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# TRACTIONAL MOTION 1692-1767 THE JOURNEY OF A MATHEMATICAL THEME FROM HOLLAND TO ITALY[1]

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§ 1. In 1765-67 the Italian mathematicians Vincenzo Riccati and Girolamo Saladini published a textbook on mathematical analysis entitled *Institutiones analyticae.*[2] In its third volume, the book contained a 60 page section on what was called the "construction of differential equations by means of a tractional curve."[3] Riccati was responsible for that section; the subject had captured his interest already in 1750. In 1752 he had published a monograph about it, entitled "On the use of tractional motion in constructing differential equations,"[4] and in 1753 the Florentine periodical *Symbolae Litterariae Opuscula Varia* carried a letter on tractional motion which Riccati had written to Maria Gaetana Agnesi in 1750.[5] I take these publications of Riccati as the end of the journey through mathematics of a curious theme: tractional motion.

Tractional motion[6] occurs when a heavy object is dragged over a horizontal resisting surface by a cord (whose motion is frictionless) of constant or variable length, the free end of which is moved along a straight line or a curve in the plane (see fig. 21). It is supposed that, because of the friction, the direction of the motion is always along the cord. The curve along which the free end of the cord is moved is called the base. The path described by the heavy object is called a tractional curve. If the end Q of a cord PQ is move along a

1. In the present article I give a non technical survey of the story of tractional motion, leaving out many of the mathematical details. In my article [Bos 1988] I have dealt more extensively with the early part of the story (till c. 1710). I hope to give a detailed description of the later part of the story elsewhere.
2. [Riccati-Saladini 1765].
3. Chapters 14-16 of book 2 of part II, vol. 3, pp. 470-531. Chapter 14 is called "Quaenam aequationes construi possunt ope tractoriae praedita tangente constante, methodo directe investigatur"; chapter 15 deals with the corresponding "inverse method" and chapter 16 gives the solution of some further inverse problems.
4. [Riccati 1752].
5. [Riccati 1753], the letter was dated Nov. 3, 1750.
6. For a more detailed explanation see [Bos 1988] pp. 10-12.

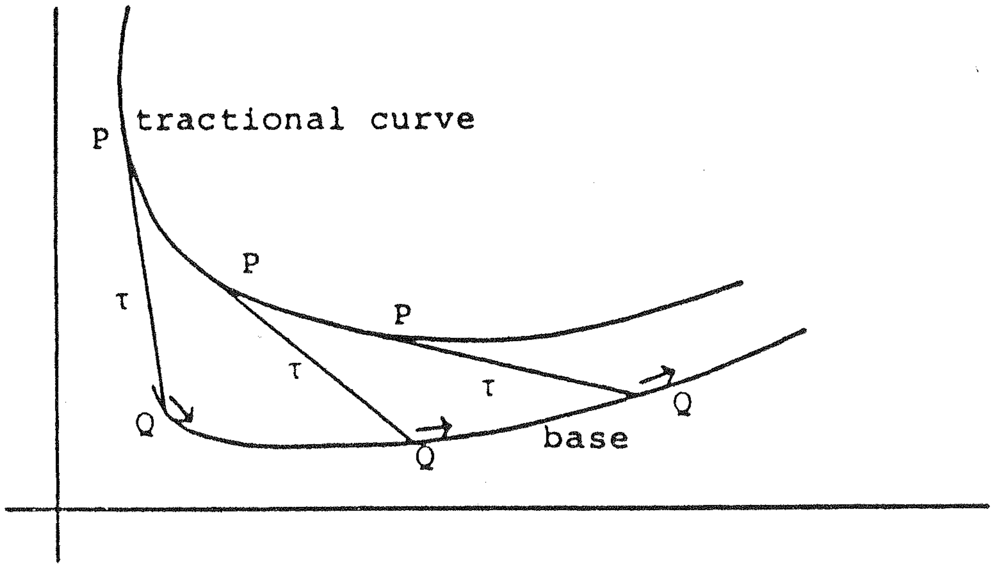



**Fig. 21 Tractional Motion.**

straight or curved base as QQQ, then the other end P of the cord (where the heavy body is assumed to be) describes the pertaining tractional curve PPP which has the property that in each of its points the cord is tangent to it.

Riccati studied the use of tractional motion in solving differential equations. Differential equations involved infinitely small quantities, the so-called differentials; such equations were at the centre of the mathematical interest in the eighteenth century. The creation of the differential and integral calculus had enabled scientists to translate mechanical problems about motion and deformation in mathematical terms. In most cases the result of such a translation was a differential equation whose solution provided the description of the mechanical process that was studied. The majority of the mathematical investigations in that period had the ultimate purpose of solving differential equations. Differential equations were problems; their solutions were functions or curves. Eighteenth century mathematicians endeavoured to solve these differential equations by deriving the equations of the solution curves or the formulas of the solution functions. Being solutions, these equations or formulas should no longer involve differentials, they should be "normal" equations concerning finite quantities only. However, mathematicians encountered types of differential equations whose solutions they could not express in such finite equations or formulas. Riccati's study concerned certain such types. He argued that, if finite equations for the solutions could not be found, the solution curves might perhaps be traced by tractional motion. To be more specific, he suggested a method for tracing the solution curves of certain classes of differential equations, and that method involved tracing auxiliary curves by tractional motion. The details of his procedure are mathematically too technical to be discussed here.

§ 2. Seemingly Riccati offered instrumental solutions of problems that could not be solved directly in a mathematical way — a timehonored procedure in applied mathematics, witness a rich history of computational instruments from abacus, via astrolabe, slide rule and mechanical integrator to the electronic computer. But did Riccati seriously consider tractional motion as a practicable means of solving differential equations? His terminology suggested actual execution of the curve tracing process, and he even referred to instruments that had earlier been made for tracing tractional curves.[7] On the other hand, in the majority of his examples the base curve was so complicated that, although its equation was known, it was hardly practicable to draw it — and tractional motion of course presupposed the base to be drawn. Despite the imagery of movement and machines, Riccati's approach has to be understood as belonging to abstract mathematics. He considered the abstract physical process of tractional motion as a solution of certain problems in pure mathematics, namely a certain class of differential equations. In other words, he considered tractional motion as a *means of construction* in abstract mathematics in the same way as the use of compass and straightedge is considered as means of construction in pure geometry; the mechanical imagery of "tracing", "using the compass", etc. serves as metaphoric description of abstract processes. The essential construction occurs not on paper but, if anywhere, in the mind of the mathematician.

