in Bos, H. e.a. Studies on Christiaan Huygens 1980 126-146

Huygens and mathematics

H.J.M. Bos

1. Introduction

There are two highlights in the development of mathematics in the seventeenth century. These are: the creation of analytical geometry by Descartes and Fermat, and the creation of the differential and integral calculus by Newton and Leibniz. Huygens had little to do with either of these highlights. From Descartes' *Géométrie* of 1637 he learned to use the new analytical geometry with ease and certainty, but he did not add to it. He was also familiar with methods in infinitesimal calculus which may be seen as precurosors of the differential calculus, for instance those of Cavalieri, Fermat or Pascal, but in this domain Huygens did not contribute much himself. When, around 1690, he learned about the new differential and integral calculus he was no longer in a position to use it in a creative way. This means that the obvious approach to my subject "Huygens and mathematics", namely to ask how Huygens fits into the story of seventeenth century mathematics, does not work well.

This is a curious state of affairs because Huygens was a great mathematician in the judgment of his contemporaries and of all those who have later acquainted themselves with his work. The reasons for this discrepancy lie in the fact that historians of mathematics tend to see the highlights of their story in *theories* and in *methods* (as analytical geometry and calculus). But there is another side to mathematics too, which in that approach tends to remain in the background, and that is the side of the *material* to which the theories and methods apply, and the *problems* which they help to solve. Huygens as a mathematician was not a man of abstract theories and methods, his preference lay towards the use of these to solve problems, preferably problems in physics. I think therefore that I should structure my report on Huygens and mathematics by taking as central themes Huygens' approach to mathematical problems and his dealings with the material of mathematics.

The material of the new mathematics worked out in the seventeenth century was first and foremost curves. In this period the number of curves investigated by mathematicians increased enormously. The curve collection of classical mathematics, consisting of the conic sections, some higher algebraic curves and two or three transcendental curves, was extended by the algebraic curves introduced by Cartesian analytical geometry, and by many new transcendental curves. The exploration of this new mathematical material of curves forms one central theme in Huygens' mathematical work. Interest in these new curves was largely caused by the confrontation with new problems that required the knowledge of such curves. The second central theme in Huygens' mathematics is his approach to these new problems.

Before reporting on Huygens' mathematics I shall have to give two somewhat technical preliminary explanations concerning these two themes; the one is about transcendental curves and the other about what I shall call inverse calculus problems.

Seventeenth century analytical geometry provided a most powerful new tool for the study of curves, namely to characterize the curve by its equation in two unknowns x and y. But these equations were algebraic, that is, they involved as operations only addition, subtraction, multiplication, division and roots. There were no equations or formulas involving sines, logarithms, exponentials or (in the early period of analytic geometry) infinite series. Therefore the tool was only applicable to curves which admit such an algebraic equation. These curves are called algebraic curves. There exist other, non-algebraic curves they are called transcendental[1]; examples are the spirals, the cycloid and logarithmic curves. Obviously these transcendental curves caused difficulties because the new methods of analytical geometry were not directly applicable to them. We will see that in Huygens' dealings with curves these difficulties around the transcendental curves played a crucial role.

There are some standard problems in the geometry of curves which we might call calculus problems because the differential and integral calculus has provided easy methods to solve them. These problems are of the following form (see figure 1): Given a curve and a point on it, to find, or construct, the tangent in that point, or the perpendicular in that point, or the area bounded by curve, axis and ordinate through that point (the so-called "quadrature"), or the arc-length from the origin to that point. These problems in general do not lead to new curves because they presuppose a given curve. With the term "inverse calculus problems" I want to indicate a type of problems that are inverse to the ones just mentioned and that do lead to new curves. They are of the form: For which curve have the tangents such and such a property? or, which curve has such and such an area? In the case that a property of the tangents was given, these problems were called inverse tangent problems in the seventeenth century; my term is a generalization.

These inverse calculus problems were a key factor in the development of the differential and integral calculus in the seventeenth century because they required new methods and because they gave rise to new

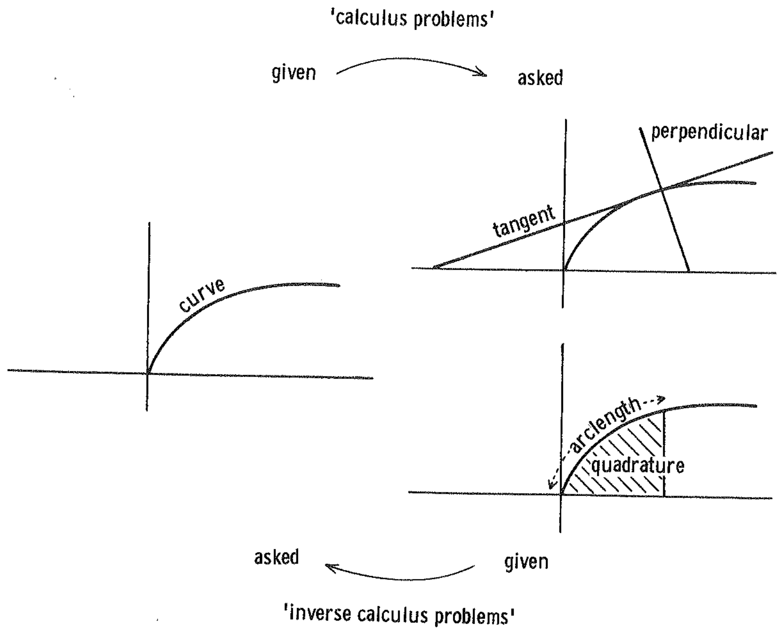


Figure 1

curves. Huygens was repeatedly confronted with these problems and his studies of them are very revealing of his approach to mathematical problems.

2. The formative period, 1645-1655

Huygens approached mathematics with a very characteristic style. He formed this style in the decade from 1645 to 1655, which may be called his formative period in mathematics. It is the period of his first studies in higher mathematics, guided by Van Schooten, of his further independent mathematical research and of his first publications.

