Pi, Ludolph van Ceulen, and the Challenge of Mathematics*

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* Address delivered (in Dutch) on July 5th, 2000, during a meeting in the *Pieterskerk* (St. Peter's Church) in Leiden, on the occasion of the unveiling, by his Royal Highness Prince Willem Alexander van Oranje, of a memorial tablet for Ludolph van Ceulen. The tablet is a recent replica of van Ceulen's tomb stone, which disappeared in the 19th century (cf. Note 1 below). The meeting was organised by the Dutch *Wiskundig Genootschap* (Mathematical Society) under the auspices of the *Koninklijke Nederlandse Akademie van Wetenschappen* (Royal Netherlands Academy of Sciences). It was incorporated in the programme of the *Fourth Algorithmic Number Theory Symposium* (ANTS IV), held in Leiden, July 2–7, 2000. The Dutch text of the address is published as [3]. Another address, on pi, was given during the meeting by Peter Borwein ([2]), and music by Sweelinck was performed on the Pieterskerk's organ by Joseph Steenbrink.

Your Royal Highness, Ladies, and Gentlemen —

It is a strange fate that befell Ludolph van Ceulen: history reduced him to the man who calculated the number pi correctly up to 35 decimals, and had the result carved on his tombstone.[1]

Ludolph used the method by which Archimedes some 2000 years before had calculated two decimals. Indeed it is Ludolph's perseverence that impresses us, not his originality — "steel-like industriousness" wrote a nineteenth-century biographer. It earned him the honour that the number pi is sometimes called the 'Ludolphian number,' but also the dubious distinction that his name now evokes the image of the cipherer, the calculations freak. This stereotypical image in turn evokes a question: What brings someone to undertake such immense calculations? Direct usefulness was not the reason; at that time ten decimals were more than enough for precision calculations. What lay beyond that precision? What was he looking for? What called him?

Ludolph did not calculate without purpose. Calculating was part of a many-sided career.[2] He was the son of a merchant from Hildesheim in Germany (not from Cologne, as his name would suggest). After a start about which we know little he moved to the Netherlands. The young republic had secured its territory in the provinces of Holland and Zeeland and stood at the beginning of a boisterous economic expansion, based primarily on trade.

In 1580 Ludolph had been living for some years in Delft; he was 40 years old and combined two professions: fencing master and teacher of reckoning. The two activities were less widely apart than one might think: The young sons of the new upper class should be good reckoners for the family's business and competent fencers for good manners.

In 1596 he published *Van den Circkel* ('On the circle'). The book contains his first calculations of pi, to 20 decimals. In 1600 Ludolph, having moved to

[1] The story of van Ceulen's tombstone in the Leiden Pieterskerk is a curious episode in the history of mathematics, until recently lacking a satisfactory ending (the stone disappeared in the 19th century and only a Latin version of the text on it was known). In fact it was dissatisfaction with the state of affairs around the stone with the 35 decimals which, on the initiative of Hendrik Lenstra, had prompted a group of Dutch mathematicians finally to close the episode with the installation and inauguration of a fitting memorial tablet. The text on the tablet is a reconstruction of the original Dutch text on van Ceulen's tomb stone. See [12] for more details.

[2] I have benefitted from two very detailed studies of van Ceulen's career and mathematics: [4] and [10].

Leiden, was appointed as teacher of mathematics at the new school for engineers. The school, connected to the university, was created at the instigation of Prince Maurits in order to train a corps of Dutch engineers. Ludolph taught there until his death in 1610.

A respected burgher with a many-sided profession; and also a rich and no doubt bustling family life; when he married Adriana Symons in 1590 he brought with him five children from an earlier marriage, and she eight; later at least two more children were born in the van Ceulen household.

