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Johann Bernoulli on Exponential Curves, ca. 1695 Innovation and Habituation in the Transition from Explicit Constructions to Implicit Functions[1]

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1. Introduction

With enthusiasm, style and dedication, our Groningen colleagues have made the Dutch mathematical community aware of the fact that here, 300 years ago, Johann Bernoulli embarked on a professorship in mathematics which he was to hold for 10 years. It was, therefore, easy to decide on a main character for this lecture: Johann Bernoulli (see Figure 1).

While in Groningen, Bernoulli lived in 'Oude Boteringestraat'. Yesterday evening I went there to experience the feeling of walking in his footsteps. I must confess I did not experience much. In Oude Boteringestraat he left no noticeable traces; but in mathematics he did. Yet there is always something elusive in the relation between a person – a historical person or a contemporary – and her or his longer-term significance for some field of knowledge. It seems such an obvious question: This mathematician, what did he do? Or conversely: This part of mathematics, who created it? But even a slight acquaintance with historical literature on mathematics makes it clear that these questions are much too simple. Not infrequently theorems are due to others than those whose name they bear, and anyway there is always a story behind the naming. These arguments of naming and origin concern the relation between mathematicians and mathematics. The relation is at the same time elusive and fascinating. On the one hand, there is the single person, the mathematician, whose writings one can read, and by reading one can form an image of this person at work. On the other hand, there is the millenia-old tradition of doing mathematics, with its long-term changes which determined how we see mathematics now, and why we find our mathematics self evident.

Considering these things I had little difficulty in finding, as well as the protagonist, a focus for my lecture; it would have to be a piece of mathematics with which Bernoulli was engaged around 1695 and which could serve to explain the fascination which, for me, lies in the relation between the single person and the great lines of the development of mathematics. Such a fragment of

[1] Opening Address at the 31st Dutch Mathematical Conference, Groningen, April 20 and 21, 1995.

FIGURE 1. Johann I Bernoulli. From: *Opera Omnia* **1** (1742). Courtesy University Library, Groningen

Bernoullian mathematics is indeed available. It is not a magnificent piece of work, but neither is it without significance. It is about exponential curves and the exponential formulas used to represent them.

2. Bernoulli

Let me start with the person Johann Bernoulli. I imagine how, sometime late January 1697, he walked through Oude Boteringestraat from his home to the lecture room. He had just written a letter to Leibniz (Figure 2) in which he mentioned exponential equations. He considered these as his intellectual property (be it shared with Leibniz) – why he did so will become clear later on. Exponential equations had appeared in publications in the previous few years in a way with which Bernoulli was decidedly unhappy. This had to do with a Dutch scholar, Bernard Nieuwentijt. Nieuwentijt (Figure 3) left hardly

FIGURE 2. Gottfried Wilhelm Leibniz

any traces in mathematics, but at that time he had published three works of mathematics which created some uneasiness about the new differential and integral calculus of Leibniz. In his books he had voiced several objections against the fundamental concept of Leibniz' new technique, namely the differential. Bernoulli considered these objections nonsensical, but they did not bother him and he gladly left the defence of the new method to Leibniz. Worse was Nieuwentijt's claim that Leibniz' new method could not deal with exponential equations. Nieuwentijt himself had presented some calculations about these equations and had even derived a single correct result. How did Nieuwentijt come to be interested in these equations? Bernoulli must have wondered about that. Nothing really informative had been published about exponential equations; only in the secure shelter of the private correspondence between Bernoulli and Leibniz had they been dealt with. Two and a half years ago Bernoulli had sent his version of the exponential calculus to Leibniz, as one of the personal treasures which he proudly showed to the admired inventor of the new differential and integral calculus. Leibniz had written back with praise,

FIGURE 3. Bernard Nieuwentijt

noting that he – Leibniz – had developed the same approach earlier but that in some respects Bernoulli had come further than he. So, exponential equations were Bernoulli's business. But now Nieuwentijt had really spoiled the matter. Nieuwentijt could not differentiate exponential expressions (of course he couldn't; it was not that easy), so he had publicly alleged that this was a weakness of the calculus. This had prompted Leibniz to publish the rules of he exponential calculus in a brief article. He had mentioned that Bernoulli had found the rules independently, but nevertheless the game had lost its suspense and Bernoulli's profit out of it was minimal.

Something had to happen – Bernoulli decided to publish his own version of the exponential calculus. The article soon appeared in the March 1697 issue of the journal *Acta Eruditorum*. It is this article ([5]) which I want to discuss in the present lecture, but first some previous history has to be told. (For a short chronology of the events around exponentials see Table 1.)