These first publications concerned very classical and prestigious problems, namely the quadrature of segments of conic sections, in particular the quadrature of the circle. His *Theorems on the quadrature of the hyperbola, the ellipse and the circle*[2] appeared in 1651, with an appendix containing a refutation of the circle quadrature by Geogory of St. Vincent. The *Inventions about the magnitude of the circle*[3] followed in 1654 with an appendix containing solutions of a number of classical geometrical problems. Before these publications Huygens had written a long work on the equilibrium position of floating bodies.[4] This required the determination of volumes and centres of gravity of many mathematical shapes like cones, paraboloids and their segments. He never published this work.

It appears from these studies and from the further manuscripts of this period that Huygens' training in mathematics was predominantly classical. It is true that, through Van Schooten, Huygens acquainted himself with the modern methods of Descartes, Fermat, Cavalieri and others, but the subjects he studied and the methods he used in most cases were directly taken from the classical mathematicians, notably from Archimedes. Indeed, even when he did use the new analytical geometry, it was to solve classical geometrical problems, and Huygens presented the solutions (in the appendix to the *Inventions*) in thoroughly classical geometrical style, omitting the analytical methods by which he had found them. In fact Huygens, like many of his contemporaries, believed that the mathematicians of antiquity had possessed an analytical method similar to the Cartesian and Fermatian analytic geometry.[5]

I want to show in some detail the central result of Huygens' first publication. This because it is a beautiful result (Huygens was rightly proud of it) and also because I can use it to illustrate the characteristics of the mathematical style that Huygens acquired in his formative period. Archimedes had proved (see figure 2) about the parabola that the area of a segment ABC is 4/3 of its inscribed triangle ABC. This is Archimedes' quadrature of the parabola. Furthermore Archimedes had proved that the centre of gravity L of the segment ABC lies on the axis BD of the segment and such that BL is to LD as 3 : 2.

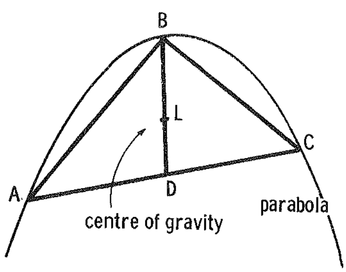


Figure 2

Huygens studied the same questions in the case of an arbitrary conic section, ellipse, circle of hyperbola. He did not find the direct determination of the quadrature or the centre of gravity (that would amount to having

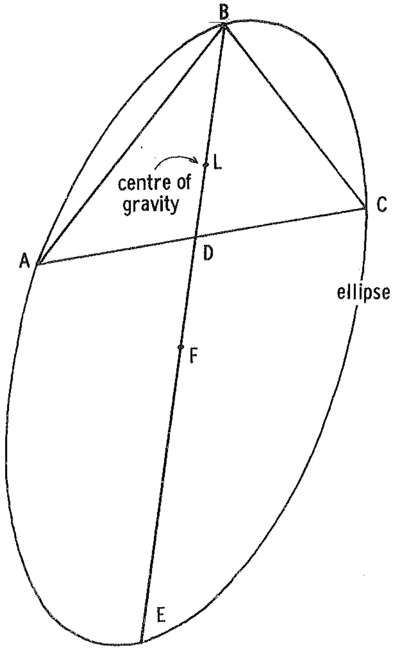


Figure 3

solved the quadrature of the circle) but he found a relation between the two. His result was this (see figure 3): Let ABC be a segment of an ellipse (the case of the hyperbola is analogous). F is the middle of the ellipse, L the centre of gravity of the segment. Then Huygens proved that[6]

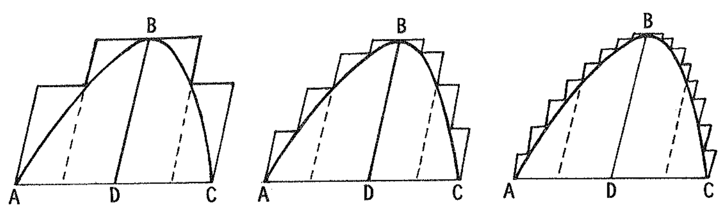
segment ABC: triangle ABC = $\frac{2}{3}$ ED : FL.

Now triangle ABC and 2/3 ED are known, so that Huygens here relates the area of the segment to its centre of gravity; if the one is known, the other can be found too.

A beautiful result. It may be called Huygens' master-proof in classical mathematics: with classical methods applied to a classical problem he found essentially new results. The result was also powerful: in his *Inventions* he used it to work out a method to approximate the quadrature of the circle (or in other words π), and later, in 1661, he used it in a method to calculate logarithms by means of the quadrature of the hyperbola.[7] In both these cases his methods were essentially better than the existing ones.

In order to show how this result illustrates characteristics of Huygens' mathematical style I shall have to say more about Huygens' proof of it. Huygens proved first[8] that the centre of gravity of the segment lies on the axis. For this he considers (see figure 4) circumscribed rectilinear figures around the segment. It can be proved easily that the centres of

Figure 4



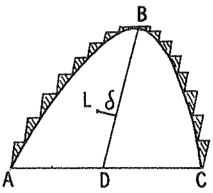


Figure 5

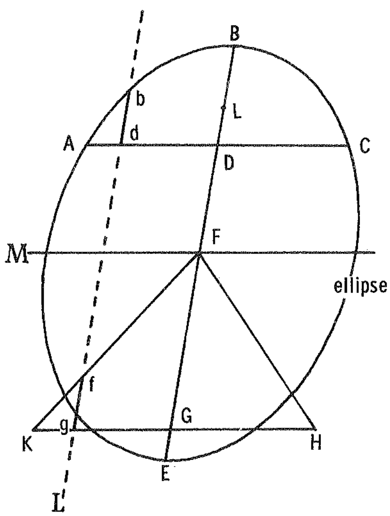


Figure 6

gravity of these rectilinear figures lie on the axis. The approximating figures now suggest the following argument: I can choose the figures ever closer to the segments; the centre of gravity is always on the axis. Therefore in the limiting case, when the approximating figure coincides with the segment, the centre must also be on the axis.

Huygens, however, following the classics, does not accept this argument as proof. Instead he argues as follows. Suppose (see figure 5) the centre of gravity lies outside the axis, at a certain distance δ. I can choose the circumscribed figure so near to the segment that the difference between them – arched in the figure – is so small that it cannot cause a shift of δ in the position of the centre of gravity. This can be proved on the basis of the known properties of the centre of gravity. Now this is a contradiction and therefore the presupposition, namely that the centre of gravity lies outside the axis, must be false. Hence it lies on the axis, *quod erat demonstrandum*.

