E.J. DIJKSTERHUIS

on

EUDOXOS' THEORY OF RATIO

<u>Source</u>

DIJKSTERHUIS, E.J., *De Elementen van Euclides*, Groningen, 1929, 1930.

Commentary by H.J.M. BOS

EDUARD JAN DIJKSTERHUIS, born 28.10.1893 in Tilburg, died in Bilthoven 18.5.1965. He studied in Groningen, obtaining his doctor's degree in 1918. From 1919 to 1953 he worked as a teacher of mathematics in Tilburg. He was also a "privaat docent" at Amsterdam and Leiden. In 1953 he was appointed as extra-ordinary professor of the history of science in Utrecht, since 1954 he combined this with a similar post in Leiden. In 1960 his position in Utrecht was changed to an ordinary professorship from which, for reasons of health, he retired in 1963.

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Chapter 8, §§ 1, 2; vol. 2 pp. 55-62, translated by H.J.M. Bos [(1)].

CHAPTER VIII.
BOOK V.
THE THEORY OF RATIO.

1. Introduction.

In the first volume of this work we have had several opportunities to notice the great influence exerted upon the development of Greek mathematics by the discovery of the possibility that two homogeneous magnitudes are to each other in a ratio which is not a ratio of any two numbers; we have also stressed the great value that has to be attached to the victory, achieved through Eudoxos' theory of ratio, over the "scandale logique" caused by this discovery. Indeed, it is no exaggeration for this theory, a supreme creation of the mind, to be counted as one of the most impressive monuments of Greek culture, and there is good reason to study attentively the elaborate and exact exposition which Euclid gives of the theory in the fifth book of his *Elements*.

It is not possible to separate with certainty the contributions of Eudoxos and Euclid in the structure of the fifth book. But we may well accept a scholium on Euclid V in which it is stated that this book not unjustly bears the name of Euclid, because, although the theory expounded in it is said to be found by another, Eudoxos, the exposition of the theory in the form of Elements, and its insertion in the system of geometry is generally thought to be due to Euclid.

The fifth book deals with magnitudes in general. It is not further stated what the characteristics of a magnitude are.

If we admit the Aristotelian distinction of quantities in magnitudes, divisible in infinity, and quantities which can only be divided a finite number of times, the magnitudes of book V comprise only lines, areas and volumes, and those physical entities, such as times, which have the infinite divisibility in common with the geometrical magnitudes. Accordingly the scholia on book V affirm that magnitude is that which can be augmented and divided in infinity, and that it occurs in three kinds: line, area, volume. If, however, we take into account another passage from Aristotle, already cited before, in which, in obvious reference to the theory of Eudoxos, numbers are mentioned on a par with lines volumes and times, then again we doubt whether we are allowed to exclude numbers from the category of magnitudes. On the other hand, the concepts of proportion and the theory of proportionality for numbers are introduced and developed separately in book VII. This, however, might be explained by the hypothesis that Euclid wanted to incorporate in his work both the theory of ratio formulated by Eudoxos and the rigorous foundation of arithmetic probably due to Theaitetos. We then have the question of the extent to which Euclid has linked the two theories for the case that both apply to numbers, and because obviously this question can be dealt with only after an exposition of both the books V and VII, we shall have to let the problem of the extension of the concept of magnitude rest for the moment.

2. Definitions.

D e f i n i t i o n I

α′. Μέρος ἐστὶ μέγεθος μεγέθους τὸ ἔλασσον τοῦ μείζονος, ὅταν καταμετρῇ τὸ μεῖζον.

I. *A magnitude is a part of a magnitude, the smaller of the larger, if it measures the larger.*

This means that it is contained in the larger an integer number of times. Hence the word μέρος is used here in a more particular sense than was the case in axiom VIII.

According to the second definition the greater is called πολλαπλάσιον (multiple) of the smaller

D e f i n i t i o n III

γ′. Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητά ποια σχέσις.

III. *Ratio is a certain relation in size between two homogeneous magnitudes.*

This definition, which would still be meaningless even if we knew what homogeneous magnitudes are, has to be considered, like so many other definitions of Euclid, as a mathematically ineffectual preliminary description of the object which has to be defined.