3. Exponential calculus

The context of Bernoulli's (as well as Leibniz') version of the exponential calculus was the differential calculus as first published by Leibniz in 1684. At present we know the differential and integral calculus as a theory about *func-*

tions, their *derivatives* and their *integrals*, which themselves are also functions. In its Leibnizian form the calculus was different; it was a theory of *variables* and their *differentials*. Variables were the variable quantities in mechanical or geometrical problem situations; thus the height and the velocity of a projectile, or the coordinates of a point on a curve, were variables. Differentials were the infinitely small increments or decrements of these variables, occurring if successive (or infinitely near) stages (of the projectile in flight, or of the position of points on the curve) were considered. Nieuwentijt's objections in particular concerned these infinitely small differentials. Leibniz' calculus featured two operations, *differentiation*, symbol d, and *integration*, symbol $\int$ (Leibniz originally called it 'summation'). In his first article on the calculus of 1684 ([12]) Leibniz had given the rules for differentiation of variable u, v:

$$\begin{aligned} d(u \pm v) &= du \pm dv \\ d(uv) &= udv + vdu \\ d\frac{u}{v} &= \frac{vdu - udv}{v^2} \; . \end{aligned}$$

The rules only concerned the algebraic operations addition, subtraction, multipliction, division and root extraction. In particular, they did not cover expressions in which the exponents were variable. The exponential calculus provided additional rules applicable to such exponential expressions. The rule that Leibniz published in 1695 (in reaction to Nieuwentijt's objections and with due recognition of Bernoulli's independent discovery) was:[2]

$$d(u^v) = u^v \log udv + vu^{v-1}du \; . \tag{1}$$

In the corresponding modern form (with functions instead of variables) it is the familiar rule

$$(u^v)' = u^v \log u.v' + vu^{v-1}.u' \; , \tag{2}$$

in which u and v are functions of some independent variable t and $'$ indicates differentiation with respect to t.

Now this was a really new result. The rule itself was new, but the essential novelty lay in the fact that variable exponents had hardly been studied before.

[2] [14] pp. 324.

c. 1640	Analytic Geometry (Descartes, Fermat); equations involving $+$, $-$, $\times$, $\div$, roots but *no* variable exponents.
1679	Leibniz writes to Huygens about $x^x + x = 30$.
1682	(Feb.) Leibniz publishes the equation $x^x + x = 30$ [11].
1684	(Oct.) Leibniz's first article on the differential calculus [12].
1690-91	Leibniz discusses exponential equations in correspondence with Huygens; he solves an inverse tangent problem, finding $\frac{x^3y}{h} = b^{2xy}$ as the equation of the curve.
1692	(July) In an article in the *Journal des Sçavans* ([13]), Leibniz mentions exponential equations and the need to study the related curves.
1692	(Fall) Bernoulli develops his exponential calculus.
1694	Niewentijt publishes *Considerationes* ([17]) in which he claims that the Leibnizian calculus cannot deal with exponential expressions.
1694	(Sept.) Bernoulli sends his treatise on the "Calculus Percurrens" to Leibniz.
1695	(June) Review ([1]) of Nieuwentijt's *Considerationes* in the *Acta Eruditorum.*
1695	Nieuwentijt's *Analysis Infinitorum* ([18]) appears.
1695	(July) Leibniz answers Nieuwentijt's criticisms, publishes the rules for differentiating exponential expressions, mentioning that Bernoulli also, and independently, had found the rules ([14]).
1695	(Oct.) Bernoulli arrives in Groningen.
1696	Nieuwentijt publishes his *Considerationes secundae* ([19]).
1696	(Febr.) Review of Nieuwentijt's *Analysis infinitorum* in the *Acta Eruditorum* ([2]).
1697	(March) Review of Nieuwentijt's *Considerationes secundae* in the *Acta Eruditorum* ([3]).
1697	(March) Bernoulli's treatise on exponentials appears in the *Acta Eruditorum* ([5]).
1697	(June) The *Acta Eruditorum* publishes excerpts from Nieuwentijt's *Considerationes Secundae* ([4]).

TABLE 1. Brief chronology of the events around exponentials.

4. VARIABLE EXPONENTS

Exponential expressions occur in formulas, and formulas are a relatively recent phenomenon in mathematics. Viète was the first (around 1600) to introduce letters to represent both unknowns and indeterminates. Descartes streamlined this approach and in his writings we find algebraic formulas much like the ones we are used to. Moreover, Descartes and Fermat elaborated the method of representing curves by equations (in two unknowns) and thereby introduced analytic geometry. However, their formulas only involved the operations $+$, $-$, $\times$, $\div$ and root extraction. In fact, for philosophical reasons, Descartes held that only these operations ought to be accepted in pure, exact geometry. Thus there were no formulas involving exponentials, logarithms or trigonometric relations.

It was Leibniz who first introduced variable exponents. In 1679 he wrote to Christiaan Huygens about the equation $x^x - x = 24$, which can easily be 'seen' to have a solution $x = 3$, but for which none of the known methods of solution of equations applied.[3] Huygens, however, saw no use for this symbolic speculation. Leibniz also mentioned the equation briefly in an article published in 1682 ([11]).