This, then, is the theme whose journey through time and place I want to follow: the use of tractional motion as an abstract means of construction in solving problems equivalent to differential equations. I find it an intriguing theme. One would expect tractional motion to be seen as a problem in mechanics: what is the curve traced by such and such a process of dragging an object? That mechanical problem would, if translated into mathematical terms, lead to a differential equation, which in its turn might be solved; its mathematical solution providing the answer to the original mechanical problem. Here, however, by a notable 'Gestalt' switch, the roles are reversed: tractional motion, rather than being a problem itself, serves as a (means of) solution. Besides this unusual role division between mathematics and mechanics, solution and problem, my theme also concerns a remarkable

7. [Riccati 1752] pp. 28 and 42. The references were to Poleni's instruments which I discuss below.

interplay of geometrical procedures (construction of lines within figures) and analytical ones (derivation of equations). We see Riccati accepting an abstract mechanical process in a construction that should serve as an alternative for such a convincingly satisfying analytical procedure as writing down the equation of the solution.

In following the theme's journey from Holland to Italy I will try to show how and why tractional motion for some time acquired such a special status in mathematics.

§ 3. Riccati's work constituted, as far as I know, the theme's journey's end — no mathematician after him seems to have studied tractional motion as an abstract theoretical procedure for solving differential equations. The theme started its journey in Holland in 1692. In November of that year Christiaan Huygens studied tractional motion and wrote down his thoughts, calculations, designs and experimental results on nine manuscript folios which he gave the general title

> The quadrature of the hyperbola by a new quadratrix of it, which is traced in one single motion.[8]

The title reflects a structure of argument similar to Riccati's: a mathematical problem, the quadrature of the hyperbola, is solved, constructed, by means of a geometrical object, its quadratrix, which is a curve that in its turn is obtained by a mechanical process, namely tractional motion.

It appears from the manuscripts that Huygens wanted to design and make an instrument that accurately traced the simplest kind of tractional curve. This was the curve that arose when the base was a straight line and the cord had constant length (see fig. 22). Huygens called it the *Tractoria*; the curve has become generally known under the name *Tractrix*. It has the property that at each of its points the distance along the tangent from the point to the base is constant.[9] Figure 23 shows the design of a tractrix instrument which ultimately satisfied Huygens best. A block B was moved along the straight rim (serving as the base) of a horizontal surface. The beam DC was connected to B by a pin at

8. [Huygens ms 1692], cf. [Bos 1988] pp. 25-32.

9. With the X-axis along the base and the initial position of the cord of length a along the Y-axis, the tractrix has the equation

$$x = a \log [(a+\sqrt{a^2-y^2})/y] - \sqrt{a^2-y^2}.$$

The curve enjoys fame in mathematics not so much for being the simplest tractional curve as for the fact, discovered by Beltrami, that its surface of revolution around the X-axis provides a model for the hyperbolic type of non-Euclidean geometry.

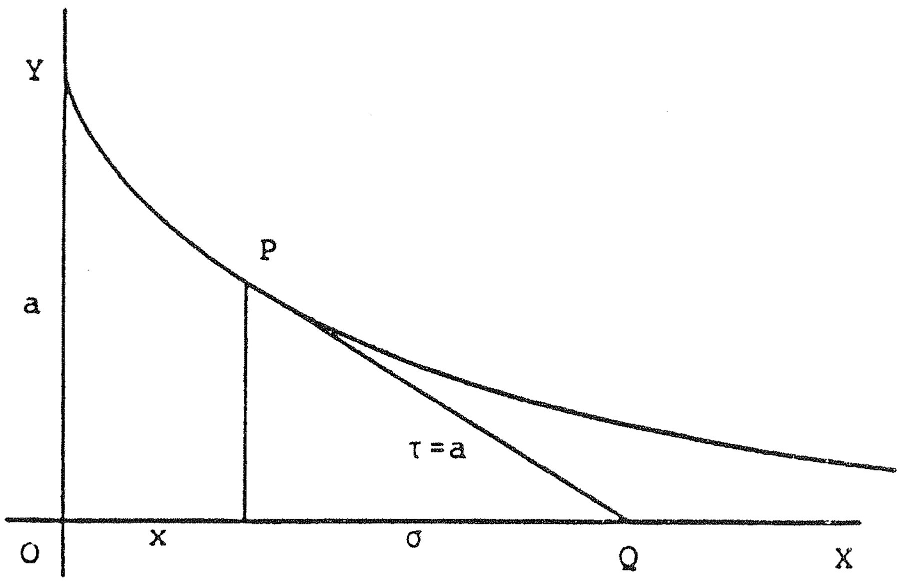



Fig. 22 The Tractrix.

