
Zedler, J.H. (ed.), *Grosses vollständiges Universallexikon aller Wissenschaften und Künste* (64 vols, 4 suppl. vols), Leipzig 1732-1754 (reprint Graz, 1961-1964), vol. 21 (1739) col. 955;
Jöcher, C.G., *Allgemeines Gelehrtenlexicon*, 4 vols, Leipzig 1750-1751;
Adelung, J.C., Rotermund, H.W., Günther, O. *Fortsetzungen und Ergänzungen zu Jöcher's allgemeinem Gelehrtenlexicon*, 7 supplement vols, Leipzig 1784, 1787, Delmenhorst 1810, Bremen 1813, 1816, 1819, Leipzig 1897 (facsimile reprints of these volumes: Hildesheim (Olms), 1960-1961), vol. 3, col. 604 and suppl. vol. 4 col. 1960.

[3]Jöcher (note 2), suppl. vol. 4, col. 1960. The title of the astronomical treatise is given there as "Methodus erigendorum thematum astronom" (sic), Frankfurt 1618. I have not tried to trace these publications.

> make a similar solid in a given ratio, by means of the second Mesolabe,[4] by which two continuously proportional means can be taken. Now at last easily and geometrically solved after innumerable attempts of the most eminent mathematicians. The history of the problem is given first and some results are added about the trisection of an angle, the construction of a heptagon, the quadrature of the circle and two very convenient designs of proportional instruments.[5]

Such wordiness, and especially the claim to have "at last" solved the problem of duplicating the cube, are likely to draw a smile from the modern reader; the tradition of mistaken solutions of the problem, announced with similar pomp, is an old one. Molther's solution, however, was not mistaken, in particular he did not claim, as several others had done before him, to have solved the problem by straight lines and circles (ruler and compass[6]), which cannot be done. And although the mathematical content of his book is definitely unoriginal, there are aspects of his arguments which I find to be of some interest, especially because they relate to questions debated by some contemporaries of higher mathematical standing, such as Viète, Kepler and Descartes.

Duplication of the cube, mean proportionals and neusis

A mean proportional between two magnitudes a and b is a term of a geometrical sequence with a and b as first and last terms respectively. The simplest case occurs when the sequence has three terms: a, x, b, with

$$a : x = x : b ;$$

x is then called the mean proportional or the geometric mean of a and b. The construction, by straight lines and circles, of the geometric mean between two

[4]The term Mesolabum was used in classical Greek geometry to denote an instrument for constructing mean porportionals. In the sixteenth and seventeenth centuries the term, also spelled *mesolabium*, no longer had a strictly instrumental connotation; it meant in general the art of constructing mean proportionals. Molther, however, used it in the classical sense. He called 'second' mesolabe the instrument for constructing *two* mean proportionals.

[5]*Problema Deliacum de cubi duplicatione, hoc est de quorumlibet solidorum, interventu Mesolabii secundi, quo duae capiantur mediae continue proportionales sub data ratione similium fabrica. Nunc tandem post infinitos praestantissimorum mathematicorum conatus expedite et geometrice solutum. Ubi historia problematis praemittitur, et simul nonnulla de anguli trisectione, heptagoni fabrica, circulique quadratura et duabus commodissimis instrumentorum proportionum formis inserentur.* The book was published at Frankfurt, "Typis ac sumptibus Antonii Hummii", in 1619. I have consulted the copy in the Bayerische Statsbibliothek at Munich (4° Math P 237) and a xerox of the copy in the library of Brown University, Providence, R.I., kindly provided by Prof. G.J. Toomer.

[6]In the following I use the terminology 'by straight lines and circles' rather than 'by ruler and compass' for constructions according to the Euclidean postulates because Euclid did not mention instruments to perform these constructions.

line segments is given in Euclid's *Elements* (II-13 and VI-14). The case of two mean proportionals arises when the sequence has four terms: a, x, y, b. Thus the problem of constructing two mean proportionals is: Given two line segments a and b, it is required to construct two line segments x and y such that a, x, y and b form a geometric sequence, i.e.:

$$a : x = x : y = y : b .$$

This problem was formulated and solved in classical Greek geometry. Tradition has it that the problem arose in connection with one of the three 'classical problems', namely the duplication of the cube: To construct a cube twice as large (in content) as a given cube. In algebraic terms, if the edge of the given cube is called a, it is required to find the edge x of a cube determined by $x^3 = 2a^3$. If we write $y = x^2/a$, the equation implies $a : x = x : y = y : 2a$, so x is the first of two mean proportionals between a and $2a$. Greek geometers also realized this fact (be it not exactly along the algebraic line above) and knew that hence a method for constructing two mean proportionals would imply a method to duplicate the cube.

Molther also reduced the duplication problem to a construction of two mean proportionals, for which he used an expedient called 'neusis'.[7] Greek geometers from the classical period had found that several problems which could not be constructed by straight lines and circles, could be reduced to the so-called 'neusis' problem. This problem was:

> **Problem** (neusis)
>
> Given: two straight lines l and m (see Figure 1), a point O and a line segment a.
>
> Required: to find a line through O, intersecting l and m in points A and B respectively such that $AB = a$.