Some ten years later, Leibniz mentioned exponential expressions again to Huygens. Their correspondence had been interrupted for some time, but in 1690 Leibniz resumed it and tried to convince Huygens of the power of his new calculus. Huygens was sceptical[4] and suggested that Leibniz should test his method on a number of problems. For this purpose Huygens took two algebraic curves and calculated their tangents by methods that had been known for some time. He introduced some rewriting of the resulting expressions so Leibniz would not be able to guess what the curves were by reconstructing the tangent calculation; then he sent the result to his correspondent with the challenge to determine which were the original curves. Problems of this type – to determine a curve from a given property of its tangents – had become important at that time; they were called 'inverse tangent problems'. In modern terms they lead to first-order differential equations. Huygens determined the tangents by calculating the so-called 'subtangent' σ, which is the distance along the axis between the points in which the tangent and the ordinate respectively intersect the axis (see Figure 4). However, he omitted to make clear that he took the subtangent to be positive if the tangent intersected the axis to the left of the ordinate. As it happened, Leibniz used the opposite convention for the sign of the subtangent. Thus Leibniz saw himself confronted with a different problem than the one Huygens had in mind, in fact a more difficult one; its solution was not an algebraic curve, it involved a logarithmic relation. Leibniz wrote the equation of the curve he had found in terms of variable exponents:

$$\frac{x^3 y}{h} = b^{2xy} ,$$

and sent this formula to Huygens. Understandably Huygens was unimpressed; whatever the formula meant, it was not the correct solution; apparently Leibniz was playing empty symbolic games which did not solve problems. It took Leibniz a good number of letters to clear up the misunderstanding and to convince Huygens that his new calculus was more than new notations for old methods, but in the end Huygens did change his mind. Meanwhile, Leibniz had once more mentioned variable exponents in an article; in 1692 ([13]) he wrote about the equations $x^x + x = 30$ and $c^x = ab^{x-1}$. The latter could be solved by logarithms ($x = \frac{\log a - \log b}{\log c - \log b}$), but for solving the former (that is, finding the obvious solution $x = 3$ by a general method), Leibniz asserted, one needed non-algebraic curves. He probably had in mind that the solution of $x^x + x = 30$ was the x-coordinate of the point of intersection of the line

[3] Letter to Huygens of Sept. 8, 1679, [10] vol. 8 pp. 214–219

[4] The arguments I summarise here started with Leibniz' letter to Huygens of July 25, 1690, [10] vol. 9 pp. 448–452.

$y = 30 - x$ and the curve $y = x^x$. In the article he called such non-algebraic curves 'transcendental' because, as he wrote, "they pass through all degrees"[5]

Probably it was this article of 1692 which induced Johann Bernoulli (and perhaps also Nieuwentijt) to study exponential expressions. Bernoulli worked out his method for differentiating such expressions in 1692. He gave some hints about it in a letter to l'Hôpital (whom he had taught the Leibnizian calculus in Paris) and it is interesting to note in the Marquis' answer how difficult the concept of a variable exponent was:

> As to your new logistic calculus I cannot imagine at all what you have in mind, for what can m^n mean if m and n represent lines? A line to the power indicated by another line?[6]

The passage shows the conceptual problem connected to exponential expressions: Mathematicians generally regarded variables in equations to represent line segments; exponents, however, indicated dimensions or powers and had therefore to be integer or at least rational numbers.

In 1694, as I mentioned, Bernoulli sent his exponential calculus, or "calculus percurrens" as he called it, in a letter to Leibniz. In the same year Nieuwentijt expressed his opinion that exponentials could not be dealt with by Leibniz' calculus, and, as we have seen, this prompted Leibniz to publish the rules for differentiating exponential expressions in 1695. But Nieuwentijt made it clear, in publications of 1695 and 1696, that Leibniz had not convinced him. And so we are back at the situation which (so I imagine) Bernoulli considered on his walk from home to his lecture in late January 1697.

5. Bernoulli's article of 1697

I now turn to the article in which Bernoulli published his version of the exponential calculus. It took up 8 pages of the journal *Acta Eruditorum*, one of the earliest European scientific journals, founded, by Leibniz and others, in 1682. The journal contained mostly reports on books on a wide range of subjects that would interest the 'Erudite'. But with some regularity there also appeared mathematical articles such as the one by Bernoulli. Its one and only figure (see Figure 7) was placed, together with two figures of another mathematical article, in a small corner of a figure sheet mostly devoted to instruments for amputations and methods for dressing amputation wounds.

Bernoulli started by explaining how exponential expressions should be interpreted. He referred them to one fundamental curve, the *Logarithmica*, which, he claimed, was "truly the simplest of all transcendental curves".[7] The logarithmica (Figure 4) had been studied from ca. 1650. Bernoulli gave two equivalent definitions of the curve:

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[5] [13] p. 279: "parceque il n'y a point de degré qu'elles ne passent" – which is an apt characterization of the exponential expression x^x.

[6] Letter of l'Hôpital to Bernoulli of May 16, 1693, [8], p. 172: "à l'egard de vôtre calcul logistique je ne m'en puis former aucune idée, car que peut signifier m^n si m et n marquent des lignes? une ligne elevée à la puissance designée par une autre ligne?"

[7] [5] p. 180: "omnium profecto curvarum transcendentium simplicissima".