D e f i n i t i o n IV

δ′. Λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται, ἃ δύναται πολλαπλασιαζόμενα ἀλλήλων ὑπερέχειν.

IV. *It is said that magnitudes have a ratio to each other, which, multiplied, can exceed each other.*

Because definition III speaks of ratio only in the case of homogeneous magnitudes, apparently the aim of definition IV is to determine what homogeneous magnitudes are, and therefore to enounce (although it does not say so) that two magnitudes have a ratio to each other if and only if for each of them a number can be given such that its multiple indicated by that number is greater than the other. Formulated in this way, the definition is apparently equivalent with what at present usually is called the postulate or the axiom of Archimedes. We have here the oldest known form of this postulate, and although its formulation is not quite as could be desired, it is used throughout book V in so deliberate a manner as the fundamental principle of the theory of ratio that there is every reason to call the fourth definition of book V the postulate of Eudoxos. Of course the fact that Archimedes makes use of this postulate, be it in a slightly different form, can certainly not be a motive to attach his name to it.

It may not be superfluous to stress once again that the comparison of two magnitudes, as meant in Definition IV, is a comparison of the magnitudes themselves, not of the numbers whereby we would express the values of these magnitudes in suitable units. A length and a time for instance are not homogeneous magnitudes and therefore we cannot speak about a proportion of a length to a time.

The actual foundation of the concept of "ratio" is given in

D e f i n i t i o n   V

ε'. Ἐν τῷ αὐτῷ λόγῳ μεγέθη λέγεται εἶναι πρῶτον πρὸς δεύτερον καὶ τρίτον πρὸς τέταρτον, ὅταν τὰ τοῦ πρώτου καὶ τρίτου ἰσάκις πολλαπλάσια τῶν τοῦ δευτέρου καὶ τετάρτου ἰσάκις πολλαπλασίων καθ' ὁποιονοῦν πολλαπλασιασμὸν ἑκάτερον ἑκατέρου ἢ ἅμα ὑπερέχῃ ἢ ἅμα ἴσα ᾖ ἢ ἅμα ἐλλείπῃ ληφθέντα κατάλληλα.

V. *It is said that magnitudes are in the same ratio, the first to the second and the third to the fourth, if arbitrary equal multiples of the first and the third are simultaneously greater than, equal to or less than arbitrary equal multiples of the second and the fourth, taken in corresponding order.*

The definition is not quite exactly formulated and it is difficult to translate it to a certain extent faithfully. Its meaning can be rendered as follows: the magnitudes A and B are in the same ratio (or: have the same ratio) as the magnitudes Γ and Δ if for each pair of numbers μ and ν:

simultaneously with $\mu A > \nu B$ also $\mu\Gamma > \nu\Delta$

simultaneously with $\mu A = \nu B$ also $\mu\Gamma = \nu\Delta$

simultaneously with $\mu A < \nu B$ also $\mu\Gamma < \nu\Delta$.

D e f i n i t i o n   VI

ϛ'. Τὰ δὲ τὸν αὐτὸν ἔχοντα λόγον μεγέθη ἀνάλογον καλείσθω.

VI. *Let magnitudes, which have the same ratio, be called proportional.*

We shall, again, introduce a symbolism, not commonly used

in modern mathematics, to be able to render the greek text in a condensed way, without, however, changing it essentially. We therefore denote the ratio of A to B as Λ(A,B) and the proportionality of A,B,Γ,Δ by

$$\Lambda(A,B) = \Lambda(\Gamma,\Delta).$$

Note that for the definition of proportionality it is only required that A and B are homogeneous to each other and Γ and Δ also. Thus it is not necessary that, for instance, Λ(A,Γ) exists.

The word ἀνάλογον which is equivalent to ἀνὰ λόγον is always used undeclined.

The next definition introduces the predicate "greater" for ratios.

D e f i n i t i o n   VII

ζ. Ὅταν δὲ τῶν ἰσάκις πολλαπλασίων τὸ μὲν τοῦ πρώτου πολλαπλάσιον ὑπερέχῃ τοῦ τοῦ δευτέρου πολλαπλασίου, τὸ δὲ τοῦ τρίτου πολλαπλάσιον μὴ ὑπερέχῃ τοῦ τοῦ τετάρτου πολλαπλασίου, τότε τὸ πρῶτον πρὸς τὸ δεύτερον μείζονα λόγον ἔχειν λέγεται, ἤπερ τὸ τρίτον πρὸς τὸ τέταρτον.