The problem requires the segment a to be placed between the lines l and m in such a way that it points in the direction of O.[8] There were variants of the problem in which one or both of the straight lines were replaced by circles. In particular cases (for instance when the distances of O to l and m are equal) the neusis construction can be achieved by straight lines and circles only, but not in general. I shall use the term 'construction by neusis' for a construction which reduces a problem to a neusis problem, and I shall call a 'neusis procedure' any procedure to actually construct the solution of a neusis problem. In his book Molther solved the problem of two mean proportionals by a neusis construction and he explained a neusis procedure.

[7]On neusis in classical geometry see for instance *The works of Archimedes* (ed. T.L. Heath), New York (Dover), 1953 (reprint of ed. Cambridge 1897-1912), pp. c-cxxii, and W.R. Knorr, *The ancient tradition of geometric problems*, Boston etc. (Birkhäuser), 1986, pp. 178-187 and index *s.v.* 'neusis'.

[8]This explains the name of the problem: the segment has to be placed such that it 'verges' towards P; neusis derives from the Greek verb 'neuein' which means 'to verge'.

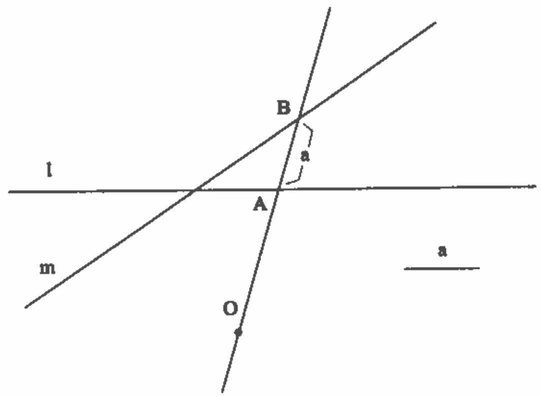

Figure 1

Neusis: the classical heritage

The mathematical techniques which Molther applied in his neusis construction were of classical origin and well known at his time. The two main sources from which early modern mathematicians learned about neusis and similar constructions and procedures were Eutocius' commentary to Proposition II-2 of Archimedes' *Sphere and Cylinder*, available since the beginning of the sixteenth century,[9] and Pappus' *Collectio*, first edited in print in 1588.[10]

In Proposition II-2 of the *Sphere and Cylinder* Archimedes assumed without further explanation the possibility of constructing two mean proportionals. In his commentary Eutocius gave twelve different constructions of this problem ascribing them to Plato, Heron, Philo of Byzantium, Apollonius, Diocles, Pappus, Sporus, Menaechmus, Archytas, Eratosthenes and Nicomedes.[11] The one by Nicomedes was a neusis construction. Several other solutions from Eutocius' list employed constructions like the neusis;

[9]The constructions which Eutocius mentioned were published in Georgius Valla, *De expetiendis et fugiendis rebus opus*, Venice 1501, book XIII, Ch. II, fols Uv^r-Xiv^r. This was a rather unsatisfactory version of the text. Johannes Werner gave a better edition in his *Libellus super viginti duobus elementis conicis (-) commentarius (-) cubi duplicatio (-)* Nürnberg, 1522, fols Civ^r - Hiv^r. The commentary of Eutocius became available in printed form in the Basel 1544 edition of Archimedes' *Opera* by Th. Geschauff. At the moment the text is easiest accessible through Vereecke's French translation: *Les oeuvres complètes d'Archimède suivies des commentaires d'Eutocius d'Ascalon* (tr. P. Vereecke), Paris, 1921.

[10]*Pappi Alexandrini mathematicae collectiones* (tr. ed. F. Commandino), Pesaro 1588. French translation: Pappus, *La collection mathématique* (tr. Paul Vereecke), Paris, 1933.

[11]There is one unattributed solution, mentioned after the one by Menaechmus.