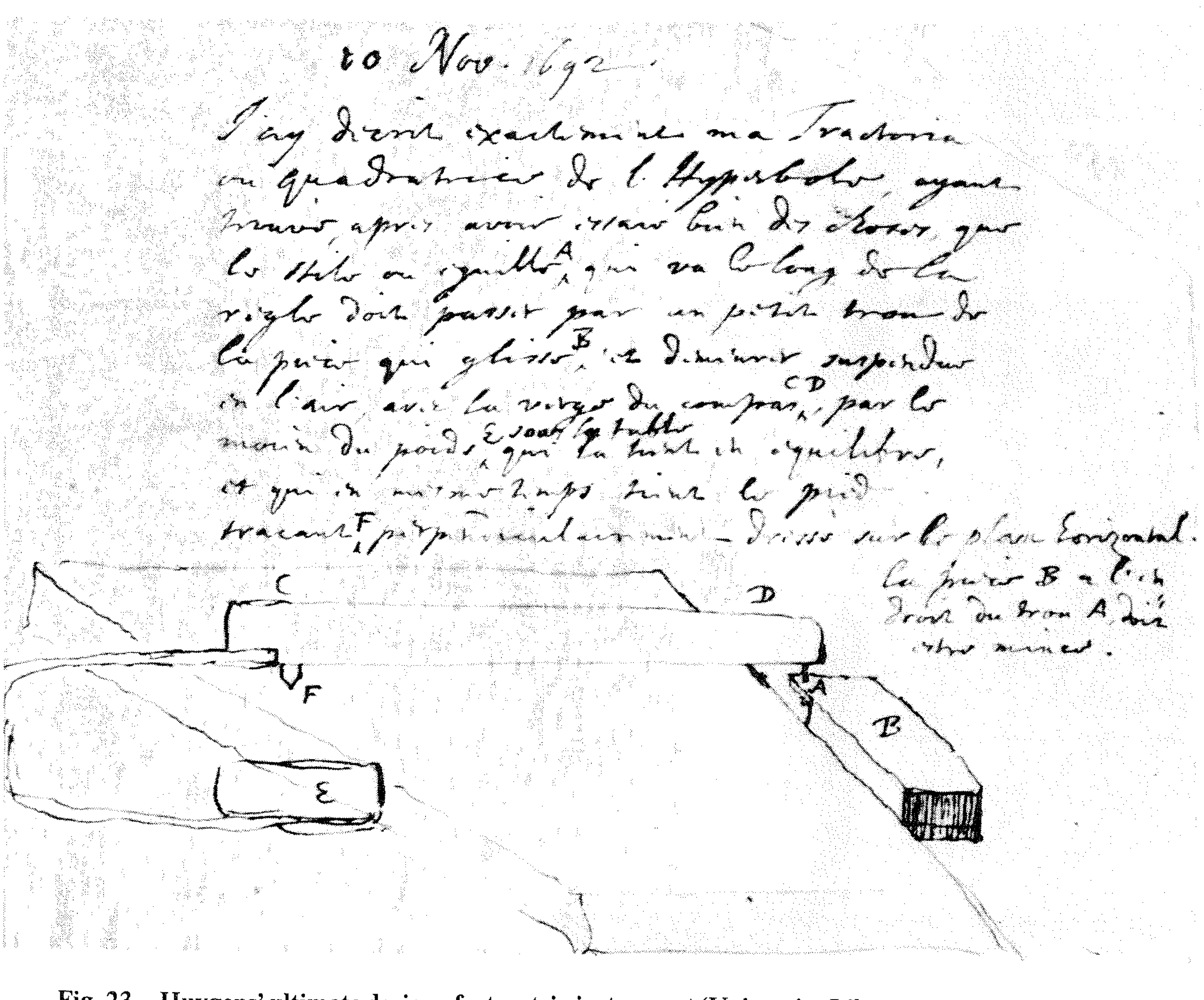



Fig. 23 Huygens' ultimate design of a tractrix instrument (University Library Leyden, Ms. Hug. 6, fol. 66[r]).

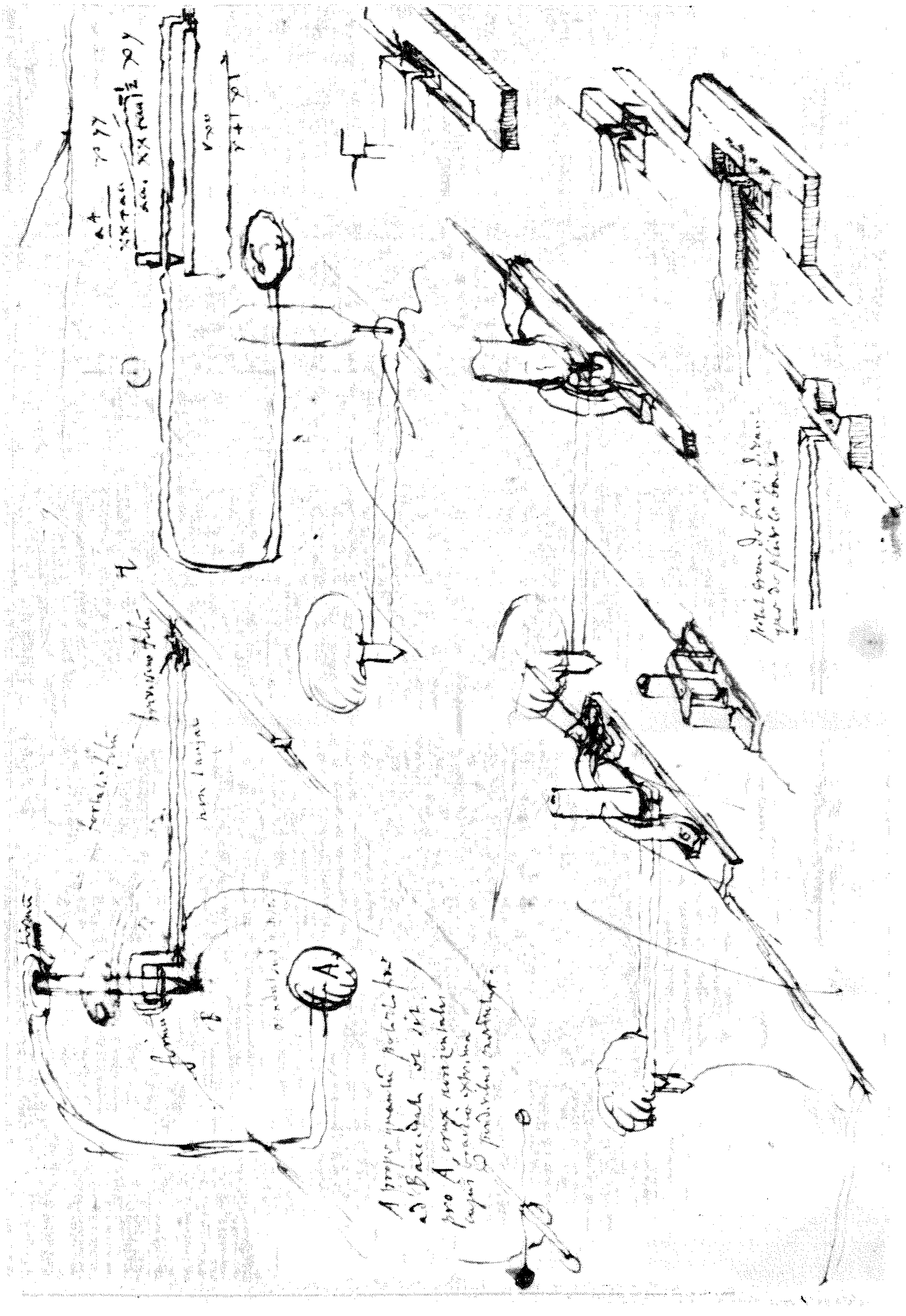


**Fig. 24** **Sketches for a tractrix instrument, from Huygens' manuscripts (University Library Leyden, Ms. Hug. 6, fol. 59r).**