Ludolph's mathematical specialism was what we would now call *large scale calculations*. The field had a solid reason for its existence: it was necessary because the earth is round, and because the sun, the earth, the moon, the planets and the stars all turn around each other. This circumstance determines night and day, the seasons, the economy, the social structures and so forth. For all civilizations we know about, the understanding and prediction of these motions was a prime necessity of life. It was also the first area of socially relevant problems that was opened up by advanced mathematics, developed by expert specialists. This had happened some 2000 years before Ludolph's time. The key was the geometry of circles and angles, trigonometry, the domain of those three Fatal Sisters governing the periodic phenomena of this world: the *sine*, the *cosine* and the *tangent*.

In Ludolph's time this branch of mathematics had acquired a new urgency: The travels of discovery and trade to the far and new continents required ever more and more precise calculations. Navigators had to calculate their positions and courses from the angles they had measured along the skies with their Jacob's staffs. The formulas for these calculations brimmed with sines. So sine tables were needed, and, if possible, tricks to shorten the calculations. The tables had to be calculated and the tricks had to be found and taught. That was the work of a small group of professional experts, the scientific top of the *large scale calculation* community. They worked in astronomical observatories, universities, and, as Ludolph, in teaching.

Ludolph's calculations of pi in record numbers of decimals belonged to what one might call the virtuoso fringe of the *large scale calculations*. They had primarily a public relations function; they showed Ludolph's professionalism and brought him fame and customers.

Pi was not crucial for large scale calculations. The central problem in the field was the calculation of sine tables. These calculations were based on the

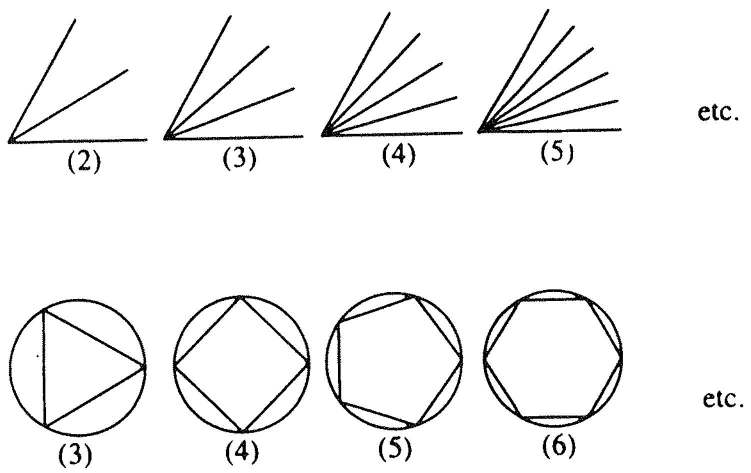

Figure 1: Angular section and section of the circle

so-called section of angles and of the circle: the division of an angle in 2, 3, 4, etc. equal parts, or the division of the entire circle in equal parts, resulting in the regular triangles, squares, pentagons, hexagons, seven-gons, eight-gons, etc. within the circle. The rest was calculation, large scale calculation.

Top research in the field centred on these problems, and explored new terrain. Two new things in particular: *irrational numbers* and *algebra*. Ludolph worked at these topics and showed there that, apart from steel-like industriousness, he also had inventiveness.

Irrational numbers had to do with square root extraction. In order to calculate sine tables it was necessary to extract square roots, indeed to do so very often and with many correct decimals. In order to understand and control these calculations it was handy first to postpone the actual calculating. You wrote only $\sqrt{2}$, or $\sqrt{3}$, or more intricately, for instance, $\sqrt{3+\sqrt{2}}$. At the end you then had a formula which provided a good overview of the calculations that still were to be done. The roots which you did not calculate, but just wrote down with root-signs, were called irrational numbers. They were new and controversial; there were even men of learning who thought they did not exist.

Ludolph accepted them with enthusiasm. With them he could write formulas for his calculations of pi. In his book on the circle they looked like this:

schreven/ van 167772160 syden/ doet √.2−√.2+√.2+√.2+√.2+√.2+√.2+√.2
+√.2+√.2+√.2+√.2+√.2+√.2+√.2+√.2+√.2+√.2+√.2+√.2
+√.2+√.2+√.2+√.2+√.2½+√1¼. Dese ghedivideert met haer Complement/

In the now common notation the formula would look like this:

$$\sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{2+\sqrt{2\frac{1}{2}+\sqrt{1\frac{1}{4}}}}}}}}$$

where the $\cdots$ indicate that I have only given the beginning and the end of the formula.