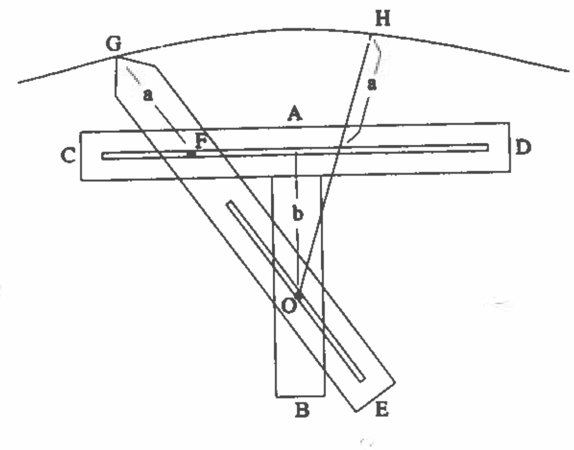

Figure 2

these were to be performed by shifting a marked ruler over the figure until, by trial and error, the correct position was found. Eutocius also mentioned a curve invented by Nicomedes by which a neusis problem could actually be solved. The curve has become known as the 'Conchoid of Nicomedes'. It is the curve traced by an instrument as in Figure 2, consisting of a system of perpendicular rulers AB, CD and a movable ruler EG. CD and EG have slots along their central lines; at O on AB and at F on EG pins are fixed which fall in the slots as shown. The distances $FG = a$ and $AO = b$ are constant (Nicomedes probably considered an instrument in which these distances were adjustable). If the ruler EG is moved the point G describes the conchoid. The pins and the slots ensure that in all its positions EG passes through the center O while F remains on the line CD. Thus any point H on the conchoid has the property that its distance to the base line CD measured along the line HO is equal to a. Eutocius showed that if for any given center, base line and segment the conchoid of Nicomedes can be drawn, the neusis problem can be solved (see Figure 3): draw the conchoid with respect to center O, segment a and base line l, let its intersection with m be B, then OB is the required line, intersecting l in A, with $AB = a$.

In his *Collectio* Pappus gave a classification of geometrical problems based on the curves used in their construction. He distinguished 'plane', 'solid', and 'line-like' problems. Plane problems were those constructible by the Euclidean means of straight lines and circles; 'solid' ones were those which, although not plane, could be constructed by the intersection of straight lines, circles and one or two conic sections. If the construction of a problem required more complicated curves than conic sections, they were 'line-like'.[12]

[12]*Collectio* (cf. note 10) book III, introduction to Proposition 5 and book IV, introduction to Proposition 34.

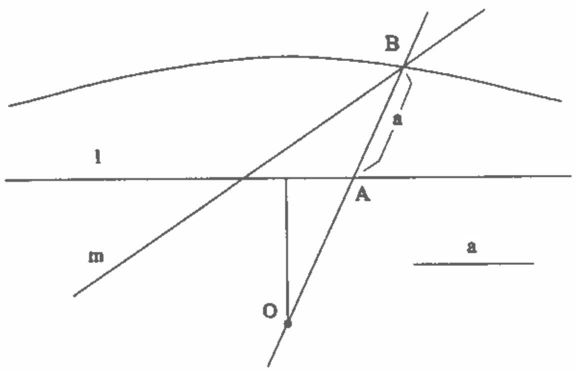

Figure 3

Pappus also discussed the neusis and its use in solving problems such as trisecting an angle or finding two mean proportionals. He was aware that the neusis problem was not plane; he proved that it was 'solid' by providing a construction by means of the intersection of a hyperbola and a circle.[13] The *Collectio* also contained a trisection by neusis.[14]

Thus the classical sources were quite informative about propositions and techniques relevant to the neusis procedure. They were, however, inconclusive with respect to the question whether procedures similar to the neusis, employing other means of construction than straight lines and circles and thereby being outside the Euclidean bounds of geometrical procedure, were legitimate. Eutocius' list of constructions suggested that the ancient mathematicians had not reached a consensus about which method was the most proper for solving the problem of mean proportionals. Eutocius himself hardly discussed the question of the geometrical legitimacy of the constructions; as far as he expressed preferences, they concerned the practical ease of the constructions, not their theoretical exactness. Pappus' classification of problems and his treatment of the neusis suggested that construction by intersection of conic sections should be considered as more geometrical than by using a conchoid or another neusis procedure.

[13] *Collectio* book IV Proposition 31.
[14] *Collectio* book IV Proposition 32.

Neusis: early modern interest

In the 1590's Viète gave the neusis construction a prominent place in his "Restored mathematical analysis or new algebra".[15] In his ***Supplementum Geometriae*** ("The Supplement of Geometry") of 1593[16] he gave neusis constructions for both the two mean proportionals problem and the trisection problem. He also showed that any geometrical construction problem which, translated into algebraic terms, led to an equation of the third or fourth degree, could be reduced to either a trisection or a determination of two mean proportionals. Thus a very large class of geometrical problems, including all the solid problems known at that time, could be reduced to a neusis.[17] Viète highlighted this result by stating that the Euclidean postulates, on which constructions by circles and straight lines were based, should be supplemented by a new one, postulating that to every center, base line and segment a neusis could be performed. This new postulate was the 'supplement of geometry' in the title of his book of 1593; Viète claimed that, with geometry thus supplemented, "no problem would be left unsolved".[18]

Legitimation

Viète formulated the neusis as a postulate; he did not propose any particular procedure for performing the construction. Nor did he argue explicitly why it should be allowed to accept neusis as a postulate; apparently he considered the postulate sufficiently legitimated by the fact that it made 'all' problems solvable. Molther was primarily interested in this legitimation question and he did not accept Viète's answer. He was not alone in this interest; the status and acceptability of constructions beyond those by straight lines and circles was widely discussed in early modern geometrical studies. I mention two examples to give some background to Molther's arguments.

Some years before Viète, Clavius had proposed another means to extend the boundaries of geometry so as to include such hitherto unsolvable

[15] In 1591 Viète published in Tours his *Isagoge in artem analyticen*; it was the first of a series of treatises which he then planned to publish and which were to form what he called the *Opus restitutae mathematicae analyseos seu algebra nova*. He did not complete this project. The *Isagoge* was republished in Viète, François, *Opera mathematica* (ed. F. van Schooten), Leiden, 1646 (facsimile reprint Hildesheim 1970), pp. 1-12. An English translation can be found in F. Viète, *The analytic art, nine studies in algebra, geometry and trigonometry* (tr. T.R. Witmer), Kent (Ohio), 1983, pp. 11-32.