By using a strictly logical argument based on a *reductio ad absurdum*, Huygens avoids the direct limit argument. This *rigour of proof* is one characteristic of the classical style which Huygens acquired in mathematics.

Another step in the proof[9] employs an argument on the equilibrium of figures. Huygens chooses (see figure 6) a triangle KFH with KH parallel to AC and

$$FG^2 = BD \times DE$$

and he shows that in that case for each line L parallel to the axis, the intercepts bd in the segment and fg in the triangle are in equilibrium around the line M. From this he proves, again by means of a rigorous *reductio ad absurdum*, that segment ABC is in equilibrium with triangle KFH around F.

I find that this episode in the proof well illustrates a second characteristic of Huygens' style in mathematics, namely his impressive familiarity with the geometrical properties of the figures he studies and his ease in using them. It is a familiarity which does not need the help of algebraic methods, equations and formulas; Huygens actually *thinks geometrically*, he sees the relations in the figures, formulas are secondary for him.

Huygens presented his study to the public in impeccable Euclidean style, theorem following strictly proved theorem. This illustrates a third characteristic of his style: his care for the logical presentation of his results. But there is a deeper characteristic behind this, which I shall call Huygens' skill in *axiomatisation*. I have in mind here in particular the cases where Huygens worked out a mathematical theory of physical phenomena. There it requires a very special skill to formulate the basic principles such that they are evident from physical considerations and at the same time serve as powerful axioms for the subsequent mathematical theory. Huygens had this skill. He acquired it from the example of Archimedes' works and showed it in his own studies on floating bodies and collision.

So, together with his care for logical rigour in proof, and his geometrical way of thinking in mathematics, I note as a third characteristic of the style which Huygens acquired in his formative period his skill in choosing powerful mathematical axioms and a care for arguing the evidence for these from physical or other principles.

3. The creative period, 1655-1660

The year 1655, the year of Huygens' first journey to Paris, marks the end of his formative period and the beginning of the most creative and prolific period in his career, the years 1655-1660. In these years he invented the pendulum clock, discovered the ring of Saturn, worked out his theory of centrifugal force and found the tautochrony of the cycloid. Also in mathematics he went in new directions. He studied the questions which Pascal had proposed about the cycloid, he studied question of probability, he worked out methods for the rectification of curves and he formulated his theory of the evolutes of curves. Here much was new. It was new material: the cycloid is a transcendental curve, probability was quite a new subject for mathematical research. The questions were new: rectification of curves, that is, the determination of the arc-length of curves, was new and much debated, if only because Descartes had claimed that the lengths of curved and of straight lines could never be comparable.[10] New methods too: for rectification Huygens found a general, non-analytic method applicable in principle to all curves[11], and Huygens' theory of evolutes involved both new questions and new methods.

In exploring the new material, the new questions and the new methods, Huygens brought the characteristics of his classical style and these determined his successes and failures. In these five years they were mostly successes. I want to mention two in particular: his work on probability and his theory of evolutes.

In Paris Huygens had heard about the probability problems discussed by Fermat and Pascal. These concerned the question how the stake in a game of chance should be divided if the game had to be stopped half-way. Huygens, at home, worked out a theory, wrote it down in Dutch, Van Schooten translated it into Latin and published it in 1657; the Dutch version appeared in 1660. The book, *Treatise on calculations in games of chance*[12], was very influential in the development of probability theory. In this study Huygens introduced what we now call the *expectation* of a stochastic variable. If at a certain moment I have the possibilities of winning either amount a, or b, and if the chances of these gains are to each other as p : q, then my expectation in that situation, or as Huygens called it the *value* which that chance situation has for me, is

$$\frac{pa + qb}{p + q} .$$

This is Huygens' result.[13] The formula has an obvious extension to the case of more than two possibilities. Now this a most powerful concept. From it, as starting point, most of the problems discussed by Fermat and Pascal are solved by obvious mathematical calculations.

For me there is no doubt that it was Huygens' classical training in mathematics, and in particular the skill in axiomatization he acquired, which enabled him to take this decisive step in understanding probability. With this skill he saw the crucial phenomenon in the mathematization of chance situations. It was not the simple concept of probability – that was obvious to most participants in the discussions. But it was precisely the situation of a player when the game is stopped half-way: the player can make out the possible events of the play, their probabilities and their gains, and he has to decide what is the value or the expectation in that situation. Moreover, in keeping with what I have said above about Huygens' skill in axiomatization, he took care to *prove* or make evident his central principle of expectation. He did that by a deep and well-found argument based on the conception of equable play which for him was selfevident. For reasons of space I cannot go into the argument but I want to stress its presence: Huygens did more than simply define expectation, he argued the evidence for his definition too.

In the same way as in his study on probability, Huygens' theory of the evolutes of curves forms an example of a mathematical success due in large measure to Huygens' classical training, in particular to his mastery of the classical proof-methods by *reductio ad absurdum*. The origin of the theory is well known, it lies in Huygens' pendulum clock. In 1659 Huygens had found that if the weight of a pendulum can be made to move in a path in the form of a cycloid, then the pendulum will be truly tautochronous, that is, all its oscillations whether large or small, will take the same time.[14] Now the weight can be made to move in a cycloidal path by means of bent metal strips (Huygens called them "cheeks") applied at the point of suspension (see figure 7), against which the cord winds up during its outward movement, thus lifting the weight above the circular path which it would normally describe. In December 1659 Huygens found the precise form which the cheeks should have in order to give the weight a cycloidal path.[15] Indeed the cheeks had to be cycloids themselves.