VII. *If of the same multiples, the multiple of the first is greater than the multiple of the second, but the multiple of the third is not greater than the multiple of the fourth, then it is said that the first has to the second a greater ratio than the third to the fourth.*

The meaning of $\Lambda(A,B) > \Lambda(\Gamma,\Delta)$ is, therefore, that there is at least one pair of numbers μ,ν such that $\mu A > \nu B$ but $\mu\Gamma \leqslant \nu\Delta$.

We shall now first consider the concept of "ratio" from the modern point of view.

When two homogeneous magnitudes are given, then this determines a partition of the positive rational numbers in two classes, of which the first (lower) class consists of the rational numbers $\frac{\nu}{\mu}$ such that $\mu A > \nu B$ and the second (higher) class consists of the rational numbers $\frac{\nu}{\mu}$ such that $\mu A \leqslant \nu B$. Neither of these classes is empty, because the postulate of

Eudoxos guarantees the existence of natural numbers p and q, such that $pA > B$ and $A < qB$. Therefore $\frac{1}{p}$ is a number of the lower class, q is one of the higher class. The partition created in this way is a Dedekind-cut. For if $\frac{\nu}{\mu}$ is a number of the lower class and if

$$\frac{\nu_1}{\mu_1} < \frac{\nu}{\mu}$$

then, because $\mu\nu_1 < \nu\mu_1$ and $\mu A > \nu B$ also $\nu\mu_1 A > \mu\nu_1 A > \nu\nu_1 B$, whence

$$\mu_1 A > \nu_1 B$$

so that $\frac{\nu_1}{\mu_1}$ is also a number of the lower class.

Now according to Definition V all pairs of magnitudes which have the same ratio determine the same Dedekind-cut, and therefore the same positive real number, so that that common ratio can be considered as a representation of that positive real number. If we postulate, in addition, that for each positive real number a pair of magnitudes can be given whose ratio represents that number, then it appears that, with respect to its extension, the concept of "ratio" is identical with the concept of positive real number, providing that in every rational cut the smallest number is reckoned to belong to the higher class. The latter proviso is added not only for choosing between two possibilities but also because of the agreement which is created in this way between the definitions of "greater" by Euclid and by Dedekind. For if

$$\Lambda(A,B) > \Lambda(\Gamma,\Delta),$$

there is at least one pair of natural numbers μ,ν such that

$$\mu A > \nu B;\ \mu\Gamma \leqslant \nu\Delta.$$

Hence $\frac{\nu}{\mu}$ belongs to the lower class of the cut represented by $\Lambda(A,B)$ and to the higher class of the cut represented by $\Lambda(\Gamma,\Delta)$, in agreement with the definition of "greater" according to Dedekind.

The consideration above makes it understandable that Greek mathematics, notwithstanding the narrowness of its concept of number, could reach so many results for which we need the help of a theory of irrational numbers, and that in a way which certainly lags behind the methods of modern mathematics as to shortness and perspicuity, but equally certainly not with respect to rigour. The role of the positive real number is taken over by the ratio; if, in particular, the real number is rational, then the corresponding ratio is a ratio of two magnitudes which are to each other as numbers.

However, we should avoid the mistake of identifying the concepts of ratio and real number with respect to their content as well. A ratio, whether corresponding to a rational or to an irrational number is not a number; hence in so far as operations have to be performed by means of ratios, they have to be introduced, studied and applied independently, even if they are apparently fully equivalent to operations which we perform with real numbers. Negligence of this insight is one of the main causes of the surprise, often bordering on vexation, with which the modern reader usually apprehends the seemingly cumbersome operations with ratios in Greek mathematics. He writes and reads these ratios as numbers, whether or not obtained as a quotient of two other numbers, and so he can often reach in one step a result which for the Greek mathematician requires the application of a number of theorems about ratios. It is necessary to adopt the standpoint of Greek mathematics in order to judge its value fairly; and to further this adoption of a different standpoint we did not introduce the symbol $\frac{A}{B}$ or even A:B for the ratio of A to B, but $\Lambda(A,B)$. If we rigorously adopt the Greek standpoint, that is, if we keep to the idea that we work only with natural numbers and with ratios of quantities, then the initial surprise soon vanishes, to make place for a feeling of great admiration for the certainty and effectiveness with which the Greek mathematicians used the theory of ratios.