[16] Tours 1593; it was the second of the series of treatises mentioned in the previous note. It is on pp. 240-257 of van Schooten's edition and on pp. 388-417 of Witmer's translation.

[17] In fact, Viète's result implies that *all* solid problems in Pappus' sense, that is, all those that can be solved by the intersection of conics, can be constructed by neusis, because such problems can be reduced to equations of degrees not higher than four. However, Viète did not explicitly draw that conclusion.

[18] "Nullum non problema solvere" (Van Schooten edition p. 12, see note 15). Remarkably, Viète did not discuss the problems leading to equations of higher degree than four, although he knew that the higher-order angular sections were of that class.

problems. He did so in a treatise on a special curve, namely the 'Quadratrix', inserted in the second edition of his Euclid.[19] From Pappus' *Collectio* Clavius had learned about that curve and its properties; in particular the fact that if the quadratrix were given, trisection, other angular sections and even the quadrature of the circle could be constructed. Ancient geometers, however, had expressed doubts about the legitimacy of this use of the curve because its definition, by a combination of motions, seemed to presuppose the quadrature of the circle, so that the curve's use in solving this problem would involve a *petitio principii*. Clavius provided a pointwise construction of the quadratrix which he considered to be fully geometrical. He then claimed with quite some emphasis (which he mitigated somewhat in later editions) that through his construction the use of the quadratrix in geometry was legitimized and the problems of the quadrature of the circle and the angular sections were truly geometrically solved.[20]

Viète and Clavius argued for an extension of the domain of geometry. Their view was opposed by others, notably by Kepler. In the same year as *Problema Deliacum* appeared, Kepler published his *Harmonices Mundi.*[21] The work contained a spirited defence[22] of Euclid's geometry against all classical and modern geometers who tried to extend the means of geometrical construction beyond straight lines and circles. According to Kepler line segments constructed by other means than straight lines and circles were not knowable, they fell outside the sphere of exact, genuinely geometrical knowledge. Also ratios involving such line segments were beyond the bounds of geometry. Kepler's reasons for this orthodox and purist conception of geometry were philosophical; in his view the Creator had shaped the world according to harmonious, knowable ratios and these were precisely the ones constructable by strictly Euclidean means; assuming that there were more such ratios would destroy the structure and the divine rationality of the creation.

Kepler was not the last mathematician to address the issue of acceptability of constructions beyond straight lines and circles. Indeed much of what Descartes' presented in his *Géométrie* of 1637 was directly or indirectly inspired by that question.[23]

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[19] *Euclidis elementorum libri XV accessit XVI de solidorum regularium cuiuslibet comparatione* (ed. C. Clavius), 2 vols, Rome, 1589, vol. 1, pp. 894-918: "De mirabilia natura lineae cuiusdam inflexae, per quam et in circulo figura quotlibet laterum aequalium inscribitur, et circulus quadratur, et plura alia scitu iucundissima perficiuntur"; Clavius inserted the treatise also in his *Geometria Practica*, Rome, 1604 (pp. 359-370).

[20] Cf. "(-) admittere (-) descriptionem hanc nostram quadratricis lineae ut geometricam" (p. 898 in Euclid edition 1589, cf note 19); in the version of the *Geometria Practica* of 1604 Clavius added 'quodammodo' in front of geometricam (p. 362, cf note 19).

[21] Johann Kepler, *Harmonices mundi libri V*, Linz, 1619, in *Gesammelte Werke* (ed W. von Dyck e.a.), München 1937–, vol. 6 (1940). There is a German translation: J. Kepler, *Weltharmonik* (tr. M. Caspar), München, 1973 (repr. of edition 1939).

[22] Book I, *Werke* vol. 6, pp. 13-64, see especially the Prooemium, pp. 15-20.

[23] Cf. my articles: "On the representation of curves in Descartes' Géométrie", *Archive for history of exact sciences*, **24**, 1981, pp. 295-338; and "The structure of Descartes' Géométrie", in *Descartes: il metodo e i saggi; Atti del convegno per il 350o anniversario della publicazione del Discours de la Méthode e degli Essais* (ed. G. Belgioioso e.a., 2 vols, Florence, 1990) pp. 349-369.

The Problema Deliacum

In his *Problema Deliacum*, then, Molther took a position in an existing mathematical debate. He proposed the use of neusis constructions to solve problems that could not be solved by straight lines and circles, and he presented in particular one such construction for two mean proportionals. The proposal was not new, as we have seen, and his construction was a modification of existing ones. Nor did Molther explicitly present them as original. What he did claim, however, as his own new and original contribution was his argument that the neusis construction was geometrical. In his opinion a convincing argument for the geometrical legitimacy of neusis constructions had not yet been given, and so, by presenting such an argument, he could claim to be the first to have really solved the problems of doubling the cube and constructing two mean proportionals. Before analysing Molther's arguments I give a brief survey of the content of the work.