Mathematical formulas have a typographical aesthetics which can be appreciated without understanding their meaning. Curiously enough this aesthetics often moves in parallel with the meaning of the formula; beautiful formula: interesting mathematics; ugly formula: boring mathematics. Those concatenated root signs look graceful, they imply a challenge: what lies behind them? Ludolph was not the only one who sensed this challenge. François Viète, the top man of the field at that time, devised a formula for pi featuring the same concatenation of square roots, in modern notation:

$$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

Viète understood the formulas somewhat better than Ludolph; but Ludolph's explorations were certainly near the frontiers of research at that time.

And then the algebra. *Algebra*, Highness, Ladies and Gentlemen, is with x. If you're doing a sum and you don't see immediately what number should come out, then pretend you know that number, call it x and start calculating with it. Sometimes you will get the answer for free, and if not, this calculating with x will often tell you something new about your problem. In Ludolph's time this approach was new and very fascinating. It has indeed something magical, for how can you solve a problem by pretending you already know the answer? The wonder of algebra!

Calculating with x leads to equations. Reckoners in Ludolph's time had learned how to solve simple equations. The division of angles and of the circle yielded new, special equations. Viète for instance, found equations like these for the division of an angle in 2, 3, 4, etc. equal parts:

$$\begin{aligned}
x &= C \\
2 - x^2 &= S \\
3x - x^3 &= C \\
2 - 4x^2 + x^4 &= S \\
5x - 5x^3 + x^5 &= C \\
2 - 9x^2 + 6x^4 - x^6 &= S \\
7x - 14x^3 + 7x^5 - x^7 &= C \\
2 - 16x^2 + 20x^4 - 8x^6 + x^8 &= S \\
9x - 30x^3 + 27x^5 - 9x^7 + x^9 &= C
\end{aligned}$$

Again they are typographically attractive, they challenge: is there a rule behind the numbers in the equations? Yes there is such a rule, and Viète found it. This was the beginning of the theory of the so-called cyclotomic equations, which is, still today, an important part of algebra.

Ludolph too was fascinated by equations. Adriaan van Roomen, another top expert of large scale calculations, suggested to Ludolph to calculate the sides of the regular triangle, square, pentagon, hexagon, 7-gon, etc., up till the eighty-gon (drawn within a circle with radius 1). Ludolph calculated them, precise up to 14 decimals (see Figure 5). On that project he probably spent more time and certainly more ingenuity than on pi. He also used cyclotomic equations, but different from Viète's. Here are his equations for the 34-, the 38-, and the 158-gon respectively:

34-gon:

$$x = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + x}}}}\ ;$$

38-gon:

$$9x - 30x^3 + 27x^5 - 9x^7 + x^9 = \sqrt{2 - x}\ ;$$

158-gon:

$$5x - 5x^3 + x^5 = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - x}}}}\ .$$

Again I can use the typography of the formulas to indicate what was going on. To the left there are chunks of the equations which Viète had found earlier. To the right is Ludolph's specialism, his signature as it were, the concatenated square roots. Thus, via circle division, Ludolph entered a new area of algebra which he explored enthusiastically, finding all kind of suggestive connections.

These examples show that Ludolph did more than just calculating decimals of pi; he functioned as a professional in a prominent branch of mathematics. He worked on a international level; but he achieved no breakthroughs. He was inventive, but not innovating. A mathematician of merit and, perhaps more to the point, an enthusiastic one.