Note

(1) In this translation I have tried to keep as close as possible to Dijksterhuis' stately classical Dutch style which characterises all his work. I have omitted the cross references and the footnotes.

HJMB

Commentary
by
H.J.M. BOS

It is appropriate that in an historical volume like the present collection of mathematical texts and commentaries, there should also be one on the history of mathematics. History of mathematics did indeed receive much interest in the Netherlands in the period 1920-1940. The choice of an author to represent this interest is not difficult, because it was without doubt E.J. Dijksterhuis who in that period was the most outstanding Dutch historian of mathematics and science; he also did much to promote the interest in the subject and to establish its position as an academic discipline.

After a first publication on the history of mechanics, [1924], Dijksterhuis' main interest, until about 1940, was in ancient mathematics. Later he turned to more recent periods (e.g. his work on Stevin, [1943] and the edition of Stevin [works]), and more general themes. His now best known book, *The Mechanisation of the World Picture*, [1950] appeared in Dutch in 1950 and was thereafter translated in German and English. It brought him international recognition as an historian of science, a late recognition, partly because Dijksterhuis published all his major works in Dutch. This, in fact, brought him another kind of recognition; his stately, detached Dutch style was universally admired and his *Mechanisation* won the "P.C. Hooft prijs", an important Dutch literary prize.

Dijksterhuis chose to bring together the results of his studies in ancient mathematics in an edition of Euclid's *Elements*, which appeared in 1929-1930. The text above is chosen from that edition. About ten years later his second great study in ancient mathematics appeared, the monograph on Archimedes, [1938]. As Dijksterhuis explained in the introduction of his edition of the *Elements*, he could not, for reasons of available space, prepare a full edition, and he therefore chose the form of editing the most important definitions, postulates, axioms and theorems of the *Elements* (of which he gave the Greek text and his own translation), and to add summaries of the remaining texts. In between, as well as in a long introductory part, he presented the relevant data on the history of Greek mathematics and the various opinions and hypotheses of historians of mathematics on that episode. This particular combination of an edition, a history and a commentary on the work of previous historians was a very happy one. I think the text reproduced above illustrates this. It is Dijksterhuis' exposition of the beginning of the fifth book of the *Elements*, and it concerns Eudoxos' theory of ratio. I have chosen it because that passage of the *Elements* is a crucial text for the understanding of the development of pre-Euclidian Greek mathematics, and because Dijksterhuis' presentation of it is a good example of his style.

As mentioned above, Dijksterhuis incorporated in his edition discussions of the main hypotheses and explanations of the development of early Greek mathematics which had been put forward by historians up to 1929. In fact Dijksterhuis did not introduce many new hypotheses or insights himself; his force lay in the clear presentation of the relevant theories and the very balanced appraisal of the evidence for and against these.

The general cadre in which Dijksterhuis himself saw the development of pre-Euclidian mathematics was based on Tannery's idea of a "scandale logique" (it is referred to in the first sentence of our text). Dijksterhuis speaks of a crisis in Greek mathematics. The same term was used by Hasse and Scholz in a monograph [1928] which Dijksterhuis saw just before completing the first volume of the edition and which he cited, in general approvingly. He was aware that the crisis is hypothetical but he called it an acceptable, even a probable hypothesis and he argued it very thoroughly. It was a twofold crisis. There was the discovery of the very general occurrence of irrational ratios, which made the foundations of the theory of proportion and similarity as used up till then, doubtful. These foundational problems were ultimately overcome by Eudoxos' theory of ratio. There was also the growing awareness of the intricacies of the continuum and the difficulties inherent in the concepts of infinitesimals and geometrical atoms. These problems were clearly exemplified in Zeno's arguments and the ultimate answer of Greek mathematics to the problems was Eudoxos' method of exhaustion.