The book opens with a dedicatory poem addressed to Maurice, landgrave of Hessen,[24] and a preface.[25] The main text starts with a history of the problems of cube duplication and constructing two mean proportionals.[26] Then follow four chapters, the first on the neusis postulate itself,[27] the second on the construction of two mean proportionals by means of the neusis,[28] the third on a particular problem in solid geometry which depends on the construction of two mean proportionals,[29] and the fourth on some related subjects.[30] The above mentioned funereal poem for Molther's father ends

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[24] Pp. 3-5.

[25] Pp. 6-9.

[26] Pp. 10-28.

[27] "Postulatum genuinum. Lineae rectae è puncto ad duas lineas requisita applicatio." Pp. 29-50.

[28] "Mesolabii secundi expositio" pp. 51-69. The chapter ends (pp. 63-69) with the remark that the first of two mean proportionals between 1 and 31, that is $\sqrt[3]{31}$, provides a good approximation for π; this is the result on the quadrature of the circle announced in the title of the book; Molther gives the values as 314138+ and 314159+ respectively (for radius 100000).

[29] "Fabrica solidorum sub data ratione similium" pp. 69-72. This was a standard problem in the early modern geometrical literature, namely: given a solid A and a ratio $a : b$, to construct a solid B similar to A and such that the volumes of A and B are in the ratio $a : b$.

[30] Pp. 73-88. Molther discussed some aspects of the determination of mean proportionals in practice, both by numbers and by instruments. He also explained the two proportional instruments announced in the title of his book. They were variants of the proportional compass. In one of them (pp. 83-86) the two sets of scales normally drawn on the two legs of the compass were drawn parallel but at some distance on a single flat surface; Molther claimed that this device, because of the absence of a hinge, was easier to make and as accurate in use as the usual proportional compass. The second (pp. 86-88) was practically

the book.[31]

In the historical section Molther critically reviewed earlier constructions of mean proportionals. He dealt with the constructions from Eutocius' list and with a number of more recent ones (Cusanus, Ramus, Oronce Fine, Peletier, Viète, Clavius, Villalpandus, Salignac, and Metius). In Molther's opinion, all these constructions failed to be truly geometrical. Some of them used curves which were traced by machines or constructed pointwise; neither method could be accepted as fully geometrical. Other constructions were unpractical, merely approximative or just false. Among all the constructions Molther preferred the one by Nicomedes which used the conchoid, because that curve could be traced more easily than the conics or the cissoid (another curve used in a construction of two mean proportionals), and also because its pointwise construction (here he referred to Clavius, who gave such a construction of the conchoid in his *Geometria Practica*[32]) was easy and practical. Yet, curves as the conchoid were not traced in a truly geometrical way and it was still an open problem

> how one should geometrically achieve the placing of the lines in these required positions, in one immediate action, with no other instrument than those the geometer is absolutely allowed to use, and with such truth and precision that the procedures indeed can bear the test of reason's criticism.[33]

The postulate

At the end of the historical introduction Molther announced his own method for finding two mean proportionals in the following terms:

> We, however, have finally realized that the matter of this difficulty, investigated through many centuries, a stumbling block for the most ingenious of mortals, is really so smooth, easy, obvious, ready and evident, that, because it meets the very terms for a legitimate postulate, it has by right to be counted as the next postulate, so that it does not at all require a belabored construction and proof, as difficult problems do, but that, as a principle clear in itself, it needs only a simple explanation, after which anyone can understand it and give it its due assent.[34]

the same as the usual instrument but Molther proposed a different use of it by which it was not necessary to have the scales on both legs of the compass.

[31] Pp. 89-102.

[32] *Geometria Practica* (see note 19), pp. 301-304.

[33] "(-) quomodo eiusmodi linearum requisitae applicationes Geometrice sine alio quam Geometrae absolute concesso instrumento, mox prima actione praestarentur tanta veritate ac praecisione ut etiam rationis censuram sustinerent." P. 25.

[34] "At vero nos rem istam exploratae per plurima secula difficultatis, in qua mortalium ingeniosissimi haesitarunt, ita expeditam, facilem, obviam, parabilem promtamque dudum

So Molther claimed that the neusis construction could be accepted as a postulate in geometry on the same level as the traditional postulates that canonized the usual Euclidean constructions, and that therefore the duplication problem could indeed be solved geometrically by means of neusis. This, as we have seen, was precisely what Viète had done in his *Supplementum Geometriae*, be it that he had not explicitly justified the postulate status for the neusis. For Molther that was the heart of the matter, which may explain how easily he dismissed Viète's work in his introduction:

> With all his subtlety Viète gathered nothing that can stand the test of criticism.[35]

The crucial part of Molther's reasoning, then, was his argument why the neusis construction was as obviously possible and acceptable in geometry as the construction of straight lines and circles by ruler and compass. The first chapter of his book was devoted to this argument. It opened with the formulation of the neusis postulate:

> Let it be postulated that, given two lines and a point in position in the same plane, a line can be drawn from that point such that the segment intercepted on that line by the two given lines is equal to another straight line given in length.[36]

The justification of the postulate

To justify the status of postulate Molther explained a procedure for the neusis which, he claimed, was legitimately geometrical. The procedure may be summarized as follows:

> **Procedure**[37] (Neusis, Molther)
>
> Given: two straight lines l and m (see Figure 4), a point O and a segment a.
>
> Required: to find a line through O, intersecting l and m in A and B respectively and such that $AB = a$.
>
> Procedure:

animadvertimus, ut quia hasce Postulati legitimi conditiones obtinet, Postulatus sit proxima meritoque annumeranda, adeo ut nequaquam ceu Problema contentiosum anxiam constructionem et demonstrationem requirat, sed tanquam Principium per se manifestum, levi contenta sit explicatione, qua adhibita à quolibet capi et assensum meriri possit." P. 27.