The result about the form of the cheeks is striking and it inspired Huygens to work out a general theory for the process of unwinding or "evolution" of curves. Let (see figure 8) a curve α be unwound and

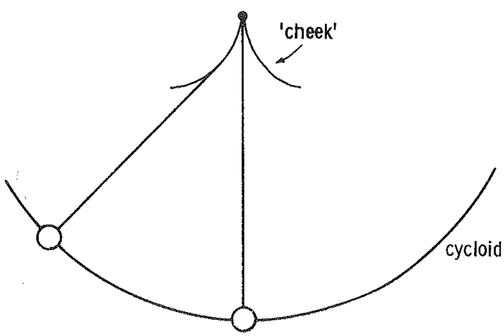


Figure 7

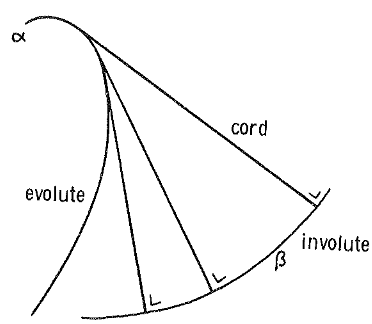


Figure 8

let a point on the cord which is wound off describe curve β. α is called the evolute and β the involute.[16] Huygens' theory deals with the relation of the two. He worked it out immediately after his discovery of the true forms of the cheeks: it was published in 1673 in his magnum opus the *Horologium Oscillatorium*.[17] I shall sketch the theory and draw attention to one central feature in it.

By definition of the process of unwinding, the cord in each of its positions is tangent to the evolute. Huygens first proves[18] that in each of its positions the cord is also perpendicular to the involute. That proof is intricate; it is based on certain inequalities derivable from Archimedes' axioms about the arc-lengths of convex curves. Huygens then goes on to prove[19] that conversely a curve which cuts all tangents of the evolute at right angles must be its involute. This amounts to showing that two curves which each cut the tangents at right angles and have one point in common, must coincide. In other words Huygens proves the uniqueness of the orthogonal trajectories of the family of tangents. Now the awareness that this is a point to be proved – rather than glossed over as obvious – is a mark of Huygens' mathematical genius and of his indebtedness to the rigorous logic of his classical examples. Indeed in modern terms he proves a uniqueness theorem for the solutions of a class of differential equations, and that is a kind of question which became current only in the nineteenth century.

His proof is equally a work of genius. I want to show this by explaining the basic lemma on which it is built. This lemma concerns (see figure 9) a convex arc AL with points A, B, C, D to L on it. In each of these points the tangent and the normal are drawn, so that along the curve a series of triangular figures AA′B, BB′C, CC′D etc. is formed. Huygens now proves that, given any length λ as small as one wishes, the points on the curve can be chosen such that the total length of all the perpendicular sides A′A, B′B, C′C, till K′K taken together is smaller than λ. His proof is basically correct. It does imply some tacit assumptions, but these could be made explicit only in a completely different context which was not developed until the nineteenth century.

The result is by no means obvious, for although the sides A′A, B′B, C′C etc. can be made very small by taking the points near to each other, their number becomes ever larger and so their sum might as well remain large. That the result is deep can also be illustrated as follows. We may consider the arcs AB, BC, CD etc., being chosen ever smaller, as arc-length differentials. What Huygens then in fact proves is that the perpendiculars A′A, B′B, C′C etc. are *second order differentials*, they

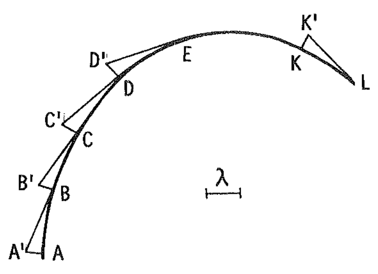


Figure 9

become infinitely small with respect to the first order differentials and, although their number tends to infinity, nevertheless their sum tends to zero. So what Huygens achieves here is to extend the techniques of approximation and inequalities which he had learned from Archimedes to the case of second order smallness – and that is a most notable achievement.

About the rest of Huygens' theory I shall be brief. He has worked out a method to calculate the evolute from the equation of the involute, provided that equation is algebraic. The proof of the correctness of that method[20] is much less rigorous than the proofs mentioned above. The method showed him that if the involute is algebraic then the evolute is algebraic too. This has a consequence which was very important to Huygens and which concerned rectification. From the configuration of evolute (see figure 10) and involute it is clear that

arc-length AB = BB′– AA′.

If both curves are algebraical this means that the arc-length AB is algebraically constructable. A consequence is that every algebraic curve has an algebraic evolute which is rectifiable. Huygens valued this result very much for it proved that rectifiability of algebraic curves is not an exceptional occurrence.

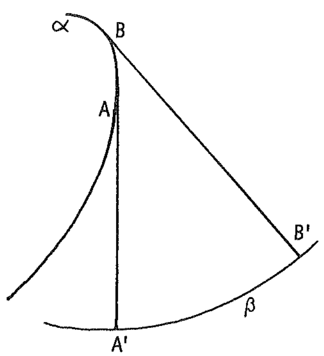


Figure 10

These two examples, probability and evolutes, show how Huygens' classical style brought him great successes in mathematics. That does not mean that the characteristics of his style were always advantageous. In his studies on rectification he found that the rigorous Archimedean style of proof had its drawbacks too: it was very time-consuming and in most cases not very illuminating; it served to prove things one was already convinced of.[21] Huygens came to solve this tension between the rigorous criteria of his original style and the tediousness it involved, by reserving that style as a sort of V.I.P. treatment for those results he valued most. Thus the *Horologium Oscillatorium* of 1673, much praised for its rigorous classical style, does indeed contain such rigorous proofs for the theory of evolutes and for the theory of fall along cycloidal arcs.[22] But in the same book the treatment of compound pendulums[23] and of the rule for calculating evolutes is much less rigorous and involves a free use of infinitely small quantities, precisely the sort of things which the classical style tried to avoid.

4. 1660-1680, transcendental curves and inverse calculus problems

In the introduction I mentioned transcendental curves and inverse calculus problems as the two central themes of new material and new problems in Huygens' mathematical work. Both themes are present in the creative period 1655-1660 which I have been discussing. The cycloid is a transcendental curve. Huygens' studies on rectification and evolutes were significant because in them he explored the boundaries between the Cartesian Geometry of algebraic curves and the wider field including transcendental curves. And the tautochrony of the cycloid, Huygens' great discovery of 1659, was in fact the solution of an inverse calculus problem: Huygens was able to translate the mechanical problem of finding a tautochronous path for the pendulum into a mathematical problem of finding a curve with a certain property of its perpendiculars. He was able to solve that problem because he happened to know that the cycloid had that property.