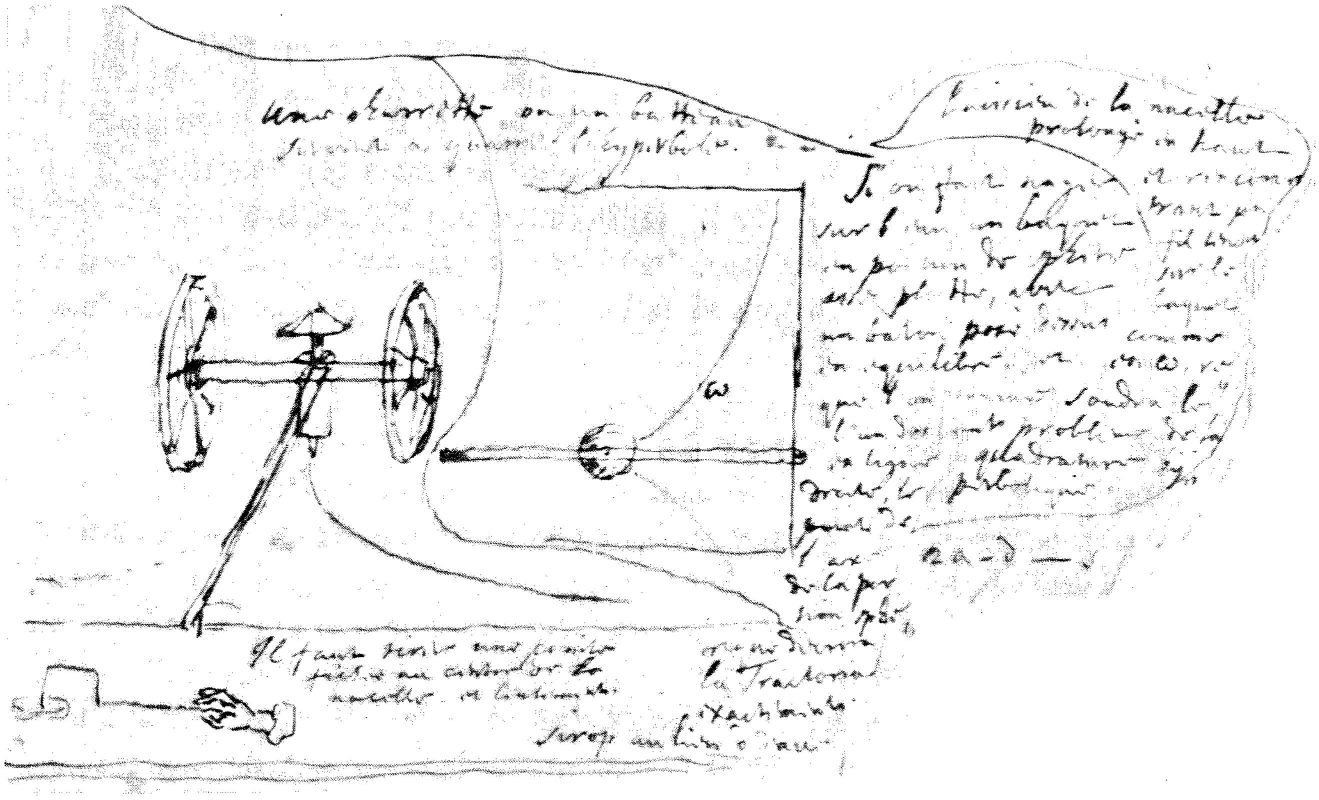


**Fig. 25 Variant designs of tractional instruments, from Huygens' manuscripts (University Library Leyden, Ms. Hug. 6, fol.64$^r$).**

A around which it could turn without friction. A pin F was placed at its other end; it traced the tractrix while B was moved along the rim. A weight E was connected with the beam by a U-shaped lamina in such a way that the centre of gravity of the whole system was below F, thus ensuring sufficient pressure as well as a vertical position of the tracing pin. In fig. 24 we see how Huygens also considered a weight on top of the tracing pin. Figure 25 shows variant designs which Huygens considered as well. One was a two-wheeled cart that was to be drawn over a horizontal surface tracing the curve by a pin between its wheels. The small drawing underneath related to the possibility of tracing by drawing a floating object. Huygens seriously considered this possibility although he finally abandoned it. The fluid would automatically ensure a horizontal position of the surface and friction could be increased by using syrup instead of water. The curve itself would be marked along a horizontal plane above the fluid's surface by a vertical pin fixed to the floating object.

§ 4. Although, as apppears from these figures and from several passages in the manuscripts, Huygens' interest in tractional motion had a decidedly technical bent, his main motive was not technical, not even theoretically mechanical, but abstractly mathematical. His ultimate aim in these studies was to overcome the Cartesian restrictive demarcation of geometry.

According to Descartes, curves belonged to geometry only if they had an algebraic equation, that is, an equation involving no other operations than addition, subtraction, multiplication, division and the extraction of roots. Other curves, for instance those involving logarithmic or trigonometric relations, were merely mechanical and had no place in geometry. I shall not here go into the question why and how Descartes had come to this conviction.[10] Suffice it to say that during the second half of the seventeenth century his work was so influential that all mathematicians were aware of his demarcation of geometry and that many felt compelled to justify their procedures if they found themselves crossing the boundaries which Descartes had prescribed. Also it should be noted that when Descartes first formulated his demarcation of geometry it was not a restriction but rather an extension with respect to earlier conceptions of geometry in which the restriction to straightedge and compass constructions induced limitations much stricter than Descartes'.

Now the tractrix does not have an algebraic equation, and so, from the Cartesian point of view, it did not qualify as an acceptable geometrical curve. But Huygens wanted to accept the curve, he wanted to study it and to use it as a "quadratrix" of the hyperbola, that is, a curve by means of which the quadrature of the hyperbola could be solved. The quadrature of the hyperbola was a problem much discussed since the middle of the seventeenth century. It is equivalent to determining to logarithmic function or the logarithmic curve.[11] This curve has no algebraic equation either and therefore it was not an acceptable curve within the Cartesian view of geometry. Huygens had found that, once one had the tractrix, the logarithmic curve could easily be constructed. Hence if the tractrix could be legitimated, the problem of the quadrature of the hyperbola would be solved.

What Huygens was doing in his manuscripts on tractional motion was to explore arguments to justify the tractrix contra Descartes; in other words he

10. See [Bos 1988] pp. 15-19; compare [Bos 1981] and [Bos 1989].

11. Quadrature of a curve meant the determination of the area bounded by the curve, the axis, a fixed ordinate and a variable ordinate. Taking $y = 1/x$ as the equation of the hyperbola, the quadrature z (fixed ordinate at $x = 1$) can be expressed as

$$\int_1^x dx/x = \log x.$$

Thus the quadrature is given by the equation $z = \log x$, or equivalently $x = e^z$. The curve expressed by these (non algebraic) equations is the logarithmic curve.

searched for a legitimation of the tractrix as a fully acceptable geometrical curve. This motivation is clear in many passages of the manuscripts. For instance:

> It should be admitted that, if my curve is supposed or given, one has the quadrature of the hyperbola. So if I find some means to draw it as exactly as a circle is drawn by an ordinary compass, would I not have found that quadrature? (...) It is true that I need the parallelism of a plane to the horizon; but that is possible, not in utmost precision, but like the straightness of a ruler. For the rest I draw my curve almost as easily as a circle and the machine I use comes very close to the simplicity of the compass.[12]

And:

> Descartes was wrong when he dismissed from his geometry those curves whose nature he could not express by an equation. It would have been better if he had acknowledged that his geometry was defective in so far as it did not extend to the treatment of these curves; for he was well aware that the properties and uses of such curves can also be investigated by geometrical methods.[13]

In an article on the tractrix which Huygens published in 1693 he wrote about the tracing of the curve:

> If this description, which by the laws of mechanics must be exact, could pass for geometrical, in the same way as those of the conic sections, performed by instruments, one would thereby have the quadrature of the hyperbola, and together with that the perfect construction of all the problems that can be reduced to that quadrature such as among others the determination of points on the *Catenaria* or Catenary, and the logarithms.[14]

12. "On doit avouer que ma courbe estant supposée ou donnée, on a la quadrature de l'Hyperbole. Si je trouve donc quelque moien de la decrire aussi exactement qu'avec un compas ordinaire on decrit un cercle, n'auray je pas trouvé quadrature? (...) Il est vray que j'ay besoin du parallelisme d'un plan a l'horizon; mais cela est possible, non pas dans la derniere justesse, mais comme la droiture d'une regle. Pour le reste je decris ma courbe presque aussi facilement qu'un cercle et la machine que j'emploie approche fort la simplicité du compas." Huygens [ms 1692] fol. 62$^r$, [Oeuvres] vol. 10, p. 412, note. (Here, as throughout the present article, the translations are mine.)