[35] "Subtillissimus Vieta nihil quod censuram sustineret venatus est" p. 26.

[36] "Postuletur, duabus lineis punctoque in eodem plano situ datis, ut è puncto isto linea recta applicetur, cuius portio a lineis illis intersecta alteri rectae longitudine datae sit aequalis," p. 29.

[37] Pp. 31-36.

1. Take a ruler and mark points C and D on it with distance a.
2. Move the ruler over the plane in such a way that it slides along the point O and that the point C on the ruler is always on the given line l.
3. Stop that motion at the moment that the point D is on the line m; then draw a straight line along the ruler; call A and B its intersections with l and m respectively.
4. OAB is the required line.

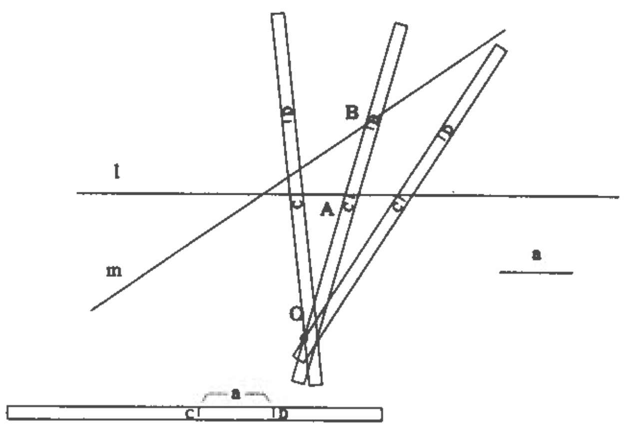

Figure 4

Molther analyzed this procedure at great length, arguing that each of its steps was legitimately geometrical. Since the first Euclidean postulate ascertained the tracing of straight lines, the ruler, by which this was done, was a geometrical instrument. Moreover such rulers could be made very precise, for instance by making them of metal, or by folding a piece of paper. The segment $CD = a$ could be marked on the ruler by a compass, which was also a geometrical instrument. The required movements of the ruler could be performed with great precision. This precision was guaranteed by our senses, which could judge whether the ruler remained along O during the motion and whether the point C moved along the given line l. It was also by the senses that the geometer decided to stop the motion at the moment that the point D lay on the second given line m. Molther argued that both motion and the testimony of the senses were implicitly assumed in the usual Euclidean postulates:

> For we have to use our sense to observe and acknowledge whether or not the ruler is placed in the way as postulated: because what can be clear by itself need not be made known through any demonstration. For at other times we acknowledge in no other way than by sense whether or not a ruler is duly placed along two points, from one of which to the other a straight line can be drawn according to the postulate; whether the given interval with which a circle is to be described is justly contained in a compass; and whether the one leg of the compass is rightly placed in the given center around which the circle is to be drawn. Indeed one sees immediately and with the same ease whether the ruler is along the point [O] and at the same time whether the point [C] on the ruler is at the line [l] and the point [D] on it at the line [m].[38]

Thus if one accepted the common Euclidean postulates one implicitly granted a legitimate place in geometry to motion and the testimony of the senses, and thereby to the neusis procedure.

Pure geometry

Although the starting point of Molther's legitimation argument was the precision of the neusis procedure as effectuated in practice, he did acknowledge geometry as a pure science, remote from the practice of drawing figures on paper by a step of abstraction or idealization. However, he argued that also if one considered pure geometry as an action of the mind alone, based on postulates, constructions still had to be performed in the mind by an inner sense, and this was done by procedures idealized from the actual physical construction procedures. Indeed the analysis of the role of motion and the senses in the actual physical procedure served to help the inner sense to perform the neusis abstractly in the mind as easily as it performed the mental operations corresponding to the use of ruler and compass. Therefore neusis should be accepted as a postulate in pure abstract geometry. Molther formulated this argument as follows:

> But also if one would judge that geometry it its most pure form should be practiced by action of the mind alone and based on its postulates, one detaches by mathematical abstraction the ideas of a material ruler and of a compass, and grasps them in one's mind, so that in the fantasy they serve as ruler and compass, guided by an interior sense. And thus it will be easy to imagine in thought the process of which we have shown how it is performed in reality.[39]

[38] "Sensu enim advertendum et agnoscendum sitne Regula ita ut postulatur applicata: quia nulla demonstratione id innotescat, quod per se liquidum esse potest. Quemadmodum alias haud aliter quam sensu cognoscimus sitne Regula ad duo puncta, è quorum uno ad alterum trahi postulatur Recta, debite applicata: sitne intervallum pro Circulo describendo datum Circini apertura iuste comprehensum: sitne pes Circini alter in centro dato circa quod gyranda est Peripheria, recte positus. Nempe eadem facilitate protinus cernitur, sitne Regula iuxta punctum A simulque Regulae punctum C in linea l, et Regulae punctum B in linea m." Pp. 33-34, in the translation I changed the letters to fit my figure.