In the two decades 1660-1680 Huygens' research in mathematics was less intensive than before. His work at the *Académie Royale des Sciences* in Paris from 1666 till 1681 caused a change in the direction of his scientific work, and also the preparation of his *Horologium Oscillatorium* took a lot of time. The book was published in 1673 but most of its contents date from the years between 1655 and 1660. Within Huygens' mathematical work in the period after 1660 the themes of transcendental curves and inverse calculus problems remained prominent and even became more so. I want to illustrate that by discussing his studies on the logarithmic curve and on fall in a medium with resistance.

It seems that before 1660 Huygens had not actively used or studied logarithms. This changed because of his interest in music. The problem of the equal division of the octave leads namely to the mathematical problem of finding mean proportionals, and that problem in turn can be solved by logarithms. So, in connection with music Huygens in 1661 took up the study of logarithms and in September of that year he studied[24] the geometrical aspects of the logarithmic relation. That is, he considered a curve (see figure 11) with the property that to each series of equidistant points along the axis (such a series represents an arithmetical series of numbers) there corresponds a geometrical series of ordinates. This, the correspondence between arithmetical and geometrical series was the way in which logarithms were first introduced; the idea of considering them as exponents to a fixed base came much

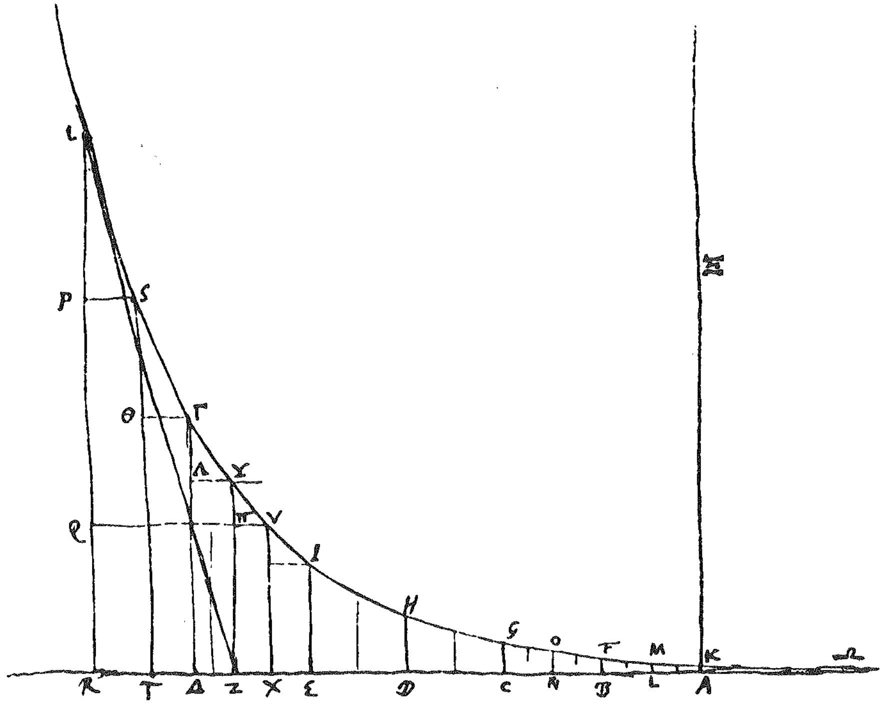


Figure 11 Huygens' figure of the logarithmic curve, 1661

later. Thus if the points on the axis are marked by numbers, these numbers behave as the logarthms of the corresponding ordinates.

The relation of the curve with musical theory is the following (see figure 12). Let AB and CD be two ordinates such that CD = 1/2 AB Consider AB as the length of a string producing a certain tone, then length CD will produce the octave. If we now divide the distance AC along the axis into twelve equal intervals and erect ordinates at the division points, then these ordinates are the mean proportionals between AB and CD and they correspond to the lengths of the string of the intermediate equal tempered half-tones between the original tone and the octave.[25]

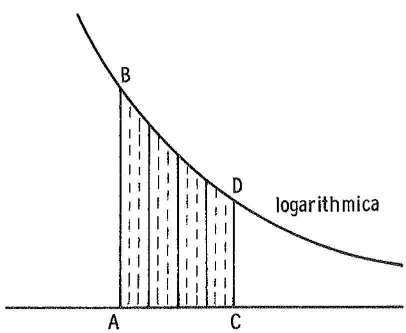


Figure 12

The importance of the curve, for musical theory and other subjects, was clear to Huygens. Indeed it has proved to be a very important curve, for it is none other than the what we call the exponential curve, with modern equation (supposing we read figure 11 from right to left)

$$y = ae^{x}.$$

To illustrate its importance in mathematics and elsewhere I need only to mention exponential growth and the Club of Rome.

Huygens was not the first to study this curve; Torricelli had done so earlier but he had not made his results public.[26] Huygens undertook a full survey of the properties of the curve. He found (see figure 13) that the subtangent of the curve is constant and he gave a numerical approximation of that subtangent from other data of the curve; this calculation is equivalent to a determination of $^{10}\log e$ to 18 decimals. He determined the quadrature and in particular the quadrature of the infinitely extended area between axis, curve and ordinate, which he found to be equal to the product of ordinate and subtangent. He also determined centres of gravity and the volumes of figures produced by revolving the curve around its axes. In short Huygens explored all the basic properties of this transcendental curve. He published his results in 1690 in the *Discourse on the cause of gravity*, which was an appendix to the *Treatise on Light*.[27]

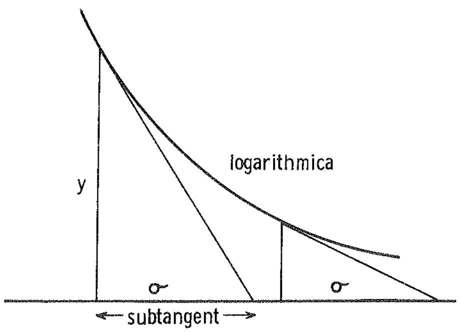


Figure 13

But it proved more than simply an interesting subject for exploration; Huygens found that he could use the curve in the solution of a number of inverse calculus problems originating in physics. One of these was the problem of determining the relation between pressure and altitude in the atmosphere. Huygens solved this problem in 1662[28] on the basis of Boyle's law and with the help of his insight into the logarithmic relation.