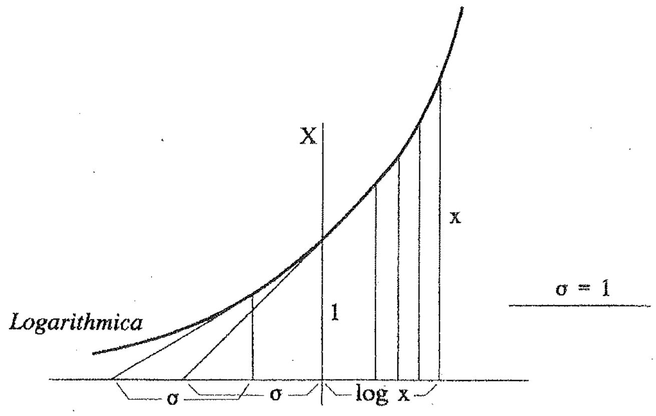


FIGURE 4. The Logarithmica

1. A Logarithmica is a curve whose subtangent σ (along an axis as in Figure 4) is constant.
2. A Logarithmica is a curve for which any arithmetical sequence of abscissas (along an axis as in Figure 4) corresponds to a geometrical sequence of ordinates (and vice versa).

Both definitions determine a family of curves; Bernoulli fixed one curve in particular by taking the vertical axis (which he used as X-axis) along the ordinate whose length was equal to the subtangent σ, and by taking the unit along the axes equal to σ. The Logarithmica defined the relation $x \leftrightarrow \log x$: the ordinate x corresponded to the abscissa $\log x$.

In modern terms the equation of Bernoulli's Logarithmica is $u = e^v$. However, it is important to note that Bernoulli did not introduce the curve on the basis of any equation, but by geometrical properties which implicitly define the relation between abscissas and ordinates. In particular, Bernoulli did not use exponentials or powers in his definitions.

6. THE RULES OF THE EXPONENTIAL CALCULUS

Bernoulli then listed the rules of the exponential calculus. The first was:

$$\log(u^v) = v \log u \; . \tag{3}$$

The rule was evident, Bernoulli wrote, from the well known logarithm tables. Such tables were indeed available since c. 1600, but it should be noted that these early tables were not based on the conception of logarithms as the inverse of exponentials. Rather, logarithm tables were conceived of in the sense of the second definition of the logarithmica mentioned above: the tables consisted of corresponding arithmetical and geometrical series, useful for transforming multiplications into additions. The second rule was

$$d(\log u) = \frac{du}{u} \; . \tag{4}$$

This rule followed geometrically from the first defining property of the Logarithmica; the familiar configuration with the characteristic triangle at a point with abscissa $\log u$ and ordinate u on the curve (cf. Figure 5) yields $du : d(\log u) = u : \sigma$, and because the unit is taken to be equal to σ, we arrive at Equation 4.

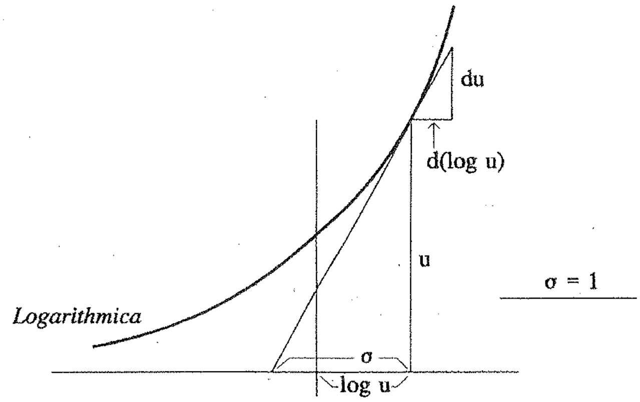


FIGURE 5. Derivation of the second rule

The third rule was:

$$d(u^v) = vu^{v-1}du + u^v \log u dv \; ; \tag{5}$$

this was the rule published earlier by Leibniz, cf. Equation 1. To derive Equation 5 Bernoulli wrote $w = u^v$, derived from this $\log w = v \log u$ by the first rule, and differentiated using the second rule, arriving at

$$\frac{dw}{w} = \frac{vdu}{u} + \log u dv \; , \tag{6}$$

from which Equation 5 follows by re-inserting $w = u^v$. Figure 6 shows the passage in Bernoulli's article containing the third rule.