13. "Non bene Cartesius e geometria sua rejiciebat curvas quarum naturam aequatione exprimere non poterat. Melius agnovisset geometriam suam hac parte mancam quod ad earum tractationem non attingeret non enim nesciebat talium quoque curvarum proprietates atque usus geometricis rationibus investigari." [mss 1692] fol. 60$^v$.

14. "Si cette description, qui par les loix de la Mechanique doit être exacte, pouvoit passer pour Geometrique, de même que celles des sections de Cone qui se font par les instrumens l'on auroit par elle, avec la quadrature de l'Hyperbole, la construction parfaite des Problemes qui se reduisent à cette quadrature; comme sont entre autres la determination des points de la *Catenaria*, ou Chainette, et les Logarithmes." [Huygens 1693a] pp. 409-412. The catenary is the curve formed by a chain or cord suspended from two points; its mathematical description was a famous problem at the time.

From the quotations it is clear how Huygens tried to legitimize the tractrix: he wanted to show that instruments could be devised, and actually built, that traced the tractrix as precisely as compasses or some other then familiar instruments traced circles or conic sections. Thus his legitimatory criterion was practical feasibility. Still in a deeper sense his argument concerned pure abstract geometry rather than practical precision. Indeed if mere practical precision was at stake, Huygens had better means at his disposal: he could use logarithm tables, which were then available in precision of ten digits, to calculate coordinates of points of the tractrix and he could then trace the tractrix through those points with a much more precise result than any instrument could give. Significantly, he did not do so; he relied on the tracing instrument for the legitimation. Thus we can characterise his approach to the justification question as idealization of practical geometrical precision with emphasis on idealization.

So at the beginning of its journey through pure mathematics, tractional motion was invoked and used with a special purpose: legitimation. Tractional motion was precise, it traced the tractrix, therefore the tractrix was a legitimate geometrical curve. Soon others took up the theme and added a new but related purpose, namely construction.

§ 5. Huygens published some results of his studies on tractional motion in an article that appeared in 1693.[15] He explained the mathematical properties of the tractrix and he expounded the idea that if the tractrix could be traced by acceptable motion, then the quadrature of the hyperbola could be solved in a legitimately geometrical manner by means of the tractrix. He did not, however, explain the instrument he had himself designed for tracing that curve.

Huygens' idea was immediately taken up by Leibniz, Johann and Jakob Bernoulli, and l'Hôpital. They explored the following argument.[16] If tractional motion, by its precision in principle, provides valid argument for justifying and legitimating the tractrix, then in fact a new means of construction is introduced in the geometry of curves. That is, we may conceive curves as known when we can indicate a tractional motion process by which they are traced.

Leibniz, the Bernoulli's and l'Hôpital were at that time engaged in developing the recently created Leibnizian version of the differential and integral calculus. By means of that calculus processes of motion could be

15. [Huygens 1693a].
16. For a more detailed discussion of this episode see [Bos 1988] pp. 32-52.

described mathematically by differential equations. These differential equations were still conceived as primarily geometrical problems, their solution therefore required a geometrical construction of the solution curve. Now if tractional motion was an acceptable means of construction, one could study several classes of such motions, derive the pertaining differential equations and classify these as constructible by tractional motion. If then, later, in dealing with some mechanical or other problem, one would encounter such a differential equation one would know that its solution could be constructed by tractional motion.

Several articles[17] with this motivation appeared in the *Acta Eruditorum* in 1693 and 1694. They centred on a problem proposed by the Bernoulli's (it became known as the "Bernoulli problem"). It concerned curves traceable by one particular kind of tractional motion, and the pertaining differential equation was one of the most complicated that were studied at all in the early period of the calculus.[18]

The interest in the subject offered Leibniz the opportunity to testify to an earlier origin of tractional curves; in an article of 1693 he told how in the early 1670's Perrault used to ask mathematicians in Parisian salons what curve was traced by his watch if he moved the end of its chain along the rim of the table and let the watch on the table follow that movement.[19] Leibniz wrote that at the time he had solved the problem (which meant that he had found a construction for the tractrix curve, because that was, obviously, the curve traced by Perrault's watch) but that he had not published it. Nevertheless I have taken Huygens' studies of 1692 rather than Perrault's question of c. 1672 as the origin of the theme of tractional motion, because I take it as essential for the theme that tractional motion was actually accepted as a means of construction. There are no signs that this was the case with Perrault's question nor with Leibniz' solution of it. Figure 26 shows a drawing from a later source on tractional motion;[20] the drawing is clearly a reference to Leibniz' story about Perrault.

§ 6. Thus legitimation and geometrical construction were the motives behind the earliest appearances of our theme. These motives were appropriate

17. [Jakob Bernoulli 1693], [Jakob Bernoulli 1696], [Johann Bernoulli 1693], [l'Hôpital 1693a], [l'Hôpital 1693b], [l'Hôpital 1694], [Huygens 1693b], [Huygens 1694a], [Huygens 1694b], [Leibniz 1693a], [Leibniz 1693b], [Leibniz 1693c], [Leibniz 1694].

18. Namely $yds = p(xdy—ydx)$; cf. my article [1988], pp. 32-43.

19. [Leibniz 1693b], pp. 296-297; This was Claude Perrault (1613-1688); his brother Charles Perrault (1628-1703) became famous as the author of the Mother Goose fairy tales.

20. [Poleni 1729], Plate BB, Figure 3.

**Fig. 26 Tracing a tractrix by means of a chain watch, from [Poleni 1729].**

and understandable around 1700. At that time, the Cartesian view of geometry was still so strong that the use of non-algebraic curves needed a legitimation. Also the conception of solution of problems was still predominantly geometrical, so a problem like the quadrature of the hyperbola required a construction for its solution as a matter of course.