[39] "Si quis autem existimet oportere puram puram [sic] Geometriam sola mentis actione, etiam secundum Postulata sua, exerceri; is quoque Regulae Circini materialis ideas

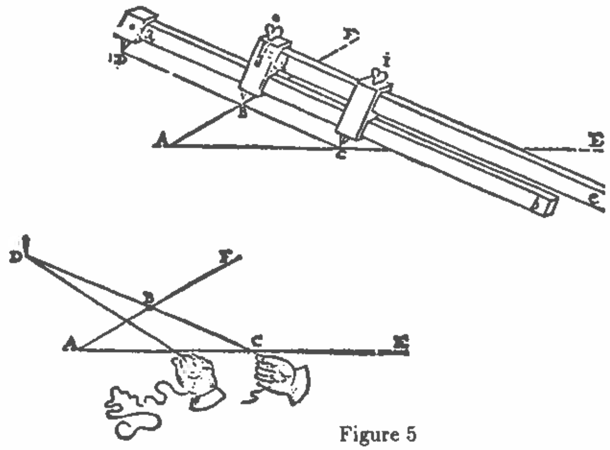

Figure 5

Molther's neusis instrument

To show the practical feasibility of neusis construction, Molther finally described two ways to actually perform the neusis, one by an instrument and one by a procedure involving a string. Figure 5 shows the drawings in his book.[40] The neusis was performed between the lines AE and AF from the point D; the given line segment was BC. The instrument was a combination of two rulers ab and de sliding along each other with adjustable points and pins. The distance oi along the ruler de could be made equal to the given segment. Ruler ab was placed with the pin at a in D and then turned around D while ruler de was made to slide along ab in such a way that the point corresponding to i moved along AE. The movement was stopped when the point corresponding to o was on the other line AF; at that position the ruler ab gave the required line.

The string procedure used a cord on which two points B and C were marked, with distance equal to the given segment. The cord, while being kept straight, was moved along the plane such that it wrapped around a pin in the given pole D and such that C followed the given line AE; the required position of the line was reached at the moment that B on the cord coincided with AF.

The construction of two mean proportionals

After these theoretical and practical arguments about the status and the execution of the neusis, Molther turned to the actual construction of solid

(afairesei) Mathematica abstrahat et mente complectatur, ut in Phantasia per sensum interiorem Regulae ac Circini opera faciant. Sic enim proclive fuerit cogitando illud fingere, quod quomodo reipsa praestetur monstravimus." P. 36, text between ⟨⟩ in Greek.

[40] Pp. 38 and 39.

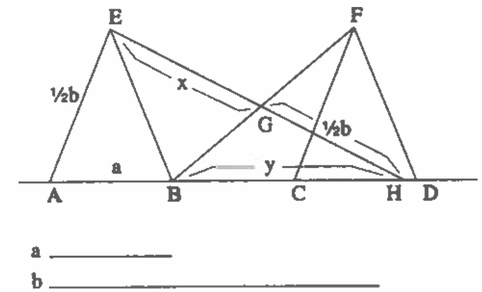

Figure 6

problems, by neusis but also otherwise. As most of these constructions are not particularly original or informative,[41] I shall only discuss the one for two mean proportionals.

The second chapter of *Problema Deliacum* was devoted to the *Mesolabium* proper, that is, to the construction of two mean proportionals. The construction was as follows:

Construction[42] (two mean proportionals by neusis, Molther)

Given: segments a and b $(a < b)$ (see Figure 6). It is required to find their two mean proportionals x and y.

Construction:

1. Mark points A, B, C, D along a straight line such that $AB = BC = CD = a$.

2. Construct two congruent isosceles triangles AEB and CFD, with $AE = EB = CF = FD = \frac{1}{2}b$; draw BF.

3. By neusis (Molther here refers to his postulate), construct a line through E, intersecting the lines BF and AD in G and H respectively such that $GH = \frac{1}{2}b$.

4. The required mean proportionals are $x = EG$ and $y = BH$.

The proof Molther gave for the correctness of this construction may be summarized as follows:

[41] Molther concluded his first chapter with a trisection by neusis (pp. 40-45) and a construction of the regular heptagon by means of a curve (introduced for this special purpose and constructed pointwise) which he called the "Helix (heptagoonografousa)" (pp. 46-48). The trisection is a variant of (and obviously inspired by) the neusis construction that occurs in Pappus' *Collectio* (cf. note 10); Molther referred to Clavius' *Geometria Practica* of 1604, where the same construction was given (on pp. 399-400, cf. note 19). Molther's change of the original construction was a slight practical improvement.