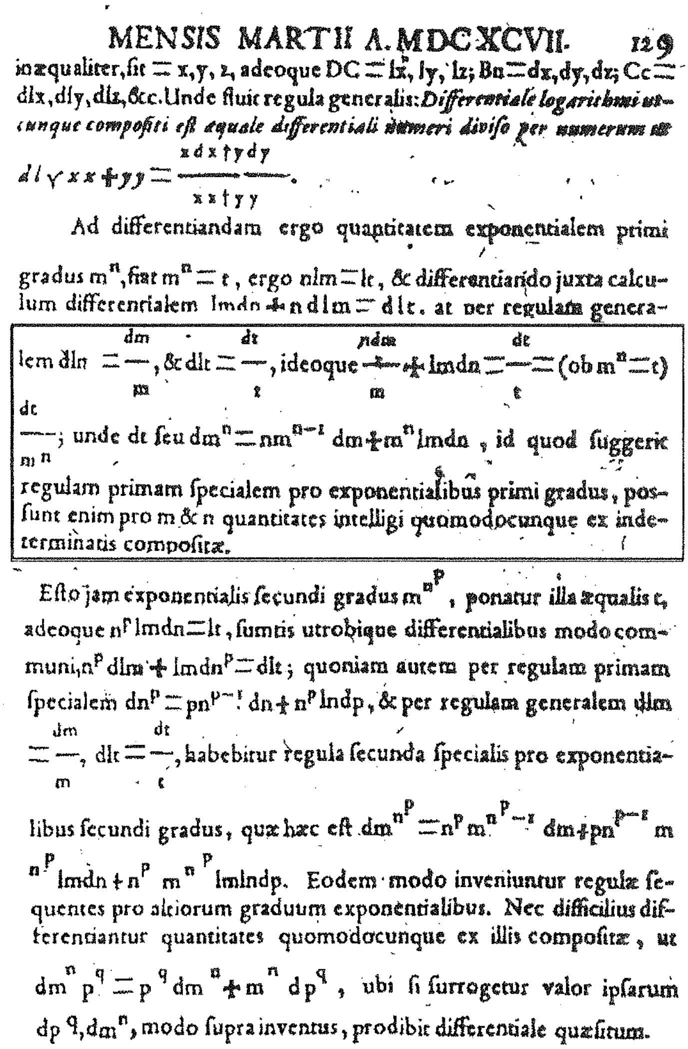
7. THE CURVE $y = x^x$

Bernoulli illustrated the use of the new rules with several examples. I discuss one of these, namely the exponential curve $y = x^x$. Bernoulli first explained what the equation meant. To do so he proposed the following procedure:

CONSTRUCTION 1 [8] – **Construction of points on the curve $y = x^x$**
Given: a 'Logarithmica' AB (see Figure 7 which gives the figure as it appeared in the Acta Eruditorum *article), with subtangent $\sigma = 1$; $AD = 1$; it is required to construct points on the curve represented by $y = x^x$.*

> **Construction**: **1.** Take an arbitrary point B on the Logarithmica with abscissa DC and ordinate BC.
> **2.** Take line segment DM along the axis such that $AD : BC = DC :$

[8] [5] pp. 184.



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inæqualiter, fit $= x, y, z$, adeoque DC $=$ lx, ly, lz; Bn $=$ dx, dy, dz; Cc $=$ dlx, dly, dlz, &c. Unde fluit regula generalis: *Differentiale logarithmi utcunque compositi est æquale differentiali numeri diviso per numerum ut* $dl\sqrt{xx+yy} = \frac{xdx+ydy}{xx+yy}$.

Ad differentiandam ergo quantitatem exponentialem primi gradus m^n, fiat $m^n = t$, ergo $nlm = lt$, & differentiando juxta calculum differentialem $lmdn + ndlm = dlt$. at per regulam generalem $dlm = \frac{dm}{m}$, & $dlt = \frac{dt}{t}$, ideoque $\frac{ndm}{m} + lmdn = \frac{dt}{t} =$ (ob $m^n = t$) $\frac{dt}{m^n}$; unde dt seu $dm^n = nm^{n-1}dm + m^n lmdn$, id quod suggerit regulam primam specialem pro exponentialibus primi gradus, possunt enim pro m & n quantitates intelligi quomodocunque ex indeterminatis compositæ.

Esto jam exponentialis secundi gradus m^{n^p}, ponatur illa æqualis t, adeoque $n^p lmdn = lt$, sumtis utrobique differentialibus modo communi, $n^p dlm + lmdn^p = dlt$; quoniam autem per regulam primam specialem $dn^p = pn^{p-1}dn + n^p lndp$, & per regulam generalem $dlm = \frac{dm}{m}$, $dlt = \frac{dt}{t}$, habebitur regula secunda specialis pro exponentialibus secundi gradus, quæ hæc est $dm^{n^p} = n^p m^{n^p-1} dm + pn^{p-1} m^{n^p} lmdn + n^p m^{n^p} lmlndp$. Eodem modo inveniuntur regulæ sequentes pro altiorum graduum exponentialibus. Nec difficilius differentiantur quantitates quomodocunque ex illis compositæ, ut $dm^n p^q = p^q dm^n + m^n dp^q$, ubi si surrogetur valor ipsarum dp^q, dm^n, modo supra inventus, prodibit differentiale quæsitum.

FIGURE 6. Rule three as published in Bernoulli's 1697 articl; the essential part is rimmed

DM.

3. Draw MN, take EF equal to MN.

4. The point F is on the curve $y = x^x$.

5. To find more points on the curve, repeat this procedure starting from other points on the Logarithmica.

Putting $BC = x$ and $DC = \log x$, it is easily seen that this procedure is the translation in geometrical terms of the relation $\log y = x \log x$ which follows from the equation of the curve by applying the first rule.