In the period after 1700 several processes occurred within mathematics that made the use of tractional motion for the purposes above obsolete — so obsolete indeed that it is something of a surprise to find them still as late as 1767 in the writings of Riccati. These processes fall under the general heading of the emancipation of analysis from geometry. Analytical methods, in the sense of calculation and manipulation with formulas, were introduced and elaborated in the seventeenth century in connection with the study of geometrical problems. But they gradually acquired independent meaning and mathematicians tended more and more to disregard their geometrical interpretation or motivation. At the same time a gradual habituation to new material occurred; mathematicians became used to the new curves such as the tractrix, and thereby they no longer felt so strongly the need for a separate legitimation of these curves. One important factor in this habituation process was that new notations were introduced that enabled mathematicians to write

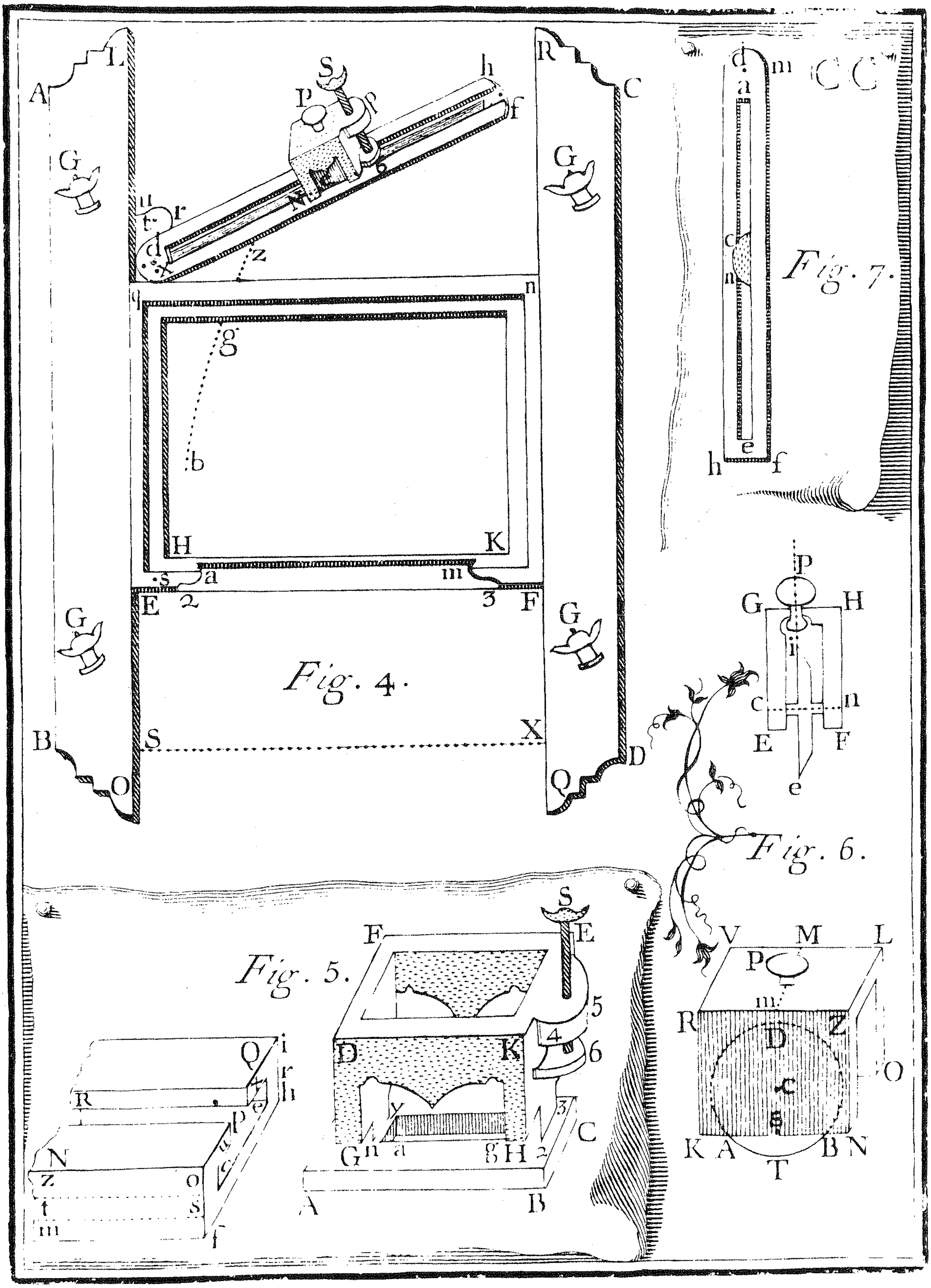



Fig. 27 Poleni's instrument for tracing the tractrix, from [Poleni 1729].