Another was a study from 1668[29] on fall in a resisting medium. I shall give some more details about that study because it well illustrates Huygens' skill in dealing with these problems. Huygens studied the case where the resistance of the medium is proportional to the velocity of the falling body. He argued as follows (see figure 14). Let the vertical axis in the figure represent time. If there were no resistance, the velocity of the falling body would, according to Galileo's law of fall, be propor-

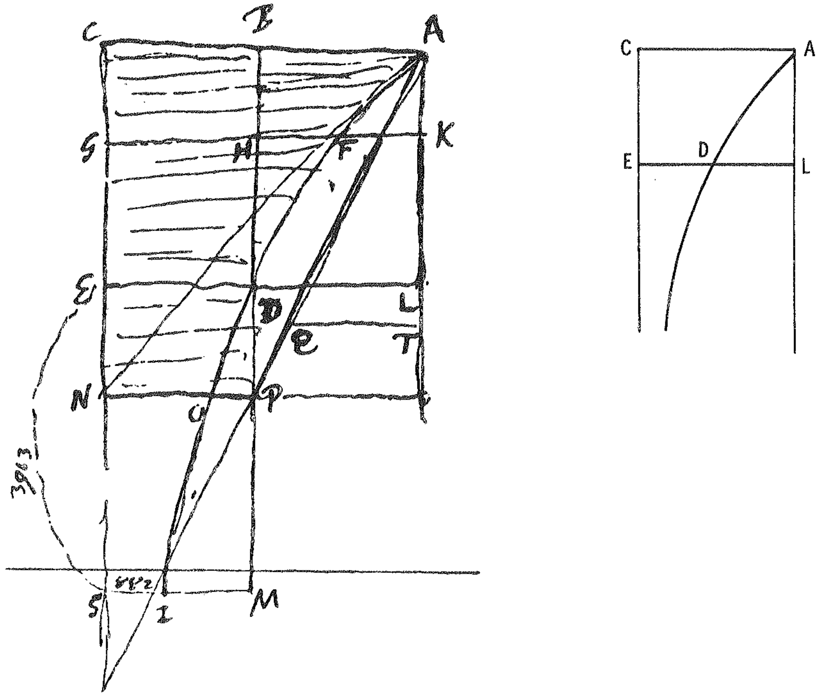


Figure 14 (left: Huygens' figure concerning fall in a resisting medium, 1668)

tional to time. Let therefore area ACEL represent this velocity at time CE. Now because there is resistance, the actual velocity will be smaller than the velocity in the case of free fall. Let us represent that actual velocity also by a area along the time axis. Then that area will be bounded by a curve as AD in the figure, and our problem is to find that curve.

The representation of the velocity as an area enabled Huygens to interpret the ordinates, as ED, as accelerations. From the given fact that the resistance, or deceleration, is proportional to the velocity, he then derived that area ACED must be proportional to the segment DL. Thus he reduced the problem to an inverse calculus problem, to find a curve with the property that its quadrature is proportional to DL. Moreover, he recognized the curve, for the logarithmic curve has precisely that property, as he had found in 1661. So curve AD is the logarthmic curve; thereby the problem was solved and Huygens could work out the further consequences of this discovery, as for intance the form of the ballistic curve in this case.

Later[30] he applied the same approach to the problem of fall with resistance proportional to the square of the velocity. Here the curve is not so easily recognizable but Huygens solved that problem too. He published his results in 1690, prompted by the fact that Newton had dealt with the same questions in the *Principia* of 1687.[31] I think that this episode well illustrates not only Huygens' skill – the idea of representing the velocity as an area is especially fortunate – but also the power of his geometrical way of thinking. Inverse calculus problems were soon to become the paradigm problems of mathematical physics. They very often lead to transcendental curves and therefore they defied analytical treatment with the methods of Cartesian analytical geometry. Hence it was Huygens' geometrical way of thinking that enabled him to deal with such problems successfully.

But at a somewhat later stage, it was this same geometrical way of thinking that prevented Huygens from following suit in the subsequent development. For the next step in the development was that analytical methods were created covering these inverse calculus problems too. This was the achievement of the new fluxional calculus and the differential and integral calculus, worked out in the years 1660-1680 by Newton and Leibniz and coming "on the market", so to speak, around 1690. By means of these methods the inverse calculus problems could be formulated analytically, that is, in terms of formulas, as differential equations, and in many cases these equations could be solved by the algorithms, transformations and tricks of the new calculus. Here Huygens' way of thinking in figures rather than in formulas, his suspicion of the introduction of new symbolisms and, no doubt, his advanced age prevented him from following. This is what characterizes the last period in his mathematical career: he was overtaken by the younger generation with the new methods.

5. The confrontation with the new infinitesimal calculus

In 1681 Huygens left Paris; he stayed in Holland the rest of his life. The year marked another turning point in his scientific career. The work in and for the *Académie* stopped; Huygens became again a private independent scholar. In these years he prepared the publication of the *Treatise on Light* with the *Discourse on gravity*, both of which contain much mathematics. In general Huygens returned more to mathematical researches in that period. In these researches inverse calculus problems, in particular inverse tangent problems became increasingly important. In 1688 the correspondence with Leibniz, interrupted since 1680, was resumed on the occasion of an inverse tangent problem publicly proposed by Leibniz[32] to test the strength of the mathematics of the Cartesians. Huygens published a solution[33], and Leibniz wrote to him about it. In the subsequent letters, Leibniz gave hints of his new methods and Huygens sent him inverse tangent problems to solve. Due to misunderstandings and to Leibniz' secretiveness about his methods, the correspondence did not give Huygens much insight into the new methods. But it did convince him that inverse tangent problems would be a test case for any new development in the mathematics of curves, and that, however much he disliked the analytical style and the flourish of new and at first sight meaningless symbols, Leibniz had somehow hit on something powerful.