And certainly also an untiring calculator. Ludolph was indeed proud of his 35 decimals and he often wrote about the pleasure he experienced from calculating. About his project to calculate the sides of the regular polygons within a circle, he wrote:

> [then I was . . .] struck by such a lust and desire to find the required numbers with the precision as proposed above [namely precise to 14 decimals] that I considered the great labour as nothing, and I attacked the work with strong resolve (and setting aside all burdens that were heaped upon me) and did not rest before (with God's help) I found what was required of me ....[3]

And so we are brought back to the questions I posed at the beginning: What brings someone to undertake such immense calculations? What was Ludolph looking for? What challenge called him?

There are two elements in the story which give a clue about these questions, the tomb stone and the calculations. Ludolph was not the first to have a mathematical result carved on his tomb stone; Archimedes, his great example, did the same. In Archimedes' case it was a brilliant new result about spheres and cylinders. Having found that result Archimedes wrote a comment which, I feel, catches the essence of the challenge of mathematics. He noted that the properties of the sphere and the cylinder he had discovered were inherent to the nature of these figures, and had therefore always been true, also before he, Archimedes, discovered them for the first time.[4]

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[3] [5, p. 17^r].

[4] [1, p. 2].

By saying so, of course, Archimedes emphasized his priority. But the remark also characterizes the mathematical experience. It points to the feeling that what you discover was already there before you, that the mathematical things are timeless, and that they should be exactly the way you find them. Moreover, for experiencing that timeless necessity, it does not matter whether the mathematics is new or already discovered by others. Neither does it matter whether you immediately see *why* the things are as you find them. On the contrary, often they are not yet really clear, but precisely for that reason challenging and seductive: they can be made clear.

And then Ludolph's calculations. For the number pi he calculated far beyond what was required for practical precision. So he gave up something, namely sense. That, of course, is why his fervent calculations raise such wonder, one feels: this has no sense anymore. Also in the case of Ludolph's division of the circle something was given up: A heptagon can be easily visualized, a 15-gon perhaps also, but a 158-gon? There visualizability is given up, the feeling that one deals with a tangible, at least mentally tangible object. What remains is merely a *rule*: 158 equal sides within a circle.

Giving up things in this way requires quite some nerve, for what safe footing is then left? As a tightrope walker, you shouldn't be prone to vertigo when following in this way the rules of calculation beyond the borders of concrete sense and meaning. You enter the abstract, you let yourself be guided by reasoning itself; your security lies in regularity, in following patterns learned or discovered in earlier practical calculations.

Put like this, it seems an unlikely and perhaps dangerous activity. But it is also a very human activity; isn't it so that our brain likes nothing better than to follow its own logics and explore the worlds in which one then is transported? The attraction of calculating, and of mathematics in general, is not so far from the more common seductions of the intellect. And if, as Archimedes emphasized, you then enter a world which invites explicit understanding, and offers the opportunity to reach that understanding, then the attraction and the challenge of mathematics is no longer so surprising.

I mentioned earlier the virtuosity of Ludolph's calculation of pi. Associating this virtuosity with the medium used to publish its result — namely a tomb stone — the image arises of an encore at the end of a concert, in which the soloist one last time plays all the tempi, trills and flageolets he can do. Impressive, exciting perhaps, but the encore should not take away the memory of the rest of the concert. Something like that has happened to Ludolph, and that is a pity, because his concert, his mathematical work, definitely had merit.

The numbers on his tomb stone may certainly evoke a memory, but not only one of virtuoso calculations. They are also an emblem of the circle division problem and of the art of large scale calculation in Ludolph's time, with all its practical applications.

And I also like to see in Ludolph's numbers a reference to mathematics in general, the domain of surprising and at the same time convincing patterns and structures, going beyond simple tangibility, precision and direct use; a domain for which the human brain appears to posses a key. There is a challenge there, the challenge of mathematics, and it is a human, and indeed a humane activity to accept that challenge and explore the domain.

One may admire them, then, Ludolph's numbers, and soon we can admire them, skilfully sculpted in a new stone. They will remind many of the Pieterskerk's visitors of mathematics — and if it were up to Ludolph, I think, it would be a mathematics explored adventurously and practised with lust.

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Figures

Figure 2: Ludolph van Ceulen, portrait on the titlepage of *Van den Circkel* 1596 [4].