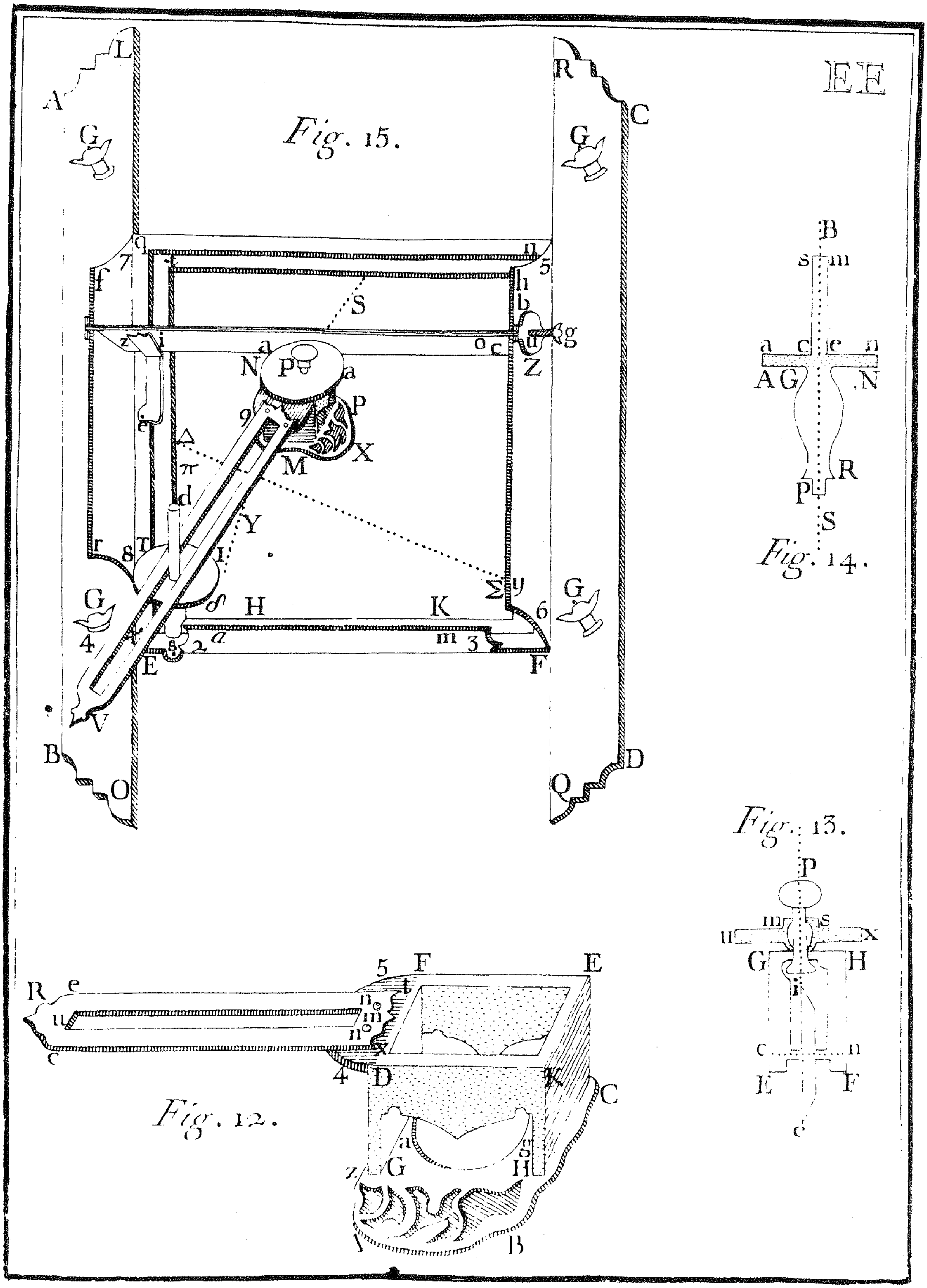



**Fig. 28** Poleni's instrument for tracing the logarithmic curve, from [Poleni 1729].

down equations for these new curves. Under the influence of these processes the earlier purposes of the use of tractional motion within pure mathematics became obsolete. Why, then, did the theme of tractional motion stay alive so long? and how, having reached Germany, Switzerland and France, did the theme continue its journey from Holland to Italy amidst a general decline of its interest? In the sequel I shall briefly sketch the answers to these two questions.

§ 7. In the period 1710-1730 our theme made three appearances in mathematical literature. Three authors from different countries independently discussed tractional motion. In each case the motivation was legitimation. These writers were: a Monsieur Bomie in France, who published an article on tractional motion in 1714,[21] John Perks in England with an article in 1715,[22] and G. Poleni in Italy in 1729.[23] The first two authors did little more than to explain a design of an instrument that could trace a tractrix or related curves and to explain the pertaining mathematics.

The case of Poleni is somewhat more interesting. Inspired by Huygens' article (apparently he was not aware of Bomie's and Perks' work) he set out to construct precision instruments to trace the tractrix and the logarithmic curve. Figures 27 and 28 show the plates of the two instruments in Poleni's publication.[24] Before publication, however, he sent copies of the machines to three famous Italian mathematicians, namely Antonio Conti, Gabriele Manfredi and Jacopo Riccati (the father of the Vincenzo whom we have already met). In his covering letter he claimed, in the same way as Huygens, that by making his precision instrument he had finally geometrically legitimized the tractrix, the logarithmic curve, and thereby all other curves that could be constructed from these by classical means. He asked his cor-

21. [Bomie 1714] cf. [Fontenelle 1714]; I have not found any biographical information about Bomie.

22. [Perks 1715]; on Perks see [Pedersen 1963].

23. [Poleni 1729].

24. [Poleni 1729] plates CC and EE. In the tractrix instrument (fig. 27) the frame EFnq is moved between the rulers LO and RQ in the direction qE. The lamina dxfh, which can turn freely around a pin at d, follows the frame. At an adjustable distance dP from d, a wheel can be fixed between the parts of the lamina. The wheel rests on the paper and traces the curve, which has the property that at each of its points the lamina dP is along its tangent — so it is the tractrix (with tangent length dP and axis ds). The design of the second instrument (fig. 28) is based on a property of the logarithmic curve, namely that at each of its points the *subtangent* has the same length. The subtangent is the segment along the axis from the ordinate to the intersection of the tangent. The instrument employs again a frame moving between rulers. Furthermore the lamina VM and the casket PM housing the tracing wheel are now designed in such a way that during the motion the direction of the wheel is towards d, so that the tangent of the curve is along Md, whereas the ruler zZ, which pushes the casket, ensures the subtangent zd to be constant.

respondents to test the instruments and to report their findings. They did so, and Poleni incorporated their answers in his publication in 1729. That publication consisted of a letter to Jakob Hermann, in which Poleni described the instruments and their use, the letters of Conti, Manfredi and Riccati, and a treatise on the mathematical properties of the tractrix.

Conti and Manfredi were generally positive in their reaction, but non-committal as regards the legitimatory purpose of Poleni's instrument. Riccati took the trouble to go into the question of legitimation more deeply. He complimented Poleni on the accuracy of the instrument but rejected the claim that practical accuracy was in any way an argument in support of the geometrical legitimacy of the curve. He also rejected Descartes' restriction of geometry to algebraic curves and claimed that, because pure geometry concerned contemplation rather than practice, legitimation could not be based on the precision of instruments. Rather, criteria for accepting or rejecting geometrical entities (as curves) or procedures ( as constructions) should relate to pure contemplation, not to practice. He mentioned in particular abstract existence and non-contradiction as criteria — a remarkably modern point of view.[25]

Thus we find in the period 1710-1730 three mathematicians, Bomie, Perks and Poleni, still interested in legitimating tractional curves in defiance of the Cartesian demarcation of geometry. By the time of publication this interest was dated and it is therefore not surprising that our three authors were of decidedly less mathematical calibre that those that were interested in tractional motion around 1700. In fact, Jacopo Riccati voiced the opinion of those who were in contact with up to date mathematical thinking: the problem of legitimation was no longer urgent.

§ 8. Although the question of legitimation had lost its urgency, the theme of tractional motion did recur in mathematics, introduced this time by one whose rank as mathematician was unquestionable: Leonhard Euler. In 1741 Euler published an article called

25. Cf. "Munus est Geometrae investigare et expendere Curvarum naturam; ideam illarum exprimere distinctam; idque palam facere, qua ratione progignantur; denique ex earum origine commonstrare, eas non contradicentium numero, sed ejusmodi rerum, quae fieri possunt, contineri. Ac quamdiu proprietatibus illarum utitur, ut propositis quaestionibus difficilioribus satisfaciat, res versatur ipsa in contemplatione sola: cum vero hinc ad usum operamque rem traducit, Mechanica saepissime opus est, et iis instrumentis, quae ad manum pertinent." [Poleni 1729] letter of Jacopo Riccati.