[42] Pp. 51-55.

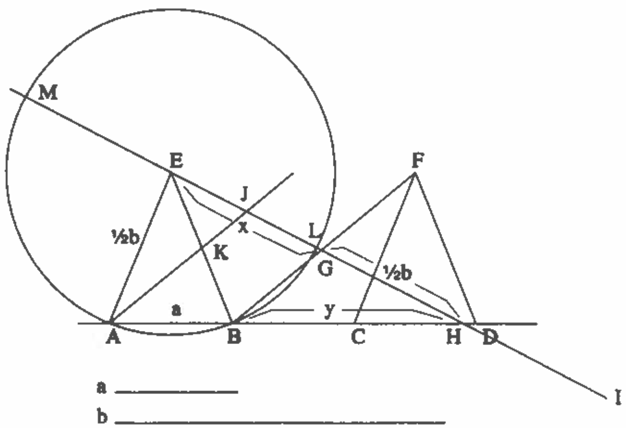

Figure 7

Proof[43]

1. Draw (see Figure 7) a circle with center E and radius $EA = \frac{1}{2}b$; prolong EH to both sides, take I on it with $GH = HI$ (and therefore $GI = b$); mark its intersections L and M with the circle, $LM = b$; draw a line through A parallel to BF, call its intersections with EG and EB J and K respectively.
2. Prove by similar triangles that $EK = KB$ and $EJ = JG$.
3. Note that, because $GH = \frac{1}{2}b = EL$, we have $EG = LH$, $MG = IL$ and $IE = HM$.
4. Prove by means of Euclid III-36 that $HL : HB = HA : HM$.
5. Because AJ is parallel to BF it follows that $AB : JG = BH : GH$; hence $AB : 2JG = BH : 2GH$; now $AB = a$, $2JG = EG = x$, $BH = y$ and $2GH = GI = b$, therefore $a : x = y : b$.
6. By similar triangles and the equalities of **2.** prove that $HA : EI = AB : EG$ and (using **4**) that $HA : EI = EG : HB$; as $AB = a$, $EG = x$ and $HB = y$ it follows that $a : x = x : y$.
7. From **5** and **6** it follows that $a : x = x : y = y : b$ as required.

If we compare Molther's construction and proof with those of Nicomedes (as given by Eutocius[44]) and of Viète (in his *Supplementum Geometriae*[45]) it appears that there is little originality in Molther's version. Without going into details here I may summarize the dependencies between the three solutions thus: Essentially they are the same, in particular with respect to the center,

[43] Pp. 55-58.
[44] Vereecke's translation (cf. note 9) pp. 618-620.
[45] Proposition 5, in Witmer's translation (cf. note 15) pp. 392-394.

lines and segment of the neusis. Viète simplified Nicomedes' proof by removing a number of inessential auxiliary elements in his figure; he introduced the isosceles triangle with sides a and $\frac{1}{2}b$ and the circle with radius $\frac{1}{2}b$ and chord a. Molther introduced more auxiliary lines, in particular the second isosceles triangle, which do not serve much function.

Conclusion – the interpretation of exactness

The justification of geometrical construction procedures beyond the use of straight lines and circles was a common concern in the early modern period. Mathematicians were unwilling to accept the restrictions of the Euclidean postulates, but the introduction of other means of construction than straight lines and circles evoked the question what exactness meant and where the borderline lay between legitimate, exact, or at any rate acceptable means of construction and inexact, unacceptable procedures. I term this issue the 'Interpretation of Exactness'.[46]

Mathematicians may adopt various strategies when confronted with the necessity to find new interpretations of exactness. They may, like Kepler, refer to authority and tradition and refuse to accept new interpretations. They may, as Descartes did in his *Géométrie*,[47] base a new interpretation of exactness on a philosophical analysis of pure understanding. They may, like Viète, motivate their choice implicitly by the quality of the resulting theory. They may also, like Clavius and like Molther, take practice as the starting point of their interpretation of exactness and argue for the legitimation of geometrical procedures by idealizing the criteria of precision that apply in practice.

All these strategies were tried out in the early modern period. They form, I find, a fascinating field of historical enquiry, not least because the various ideas developed in response to the questions concerning exact, acceptable constructions, although themselves of meta-mathematical nature, decisively influenced the development of mathematics in the early modern period. Although the idealization of practice, such as adopted by Clavius and Molther, appealed to a strong common experience of the practitioners of geometry, it was the least successful among the strategies mentioned above. Its arguments failed to convince and, contrary to the approaches exemplified by Viète and Descartes respectively, it led to no new mathematical results or techniques. Not surprisingly, then, Molther's *Problema Deliacum* met with little response and exerted no noticeable influence. Still, I think the book is of interest, because Molther worked out his legitimation in greater detail than we usually find. Thereby the *Problema Deliacum* offers us an instructive archetype of the approach to the interpretation of exactness which favours

[46] Cf. my 'On the interpretation of exactness', to be published in the Proceedings of the fifteenth international Wittgenstein symposium, held at Kirchberg am Wechsel (Austria) August 1992.
[47] Cf. the articles cited in note 23.

the idealization of practice.

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