Figure 3: Ludolph van Ceulen, portrait published 1625 [10, p. 343]

VANDEN CIRCKEL.

Daer in gheleert werdt te vinden de
naeste Proportie des Circkels-diameter tegen synen Omloop / daer
door alle Circkels recht (met alle Figueren/ ofte Landen met cromme Linien besloten)
ghemeten kunnen werden. Item aller Figueren-syden in den Circkel beschreven/
Beginnende van den 3/4/5/15 hoeck/in Irrationale ghetallen te brengen/
al hadde de Figuer veel hondert-duysent hoecken. Item des
7/11/13/17/19/23 hoecx-syden / ende wat syden ofte
Coorden men begeerdt/welcker Boge groot zijn
Graden / Minuten / Secunden / &c.
Naer elcx behaghen.

Noch de Tafelen SINVVM, TANGENTIVM, ende SECANTIVM, met het gebruyck van dien, hoogh-noodigh voor de Land-meters: Met veel andere konstighe stucken, dierghelijcke noyt in druck uytghegheven.

Ten laetsten van Interest/met alderhande Tafelen daer toe dienende / met het ghebruyck / door veel constighe Exempelen gheleerdt/ ende door't gheheele werck bewesen / ende gheproeft.

Alles door LVDOLPH van CEVLEN, gheboren in HILDESHEIM, beschreven, ende inden druck ghebracht.

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zyn deel 12 B 10 C 9 D 8 E 6 F 5 G 3 maent betaelt elck ten einde [illegible]
tuts voor geleent gelt ende gewin A 100 B 250 C 240 D 256 E 244 F 240 G 220 f
vrage na het geleent gelt van elck ende na den intrest ten 100 int iaer

TOT DELF,

Ghedruckt by Jan Andriesz. Boeckvercooper/woonende aen't
Marct-veldt/in't Gulden A B C. Anno 1 9 6.

Figure 4: Title page of *Van den Circkel*, 1596 [4].

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Houck.		Houck.	
3	173205080756887.	42	14946018717284.
4	141421356237309.	43	14599062952181.
5	117557050458494.	44	14267836639846.
6	100000000000000.	45	13951294748825.
7	86776747823511.	46	13648482672934.
8	76536686473017.	47	13358526749024.
9	68404028665133.	48	13080625846028.
10	61803398874989.	49	12814043996142.
11	56346511368285.	50	12558103905862.
12	51763809020504.	51	12312181226788.
13	47863132857511.	52	12075699484457.
14	44504186791262.	53	11848125578742.
15	41582338163551.	54	11628965785095.
16	39018064403225.	55	11417762161551.
17	36749903563314.	56	11214089447438.
18	34729635533386.	57	11017552071173.
19	32918918056146.	58	10827781717083.
20	31286893008046.	59	10644434968435.
21	29808453235254.	60	10467191248588.
22	28462967654657.	61	10295750954069.
23	27233329819249.	62	101298537677425.
24	26105238444010.	63	99691771321395.
25	25066646712860.	64	9813534865483.
26	24107336051064.	65	9662759051014.
27	23218582850460.	66	9516383164748.
28	22392895220661.	67	9374452493988.
29	21623803484788.	68	9236691729147.
30	20905692653530.	69	9102919826592.
31	20233664397486.	70	8972966970063.
32	19603428065912.	71	8846669345075.
33	19011208660836.	72	8723877473067.
34	18453671892660.	73	8604446600906.
35	17927861780608.	74	8488240639229.
36	17431148549531.	75	8375130745839.
37	16961184895101.	76	8264994849762.
38	16515869094466.	77	8157717212317.
39	16093313743345.	78	8053188021883.
40	15691819145568.	79	7951303019385.
41	15309850567299.	80	7851963151813.

Nota

Godt alleen de eer.

Figure 5: The sides of regular polygons up till 80 sides, from *Van den Circkel* [4, p. 19^v].

Figure 6: Ludolph van Ceulen, portrait publ. 1615 [7].