"On the construction of equations by means of tractional motion and other matters concerning the inverse method of tangents."[26]

Euler was the giant of eighteenth-century analysis, the mathematical science that had emerged through the emancipation from geometry of such seventeenth century methods as analytic geometry and the early versions of the differential and integral calculus. Euler's mathematics, to put it briefly, was about formulas, not about figures. And so in dealing with differential equations his approach was to try and find the formulas for the functions that were the solutions of the differential equations, or the equations of the solution curves. The usual way to do so involved a technique called "separation of the variables". Until c. 1740 this was practically the only way to find the equations of the curves or the formulas for the functions. In his article Euler dealt with a class of differential equations for which separation of variables failed. And for all the primacy of analytical methods in Euler's style of mathematics, when analysis failed, when no formulas could be found, Euler reached back to earlier geometrical canons of solution, namely construction. In the article he dealt with a special class of differential equations[27] which did not admit separation of variables. Therefore he wrote:

its construction by means of tractional motion merits our attention. Moreover, as that construction is rather simple and easy, it will be worthwhile to reduce the construction of such a difficult equation to tractional motion.[28]

Again, the format of this article does not allow the presentation of the technical mathematical details of Euler's study. Suffice it to say that for a considerable class of differential equations he described a procedure, involving tractional motion, by which the solution curves could be traced.[29] Euler did not present his use of tractional motion as feasible in practice, nor

26. [Euler 1741].

27. They were the differential equations of the form $ds + ssdz = Z(z)dz$. The so-called Riccati differential equation (named after Jacopo Riccati who had published about it in 1724 — not in connection with tractional motion) belongs to this class. About that equation it was known that only in special cases could the variables be separated.

28. "eius constructio ope tractorii motus attentionem meretur. Quae constructio cum sit praeterea admodum simplex et facilis, operae pretium erit aequationis tam difficilis constructionem ad motum tractorium reduxisse" [Euler 1741, p. 84].

29. Euler's construction of the solutions of the differential equation $ds = Z(z)dz - s^2dz$ was as follows (see the Figure): Draw a curve BN with coordinates (expressed in terms of the parameter z) $u = QN = \frac{1}{2} b \log Z$ and $t = AQ = 2b/\overline{\sqrt{(Z(z))}dz}$, where b is an arbitrary constant (and the logarithm and the integral are assumed to be constructable!). With this curve as *base* apply tractional motion (cord length b) to trace a new curve AM. Call $w(z) = \angle MNQ$ and $q(z) = \tan\frac{1}{2} w(z)$. Then $s = q(z)\sqrt{(Z(z))}$ is the solution of the differential equation.

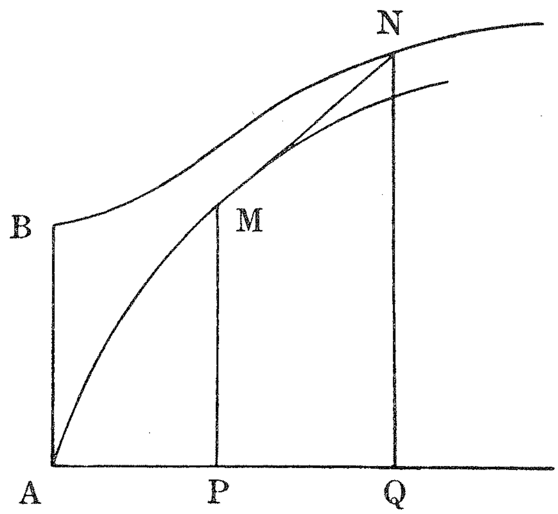
did he attach any legitimatory force to it — for him it was exclusively a means of abstract geometrical construction, to be invoked when analytical methods of representing the solutions failed.

§ 9. In its open recourse to, by then rather dated, geometrical methods, Euler's article stood outside the main stream of mathematics. Still, others acknowledged and used it. For instance, when Clairaut in an article published in 1745 dealt with a problem in mechanics and encountered a differential equation of the type Euler had studied, he explained that it was

> ... famous among geometers who have applied themselves to the integral calculus, and nobody has yet been able to separate its indeterminates except in a small number of very special cases. Mr Euler is the only one, as far as I know, who has given a general construction of the equation, and although that construction does not have all the advantages of those that are based on the separation of the indeterminates, it nevertheless is worthy of its learned author. As the construction is very little known among geometers I shall give it here.[30]

And he spelled out Euler's construction.

Clairaut did not provide the proof that the solution gained by Euler's construction procedure using tractional motion actually satisfied the differential equation, and this omission may have caused the final move in the journey of our theme. For, as Vincenzo Riccati explained in the preface of his




30. "fameuse parmi les Géomètres qui se sont appliquez au Calcul Intégral, et dont personne n'a pu séparer les indéterminées, que dans un petit nombre de cas très-particuliers. M. Euler est le seul, que je sçache qui en ait donné une construction générale; et quoique cette construction n'ait pas tout l'avantage de celles qui sont fondées sur la séparation des indéterminées, elle est cependent digne de son sçavant Auteur. Comme elle est très-peu connue des Géomètres je le mettrai ici." [Clairaut 1745] p. 9.

1752 monograph, he had hit upon Clairaut's article and, because no proof was given, he tried to find the proof, and succeeded (apparently Riccati had no access to Euler's original article to which Clairaut did refer). The exercise spawned a great enthusiasm for the method and Vincenzo Riccati started to generalize and elaborate. His results in that enterprise are the subject of the publications I mentioned at the beginning of this article; indeed Riccati's contributions are essentially no more than generalizations and clarifications of Euler's procedures.

§ 10. This brings us back to the endpoint of our theme's journey through mathematics, from Huygens' perhaps somewhat overconscientious attempts to legitimate the tractrix to Riccati's overdiligent elaboration of Euler's construction. It is not one of the grand themes in the history of mathematics. But it is, in my view, a charming, curious and revealing theme. Its journey testifies to the persistence of geometrical ideas even amidst the general emancipation of mathematics from geometrical ties; such ideas still emerged naturally in cases where the new methods of analysis failed. In this way the story of tractional motion documents a way of thinking that remained present for a long time and that occasionally had an influence on the direction of mathematical research.

The theme also points to a more general question which has always been present in the development of mathematics. This is the question: When is a problem solved? When do we know a mathematical object? Tractional motion for some time served as an adequate solution of a certain type of problem. Giving a construction by tractional motion meant that the problem was solved and that henceforth the curve in question was known. But the issue was dubious and ultimately tractional motion lost its conviction as a means of solution. Similar processes of the interpretation and re-interpretation of what constitutes a solution and what constitutes proper knowledge of mathematical objects have played — and still play — an important role in mathematics, and I find these processes very instructive objects of historical study.

Finally, I think the theme has a certain charm in its seemingly naive attitude to the mechanical intricacies of instruments and the mathematical complexities of differential equations. To me, this charm adds to the pleasure I have in following the theme's journey, in noting the variations and echoes of general ideas and in finding links between mathematical activities of individuals in Holland and Italy.

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