Assignment theory due 01/01/2017 at 11:59pm EST

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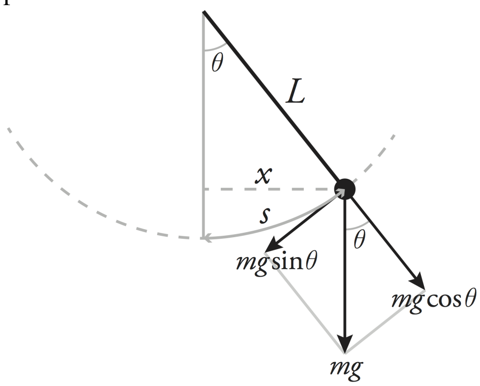
1. (1 point) † Let's investigate the differential equation for pendulum motion.



We wish to find $s(t)$, the elevation of the pendulum measured along its arc. As always we must start with Newton's law $F = ma$. The force involved is the component of gravity that pulls in the direction of the tangent; this is $-gm\sin\theta$ (negative because it acts to decrease s), so $F = ma$ says $ms'' = -gm\sin\theta$. Since we are looking for a differential equation for $s(t)$ we want s and t to be the only variables. But θ is also variable, so we must get rid of it. We could make it $\sin\theta = x/L$, which doesn't seem much better since x is also variable. But here is the trick: horizontal displacement is almost equal to displacement along the arc, i.e., $x \approx s$, at least for small θ. With this approximation we can get rid of all unwanted variables and obtain $ms'' = -\frac{gm}{L}s$. In other words, $s(t)$ is a function such that when you differentiate it twice you get back the function itself times $-g/L$. Which functions behave like this? Well, $\sin(\sqrt{g/L}t)$ does, and you could also put a constant A in front and it would still work. And $B\cos(\sqrt{g/L}t)$does too. So $s(t)$ must have the form $s(t) = A\sin(\sqrt{g/L}t) + B\cos(\sqrt{g/L}t)$ for some constants A and B.

Select all that are true:

- A. All derivatives above are with respect to time.
- B. The smaller the swings, the more accurate the solution.
- C. The fact that the solution has precisely two undetermined constants in it corresponds to the fact that the highest derivative in the differential equation was of order two.
- D. The derivation assumes that θ is in radians.
- E. This gives an easy way to estimate g.
- F. The higher the starting point, the greater the swing time.
- G. None of the above

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2. (1 point) † A diagonal matrix is a matrix with all zeros except along the diagonal. Diagonal matrices are very convenient because

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$$

If we have some non-diagonal matrix A we may want to diagonalise it, so that we can take its powers in a convenient way. I claim that in fact

$$M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{-1} A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors of A written as columns. This is a splendid fact, because if we solve for A in this equation we obtain

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{-1}$$

and therefore

$$A^n = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^n \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{-1}$$

In this way we need only three matrix multiplications instead of a hundred to compute A^{100}.

To prove my claim I only need to compute:

$$\begin{aligned} M\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{-1} A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{-1} A\mathbf{v}_1 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{-1} \lambda_1\mathbf{v}_1 \\ &= \lambda_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 0 \end{bmatrix} \end{aligned}$$

Justify the steps in this calculation:

[?] 1. first equality
[?] 2. third equality
[?] 3. fourth equality
[?] 4. second equality

A. matrix multiplication computation
B. reasoning backwards: what input gives this input?
C. simplifying by column operations
D. definition of A
E. definition of eigenvector
F. rule for scalar product of vector with itself
G. definition of M
H. formula for inverse matrix

In the same way one finds that $M\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\\lambda_2\end{bmatrix}$, so M must be $\begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix}$, as claimed.

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3. (1 point) † The scalar product can be written in two equivalent ways:

$$\mathbf{a}\cdot\mathbf{b} = \underbrace{|\mathbf{a}||\mathbf{b}|\cos\theta}_{\text{cosine form}} = \underbrace{a_1b_1 + a_2b_2 + a_3b_3}_{\text{coordinate form}}$$

What is the reason behind this magical harmony of geometry and algebra? To see this it is useful to express **a** and **b** in terms of the unit coordinate vectors $\mathbf{i},\mathbf{j},\mathbf{k}$. By breaking up **a** and **b** into their coordinate components we get

$$\mathbf{a}\cdot\mathbf{b} = (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$$

Now, the projection properties of $\mathbf{i},\mathbf{j},\mathbf{k}$ are particularly simple: any one of them projected onto itself gives [/0/1/the vector itself], and projected onto each of the other two gives [/0/1/the vector itself]. Therefore when we multiply out the parenthesis all the cross terms go away and only the "like with like" terms survive. So the result is $a_1b_1 + a_2b_2 + a_3b_3$, as claimed.

Which of the following are assumptions made in the above proof?

- A. $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$ for any vectors $\mathbf{u},\mathbf{v}$.
- B. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$ for any vectors $\mathbf{u},\mathbf{v}$.
- C. $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$ for any vectors $\mathbf{u},\mathbf{v},\mathbf{w}$.
- D. $(k\mathbf{u})\cdot\mathbf{v} = k(\mathbf{u}\cdot\mathbf{v})$ for any vectors $\mathbf{u},\mathbf{v}$, and constant k.
- E. None of the above

To complete the proof any such assumptions would have to be proved

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- geometrically (using cosine form)
- algebraically (using coordinate form)