This was confirmed in the discussions about the *catenary*. Jakob Bernoulli had proposed in 1689[34] the problem of determining the mathematical form of a freely hanging chain. The problem was very familiar to Huygens for in his early youth he had studied it[35] and proved that the form was not a parabola – an opinion then current. Now he studied it again and applied all his geometrical brilliance to its solution. He determined several properties of the curve and sent in his findings. His results appeared in 1691[36] together with the solutions of Leibniz[37] and Johann Bernoulli.[38] And Huygens had to acknowledge the superiority of these solutions, for Bernoulli and Leibniz determined the curve in a better way. They showed that it was transcendental and that it depended on the logarithmic curve, whereas Huygens had not been able to determine to which class of transcendental curves the catenary belonged. Immediately after seeing these solutions Huygens derived the results with his own methods. His final solution, published in 1693[39], is a true gem of his geometrical style.[40] But the circumstances of its origin are significant: he had to receive the crucial hint from the younger mathematicians with the new methods.

In 1687 Fatio de Duillier, the young Swiss mathematician who was later to trigger off the priority dispute on the calculus, visited Huygens and they studied together. In 1691 Fatio returned and again they worked together for some time, studying an ambitious project of Fatio, namely an analytical method for the solution of inverse tangent problems. From the manuscripts[41] that have survived from this encounter Huygens' great interest in the matter is quite clear. Together they calculated through many examples. The result was disappointing, the methods appeared to be applicable only to a restricted class of algebraical curves. The episode shows Huygens' keen interest in the problem; most likely it also somewhat strengthened his distrust of analytical methods.

By 1693, however, Huygens had seen so much of the new calculus that he began to acknowledge its force. Also, by that time Leibniz had provided more information and Huygens was helped very much by the clear and careful explanations in the letters he received from l'Hôpital (who himself had learned the subject in 1691 from the eminent teacher Johann Bernoulli). And so, near the end of his life, he learned the calculus and was able to use it, but he never felt quite at ease with it.

6. Summary and conclusion

I have tried to review Huygens' achievements in mathematics and his place in the development of that science in the seventeenth century. I have stressed that he was, in the practical style that pervades all his scientific work, more concerned with material and problems in mathematics than with methods and theories. I have indicated the three important characteristics of his classical mathematical style, and have shown how these characteristics often helped and sometimes restrained him in his research. The care for Archimedean rigour earned Huygens the beautiful and deep proofs in his theory of evolutes, but later he felt that he should not spend too much time in bringing the presentation of his results up to the standards of this rigour. His skill in axiomatizing enabled him to introduce and justify the concept of expectation in probability. Moreover, it helped him in setting up his theories of collision, floating bodies, the compound pendulum and other parts of mathematical physics. His geometrical way of thinking enabled him to explore the new realm of transcendental curves and the important new use of mathematics in physics through inverse calculus problems; for example in his results on tautochrony of the cycloid, the study of fall in a resisting medium and the catenary. But later that same geometrical way of thinking prevented Huygens from following in the next step in the development, the beginning of infinitesimal analysis. Such are the formative forces I see in Huygens' mathematical career and I hope that through the examples I have given I have been able to clarify his style and ways of thinking in mathematics.

But what did he achieve? That is not easy to pin down. Apart from the theory of evolutes and probability – which are indeed great achievements – he did not work out self-contained theories or methods. Several of his other explorations were published rather late and they were sometimes duplicated by other mathematicians. But one great achievement stands out and was recognized by his contemporaries too: Huygens showed the applicability of mathematics to the natural sciences. It is the impressive proof of the explanatory power of the mathematical method in the natural sciences which is Huygens' great achievement, with regard to mathematics and to science in general.

But speaking about Huygens' achievements in this way I feel that I present them too much as things in the past. And when we now, 350 years after Huygens' birth, experience the encounter with his mathematical work, it is not so much the achievements that are impressive, but the genius of his mathematical mind, his brilliance in handling figures and argument, his inventiveness and deep knowledge of his material. Independently of the final measure of his achievements I experience a real pleasure in the genius and beauty of his mathematical thought. I hope that I have been able to transmit something of that feeling.

Notes

1. The terms "algebraic" and "transcendental" for these classes of curves are modern. Descartes spoke about "geometrical" and "mechanical" curves. The term "transcendental" was introduced by Leibniz.
2. *Theoremata de quadratura hyperboles, ellipsis et circuli ex dato portionum gravitatis centro, quibus subjuncta est ἐξέτασις cyclometriae cl. viri Gregorii à St. Vincentio editae MDCXLVII*, Leiden, 1651; C. Huygens, *Oeuvres Complètes* (The Hague, 1888-1950, 22 vols.), *11*, 271-337.
3. *De circuli magnitudine inventa. Accedunt ejusdem problematum quorundam illustrium constructiones*, Leiden, 1654; *O.C. 12*, 91-237.
   The editors of the *O.C.* translate this title as "l'invention de la grandeur du cercle". However, Huygens knew very well that he had not found the magnitude, that is the quadrature, of the circle. Hence the translation "inventions about the magnitude of the circle" seems better. Moreover, in a letter to Gregory of St. Vincent of 3 July 1654, accompanying a copy of the book, Huygens wrote "haec quoque inventa examini tuo subjicerem" (*O.C. 1*, 288), which corroborates my translation.
4. *De iis quae liquido supernatant*, first published in *O.C. 11*, 81-210.
5. Cf. *O.C. 1*, 237 and *O.C. 12*, 5.
6. *Theorems* (note 2), th. 6 and 7.
7. *O.C. 14*, 451-459.
8. *Theorems* (note 2), th. 4.
9. *Theorems* (note 2), th. 5.
10. R. Descartes, *Géométrie*, 1637, 340-341. The claim was not new; it is, in fact, an Aristotelean doctrine. But Descartes had based on it a sharp distinction between "geometrical" and "mechanical" curves (see note 1). Hence the first rectifications of algebraic curves undermined one of the cornerstones of Descartes' theory of geometry.
11. *O.C. 14*, 314. I have discussed Huygens' method in some detail in my "L'élaboration du calcul infinitésimal, Huygens entre Pascal et Leibniz", to appear in *Huygens et la France* (ed. R. Taton, Paris, 1980). – About the same time van Heuraet and others, independently of Huygens, found similar rectification methods.
12. *Tractatus de ratiociniis in ludo aleae*, published in F. van Schooten, *Exercitationum mathematicorum libri quinque*, Leiden, 1657. The Dutch version, entitled *Tractaet handelende van Reeckening in Speelen van Geluck*, appeared in F. van Schooten, *Mathematische oeffeningen begrepen in vijf boecken*,