It should also be noted that the procedure does not in fact provide the curve as a whole, but only an arbitrary number of points on the curve. Such constructions of curves were called "pointwise". During the seventeenth century, mathematicians came to accept pointwise constructions as adequate representations of curves, be it that most mathematicians prefered continuous procedures

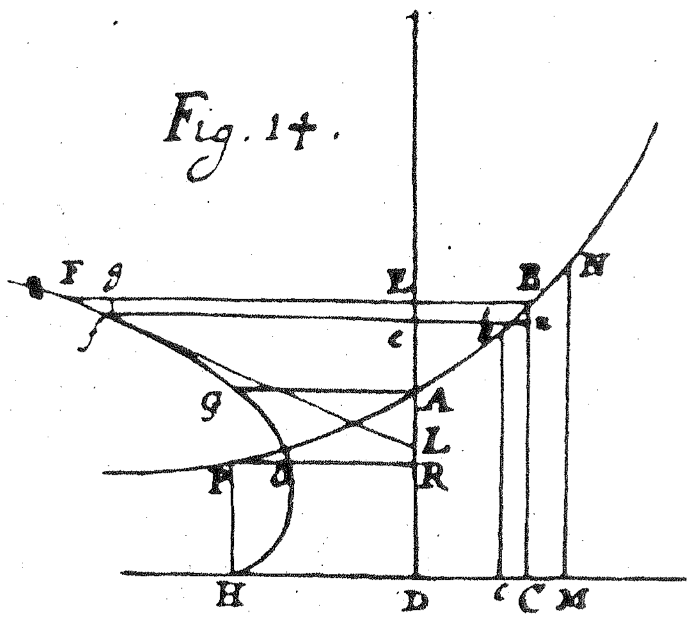

FIGURE 7. The figure from Bernoulli's 1697 article

for tracing curves.

Having thus fixed, by a pointwise construction, the meaning of the expression $y = x^x$, or in other words, having actually *given* the curve by means of this construction, Bernoulli went on to derive further results about the curve. He first dealt with drawing tangents, explaining the following construction:

CONSTRUCTION 2 [9] – **The tangent to** $y = x^x$
Given: a Logarithmica AB with respect to perpendicular axes through D (cf. Figure 7), AD = 1, further the curve $y = x^x$, drawn in the left-hand quadrant with respect to a vertical X-axis and a horizontal Y-axis, F is a point on the curve; it is required to construct the tangent at F .

> **Construction: 1.** Draw the horizontal ordinate FL of F, L is on the vertical axis; FL prolonged intersects the Logarithmica in B; draw the ordinate BC of B.
> **2.** Take EL along the vertical axis with $(AD + DC) : AD = AD : EL$.
> **3.** LF is the required tangent at F.

If we put $DE = x$, $EF = y = x^x$ and $EB = DC = \log x$, it is easily seen that the procedure in step **2** is the geometrical translation of the formula

$$(1 + \log x) : 1 = 1 : y\frac{dx}{dy} , \tag{7}$$

which follows directly from a calculation of dy according to the third rule:

$$dy = d(x^x) = x^x(1 + \log x)dx = y(1 + \log x)dx . \tag{8}$$

[9] [5] pp. 184.

It is noteworthy that Bernoulli felt that Equation 8 was not sufficient in this case, but that the relation it expressed had to be presented in geometrical form, that is, by an explicit construction.

I mention briefly three further properties of the curve $y = x^x$ that Bernoulli explained in his article[10] (cf. Figure 7). Two of these relate to special points on the curve. He determined the x-value DR of the point O on the curve with minimal distance from the vertical axis. His (correct) result is equivalent to the equality $DR = e^{-1}$. He also stated that the curve meets the horizontal axis in a point H with $HD = AD$, which in modern terms implies the result $\lim_{x \to 0} x^x = 1$.

The third property of the curve was a virtuoso result, a fitting proof of Bernoulli's mastery; it concerned the quadrature of the curve. Bernoulli claimed that the area bounded by the curve, the ordinate GA, the segment AD of the vertical axis and the initial ordinate DH could be expressed as the following series:

$$1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} \cdots . \tag{9}$$

Bernoulli did not explain how he derived his result. From his correspondence with Leibniz[11] we know that he developed $\int_0^x x^x dx$ as a double series in x and $\log x$. In the special case of $\int_0^1 x^x dx$ (which corresponds to the area $GADH$) the series reduces to the one in Equation 9. Leibniz had these results on the 'quadrature' of the curve $y = x^x$ in mind when he acknowledged that Bernoulli had pursued the study of exponentials further than he himself had done.

8. A COROLLARY

At the end of his exposition on exponentials, Bernoulli added a corollary which is illuminating. He considered the exponential curve $y = a^x$ (a a constant) and calculated its subtangent $\sigma = y\frac{dx}{dy}$. Applying the third rule gave $dy = a^x \log a dx = y \log a dx$, and hence $\sigma = \frac{1}{\log a}$. So the subtangent was constant which meant that $y = a^x$ represented a Logarithmica. The result, as it were, closed a circle in Bernoulli's treatment of exponential expressions: it proved that the curve he took as the basis for his theory, and which he introduced by its geometrical properties, independently of powers or exponents, was itself also an exponential curve. Thus only at the end of his treatment we find what from our modern point of view we would expect to be placed at the beginning, namely the explanation of the nature of the simplest exponential curve, $y = a^x$. The fact that this result comes as a final corollary underlines once more the primacy of the geometrical, constructional definition of the logarithmic relation in Bernoulli's exposition of the theory.