This cadre was more or less generally accepted around 1930 and within it there was still much room for further hypotheses and speculations, especially on the development of pre-Eudoxian theories of ratio and on the roles of Theodoros and Theaetetos. Here Dijksterhuis was inclined to agree with Zeuthen and to ascribe to Theaetetos the exact foundation of arithmetic in *Elements* VII-IX, leading up to the theory of irrationalities in Elements X.

Dijksterhuis came to a more critical judgement on various other hypotheses and usages in the historiography of Greek mathematics. For instance he rejected Frank's hypothesis which ascribes the foundations of the new exact style in mathematics to the later Pythagoreans, he opposed the view of Taylor and others that the Greek had a concept of irrational number and he avoided the usage, introduced by Zeuthen, of the term "geometrical algebra" for the contents of *Elements* II and other passages in the *Elements* of similar nature.

The hypothesis of a crisis, has in later years been modified. Van der Waerden, in an article [1940] on Zeno's arguments, accepted a crisis but only over the irrational, not over the continuum and infinitesimals. Freudenthal [1966] has argued for a continuous development with respect to foundational questions rather than a crisis. Knorr takes in a recent work on the *Elements*, [1975], also the view of a more continuous development, caused by internal mathematical problems rather than external philosophical ones, which enables him to discuss in more detail the origins of the foundational problems.

However, despite changed views on the general cadre in which Dijksterhuis places the development of pre-Euclidean mathematics, his work is still of great value. I want to mention three aspects of it which are highly relevant today.

There is first of all Dijksterhuis' stress on philological arguments. Dijksterhuis had acquired, after leaving secondary school, partly autodidactically and partly through private instruction, a thorough knowledge of classical philology and a love of classical literature. His care for the correct rendering of the texts and his insight in the value of linguistic and general philological arguments for understanding the development of Greek mathematics is manifest in his edition. In recent years A. Szabó (e.g. [1969]) and others have shown the fruitfulness of this approach for the history of ancient mathematics.

Secondly there is Dijksterhuis' insistence on the danger of anachronistic interpretations of Greek mathematics. This comes out very clearly in his discussion of the concept of ratio in the text reproduced above. Dijksterhuis carefully explaines the relation of the modern concept of a Dedekind-cut to Eudoxos' theory of ratio, but after that he even more carefully explains the difference between Eudoxos' theory of ratio, based on a concept of magnitude, and the modern theory of real numbers based on the concept of a Dedekind-cut. An Eudoxian ratio is not a real number, and Dijksterhuis wants to preclude any identification of the two in the mind of his modern reader. Therefore, rather than using the symbol $\frac{A}{B}$ which is so much loaded with algebraical connotations, he

introduces the new symbol Λ(A,B) for it. He used the same simple but crucial expedient of introducing new symbols in dealing with the so-called "geometrical algebra". He disapproved of that term, preferring the term "area calculus" for it, and he introduced special symbols such as T(α) in stead of α^2 for the square on the linesegment α. In this way he could render succinctly and without distortion the relevant parts of Euclid's text. Dijksterhuis carefully and convincingly explained his reasons for doing so. The question of this Greek "geometric algebra", its origins, its connections with Babylonian algebra, the reasons for its geometric character and the extent to which it is algebra, has received much attention afterwards. It appears still to be capable of dividing historians into very opposite camps (cf. Szabó [1969] pp. 456-599, Unguru [1975], Van der Waerden [1976] and Freudenthal [1977]), but one cannot help thinking that at least some of the misunderstandings in that debate could have been avoided if Dijksterhuis' approach had been more commonly accepted. In fact, Knorr's balanced comments on the question ([1975], pp. 10, 11) are very much in Dijksterhuis' style.

Finally, I should say that Dijksterhuis' edition is relevant today because the form of edition he chose, though originally a compromise, is a very fortunate one. It is indeed a great pity that the work has not been translated in a more accessible language shortly after its publication. It would have done then on a large scale what it certainly did in Holland on a small scale: to give the reader, who finds going through the whole of Euclid too tedious and for whom the various historiographical theories about the development of Greek mathematics are not easily accessible, a fascinating account of both Greek mathematics and the problems of its interpretation.

Although a publication of a direct translation of Dijksterhuis' edition would now be out of date, it would be very welcome if an edition in the same style as Dijksterhuis' could be produced in the near future to compile the state of knowledge about Greek mathematics as it is about fifty years after Dijksterhuis' edition.

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