Amsterdam, 1660. *O.C. 14*, 1-179, contains the Dutch text with a French translation by the editors of the *O.C.*

13. Prop. 3 in the treatises mentioned in note 12.
14. *O.C. 16*, 392-413; see also P. Costabel, "Isochronisme et accélération 1638-1687", *Arch. intern. hist. sci. 28* (1978), 3-20, and the article by Mahoney in this volume.
15. *O.C. 14*, 387-406; see also *O.C. 17*, 142-148, in particular note 2.
16. The latter is a modern term. Huygens spoke about "curva evoluta" (unwound curve) and "curva descripta ex evolutione" (curve described by the unwinding); *Horologium Oscillatorium* (note 17), part 3, def. 3 and 4.
17. *Horologium oscillatorium, sive de motu pendulorum ad horologia aptato demonstrationes geometricae, Paris*, 1673; *O.C. 18*, 27-438. The theory of evolutes forms part 3 of this book.
18. *Horol. Osc.* (note 17), part 3, prop. 1.
19. *Horol. Osc.* (note 17), part 3, prop. 2 and 3.
20. *Horol. Osc.* (note 17), part 3, prop. 11.
21. Cf. *O.C. 14*, 337. The editors of the *O.C.* have given a French translation of this Latin text on pp. 191-192 (note 14) of the same volume. This translation is as follows:

    'Quelquefois par les indivisibles. Mais on se trompe, lorsqu'on veut faire passer leur emploi pour une démonstration. D'ailleurs, pour convaincre ceux qui s'y connaissent il revient presque au même de donner une démonstration formelle ou bien le fondement d'une telle démonstration, de sorte, qu'après l'avoir examiné, ils ne sauraient douter de la possibilité d'une démonstration rigoureuse. J'avoue, il est vrai, que c'est aussi à la façon de donner à cette dernièreune forme convenable afin qu'ell soit claire, élégante, et plus appropriée que toute autre, qu'on reconnaît la science et la sagacité de l'auteur, comme dans toutes les oeuvres d'Archimède. Néanmoins, ce qui vient en premier lieu, et ce qui importe surtout, c'est la manière même dont l'invention a été obtenue. C'est cette connaissance qui réjouit le plus et qu' on demande aux savants. Il semble donc préférable de suivre la méthode par laquelle elle est aperçue le plus vite et le plus clairement, et comme posée devant les yeux. Nous nous épargnons ainsi du travail en écrivant, et les autres en lisant; il faut considérer, en effet, que les savants finiront par ne plus trouver le temps de prendre connaissance de la grande quantité des inventions des Géomètres (quantité qui va en croissant de jour en jour et qui semble dans cet âge de science devoir prendre des développements immenses) si les auteurs continuent à se servir de la méthode prolixe et rigoureuse des anciens."
22. *Horol. Osc.* (note 17), part 2.
23. *Horol. Osc.* (note 17), part 4.
24. *O.C. 14*, 460-473.
25. See further the article of Cohen in this volume.
26. Cf. G. Loria, "Le ricerche inedite di Evangelista Torricelli sopra la curva logarithmica", *Bibl. math. (ser. 3) 1* (1900), 75-89.
27. *Traité de la lumière, où sont expliquées les causes de ce qui lui arrive dans la reflexion et dans la refraction, et particulièrement dans l'étrange refraction du cristal d'Islande. Avec un discours de la cause de la pesanteur*, Leiden, 1690; *O.C. 19*, 451-548 (*traité*) and *O.C. 21*, 427-499 (*discours*). The section on the logarithmic curve is on pp. 176-180 of the *discours*; *O.C. 21*, 484-488.
28. *O.C. 14*, 483-497.

29. *O.C. 19*, 102-119. The manuscript is dated 28 October 1668. The results were published in the *discours* (note 27), 168-173; *O.C. 21*, 478-482.
30. *O.C. 19*, 144-157. The study dates from 1669, its results are mentioned in the *discours* (note 27), 173-176; *O.C. 21*, 482-484.
31. I. Newton, *Philosophiae naturalis principia mathematica*, London, 1687. Motion in resisting media is treated in book 2.
32. Leibniz proposed the problem in an article "Réponse de M.L....", *Nouvelles de la république des lettres* (Sept. 1687), 952-956; G.W. Leibniz, *Philosophische Schriften* (ed. C.I. Gerhardt, Berlin, 1875-1890), vol. *3*, 49-51. I have discussed Huygens' solution of the problem in my article cited in note 11.
33. "Solution du problème...", *Nouvelles de la république des lettres* (Oct. 1687); *O.C. 9*, 224-228.
34. Jak. Bernoulli, "Analysis problematis...", *Acta Eruditorum* (May 1690), 217-219; Jak. Bernoulli, *Opera*, (Geneva, 1744), 421-426.
35. *O.C. 11*, 37-44, manuscript dating from 1646; see also *O.C. 1*, 34-44.
36. "Solutio eiusdem problematis", *Acta Eruditorum* (June 1691), 281-282; *O.C. 10*, 95-98.
37. G.W. Leibniz, "De linea...", *Acta Eruditorum* (June 1691), 277-281; G.W. Leibniz, *Mathematische Schriften* (ed. C.I. Gerhardt, Berlin, 1849-1863) vol. *5*, 243-247.
38. Joh. Bernoulli, "Solutio problematis funicularii...", *Acta Eruditorum* (June 1691), 274-276; Joh. Bernoulli, *Opera Omnia* (Lausanne, 1742) vol. *1*, 48-51.
39. "Lettre...", *Histoire des ouvrages des sçavans* (Febr. 1693), 244-257; *O.C. 10*, 407-417, see also *O.C. 10*, 135-138.
40. I have explained this solution in my "Christiaan Huygens", *Dictionary of scientific biography* (ed. C.C. Gillispie, New York, 1968-) vol. *6* (1972), 597-613, esp. 601-602.
41. *O.C. 20*, 506-541.

This article was written partly during my stay at the *Institut des Hautes Études scientifiques* in Bures-sur-Yvette, France, November 1979. I wish to thank that institution for its hospitality and support.