This is not to say that Bernoulli underestimated the significance of the corollary. He commented (not missing the opportunity to refer to Nieuwentijt's insufficient understanding of the matter):

[10] [5] pp. 184-185.

[11] Cf. letter to Leibniz of September 2, 1694, [15] vol. 3 pp. 143-152.

> Thus by one stroke of the pen I have disclosed the clear and royal road which Mr. Nieuwentijt asked for, namely the way to reduce the logarithmica – a curve which at present anyone knows – to an exponential equation.[12]

9. Assessment

I have dealt with Bernoulli's article it in some detail in order to show how much was involved in the first appearance of such a simple rule as the one for differentiating an exponential expression. However, I opened this lecture with the question of the relation between the mathematician and mathematics, between the single person and the main lines of development, so I should turn now to these more general considerations. Bernoulli's article is one of the traces he left in mathematics, and we want to assess its significance in the global development of mathematics. To do so we need an outline of this global development which may serve as background or framework for interpreting the significance of Bernoulli's contribution. As the article contained an important innovation in analysis, a natural choice for such a framework is the global history of Analysis.

During the 100 years of which Bernoulli's Groningen period roughly forms the middle, three key processes occurred in the history of analysis (cf. Table 2): the creation of analytic geometry, c. 1640, by Descartes and Fermat, the creation of the differential and integral calculus, c. 1690, by Newton and Leibniz, and the creation of the specific 18th-century style of analysis by Euler c. 1740. In their new analytical approach to geometry, Descartes and Fermat developed the use of algebra in solving geometrical problems. Most of these were construction problems, but some also involved curves, and in this connection the two mathematicians pioneered the method of representing curves by equations. The new methods of Newton and Leibniz were developed to deal with more complicated problems, such as the determination of tangents and areas of curvilinear figures and, most importantly, the "inverse tangent problems" mentioned above. For these problems the algebra of analytic geometry was insufficient. Newton and Leibniz created new concepts and symbolisms which made it possible to deal with the infinitesimal processes, implicit in the determination of tangents and areas, by means of symbolic procedures. Their new methods were an extension of algebra by the addition of new operations equivalent to differentiation and integration. Finally, Euler was the key figure in establishing of the characteristic 18th-century style of analysis. Whereas in Newton's and Leibniz' period the problems of analysis were still primarily geometrical, now these problems had themselves become analytical: 18th-century analysis centred on the solution of differential equations.

In the course of these hundred years the very meaning of the term 'analysis' changed fundamentally. In the 1640's it designated algebraic techniques for finding solutions of geometrical (and number theoretical) problems. The

[12] [5] p. 186: "unico ergo ductu calami *planam regiamque* qualem Dn. Nieuwentiit desiderat, *viam aperui, Logarithmicam, hoc tempore nulli non cognitam, ad aequationem exponentialem reducendi*."

	Protagonists	Problems	The growing arsenal	The changing challenge
c. 1640	Fermat, Descartes	Geometrical construction problems	Algebra	Construction
c. 1690	Newton, Leibniz	Tangents, Quadratures, Inverse tangent problems	Differentiation and Integration	Explicit construction of curves
c. 1740	Euler	Differential equations	ANALYSIS	Best analytical expression

TABLE 2. Analysis c. 1640 – c. 1740

new techniques of Newton and Leibniz were first called "infinitesimal" analysis, to distinguish them from the earlier analysis which only dealt with finite quantities. Then, in the eighteenth century, the indication "infinitesimal" was dropped, analysis came to mean the study of formulas, "analytical expressions", involving infinitesimal procedures such as differentiation, integration and series expansion. The terminological change accompanied an impressive process of growth; the analytical textbooks of Euler, written in the middle period of the eighteenth century, display a richness of material and methods which is all the more imposing if one considers that 100 years before there was nothing of its kind available in mathematics.

Taking this very rough sketch of the developments in analysis at the time as a framework for an assessment, we recognise Bernoulli's achievements with respect to exponential expressions and curves as a considerable innovation and extension; he (and Leibniz likewise) created a new and powerful kind of analytical expressions, and rules for their manipulation.

10. THE ROLE OF THE FRAMEWORK

This assessment of Bernoulli's achievement is sound enough, but it leaves little room for the aspects which are so striking when one studies the article in detail, in particular, the strongly geometrical approach and the absence of the interpretation of m^n as m to the power n. The chosen framework, namely the development of analysis seen as evolution towards the later 18th-century style, leaves little room for interpreting these remarkable aspects; rather they appear as deplorable obstacles which temporarily blocked the smooth progress of mathematics and thereby produced a more haphazard process of development than would have been necessary or logical. In particular, the significance of the actual form of the article, the style in which Bernoulli wrote it, is lost within this framework. As a result, the connection between the person Bernoulli, who composed the article in his own style, and the main lines in the development of mathematics, remains underexposed and elusive. The framework entails the danger that the development of mathematics is seen as a sequence of great inventions, larded with endearing anecdotal stories of strange, inefficient behaviour of mathematicians.

11. An alternative framework: the challenges

When we put the main lines of the development of analysis central, we look at history with foreknowledge of the present. But, of course, that was not how Bernoulli saw his mathematics. For him the contemporary challenges of mathematics determined his moves, not the endpoints of the lines of development. So, if we look for an alternative framework in which to assess Bernoulli's treatise on exponential expressions and curves, we may consider how the challenges of mathematics changed.

Throughout the period c. 1640 – c. 1740 (cf. Table 2) the main challenge of mathematics was to find methods for solving problems. What were these problems and what did it mean to solve them? The problems which, around 1640, led to the analytical inventions of Descartes and Fermat were geometrical problems. Characteristically, they presupposed a given geometrical figure, and it was required to construct additional lines or figures satisfying certain given conditions. Thus, in the famous problem of angle trisection, an angle was supposed to be given and two straight lines should be constructed through the vertex, dividing the angle in three equal parts. Constructions of such problems should preferably be performed by 'ruler and compass' (that is, by straight lines and circles), but for problems that were not solvable in that way (such as the trisection problem) early modern mathematicians searched for other, more potent means of construction.[13]

Fifty years later, at the time Bernoulli wrote his treatise on exponentials, the challenges had changed. New problems had arisen, notably those for whose solution the methods of the differential and integral calculus were created. The problems were mentioned above: the determination of tangents and areas of curvilinear figures and the "inverse tangent problems". Solutions of such problems were still constructions, which meant that inverse tangent problems, or, in general, problems that required the determination of an hitherto unknown curve, were considered solved only if the curve in question was somehow constructed. We have seen one way of constructing new curves in Bernoulli's explanation of the meaning of the formula $y = x^x$, namely, a pointwise construction using another curve (the Logarithmica) which was assumed given.

Around 1700 a crucial change occurred in this constellation of problems and their solution; the explicit geometrical construction of the solution was less and

[13] It is indeed a sign of how strongly the challenges of mathematics have changed, that at present the main interest of the trisection problem is in the proof that it cannot be constructed by ruler and compass. For 16th and 17th century mathematicians the challenge was not to prove the impossibility of trisecting by ruler and compass, but to find other procedures of construction by which it could be done.

less considered to be a real challenge. The differential and integral calculus offered a new means to attack problems involving unknown curves, namely differential equations. Soon mathematicians saw these differential equations themselves, rather than the geometrical problems they represented, as the main challenge. At the same time they no longer required an explicit construction of the required curve as solution. Rather, they considered differential equations solved if an analytical expression (an equation) was found which in some sense expressed the nature of the curve in the best possible way.

This development in mathematics is easily overlooked from a modern point of view. It was, however, a most remarkable one. It concerned the role of equations in analysis. In the mid seventeenth century mathematicians saw an equation $F(x, y) = 0$ primarily as a challenge: make clear which curve is represented by this equation; do so by explicitly constructing the curve. A hundred years later, equations were primarily seen as objects, with no challenge attached to them other than to determine, if possible, a more appropriate equivalent equation. Explicit construction as basis for understanding the objects of mathematics was replaced by a trust in the formula, based on a gradually established conviction that the equations of analysis always, explicitly or implicitly, defined an object, and that therefore this object could be accepted as given or as existent. A process of habituation to the world of formulas and equations finally eliminated the demand for a geometrical explanation.

12. Significance

I now return to Bernoulli's article. The considerations above provide us with an alternative framework to assess the significance of his treatise on exponentials, namely the contemporary mathematical challenges. Bernoulli took up the challenges presented by exponentials. They were: 1) to use the exponential expressions, but also, 2) to corroborate the security of their use according to the requirements that were then current, that is, by explicit construction of the objects represented by the expressions. By accepting both challenges he did more than creating a new analytical method, he also contributed decisively to the habituation of the mathematical public to exponential expressions, and analytical expressions in general, as reliable mathematical objects.

13. Conclusion

By realizing what the challenges were in mathematics around 1700, we understand the unfamiliar geometrical approach to the material and we can recognize Bernoulli's treatise as one step in the transition from explicit construction to implicit function. To realize this, an an interpretational framework was needed such as the second one I discussed. This framework was informed by the question of the interaction between mathematics and the mathematician – how mathematics appeared in the eyes of mathematicians, which challenges, in their perception, mathematics posed. For me the fascination with the mathematical enterprise lies primarily in this interaction and in its changes over time.

I think that the interaction between mathematics and the mathematician is too often overshadowed by the attention to *results*. Results are tangible, we work with them, we know them. In historical accounts of mathematics there is usually more emphasis on the results of mathematicians from earlier times than on the challenges that provoked these results. This imbalance often stands in the way of a real understanding of past mathematics and its development.

But the imbalance is not restricted to the understanding of past mathematics. Even now, the interest in the interaction between mathematics and the mathematician is easily overshadowed by a preoccupation with results. As in the case of past mathematics, this may stand in the way of a full understanding of our field. I feel that this is an issue of some importance, and therefore it has been a privilege for me to have the occasion, in this opening lecture of your congress, to call your attention, by an example from history, to the interaction between mathematics and the mathematician.

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