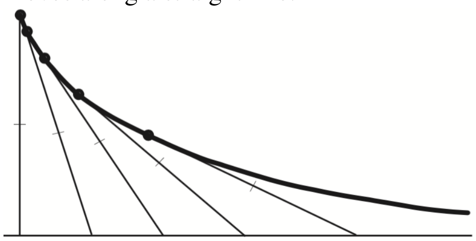
Assignment story due 01/01/2017 at 11:59pm EST

1. (1 point) The tractrix is the curve traced by a weight dragged along a horizontal surface by a string whose other end moves along a straight line:



In the physique de salon of 17th-century Paris, a pocket watch on a chain was the preferred way for gentlemen to trace this curve. The same curve can also be interpreted as the "pursuit path" of a predator

-
- on a leash
- running freely

chasing a prey that is running

-
- in a straight line.
- straight away from the predator.

Let's say that the length of the string is 1. Consider it as the hypotenuse of a triangle with its other sides parallel to the axes. Draw a figure of this triangle and write in the lengths of its sides (1 for the hypotenuse, y for the height, and the last side by the Pythagorean theorem). Find a differential equation for the tractrix by expressing the slope of the curve in terms of this triangle.
$dy/dx =$ __________
Separate the variables and make the substitution $u^2 = 1 - y^2$. Then:
$dx =$ __________ du
Split into partial fractions, and integrate each using the substitutions $s = 1 - u$ and $t = 1 + u$. The solution in terms of s and t is then:
$x =$ __________ $+C$
Substitute back to get x as a function of y, and choose the constant of integration so that the asymptote (along which the free end of the string is pulled) is the x-axis and the point $(0,1)$ corresponds to the vertical position. Then:
$x =$ __________

2. (1 point) Rashevsky (Looking At History Through Mathematics, M.I.T. Press, 1968) proposed the following model of the increase of agnosticism on a historical timescale. Assume that most people are receptive to common faiths while a small fraction pN of the total population N is naturally agnostic. Let's say that the birth rate is b and the death rate is d, so that $N' = (b-d)N$. The agnostic population A will grow because the agnostics bring up their children to be agnostic, while a fraction p of other births are naturally agnostic. Thus $A' =$ __________

The agnostics constitute a growing fraction $y(t)$ of the population, so that $A = yN$. Take the derivative of both sides in this equation with respect to time t. (Note that both N and y are functions of t.) From this we see that

-
- `y'=pb(1-y)`
- `y'=pb(y-1)`
- `y'=p(b-d)(1-y)`
- `y'=p(b-d)(y-1)`

Therefore, with the initial condition $y(0) = 1/1000$, $y(t) =$ __________

According to Rashevsky, "If we roughly assume that for religious beliefs ... only about one person in a thousand in early antiquity was a natural agnostic" (i.e., $y(0) = p = 1/1000$) and that "the order of magnitude of b is about 10^{-2} individuals/year, then "in about the last 10,000 years, the time that has elapsed since the emergence of mankind from a primitive state, we find an increase of y from 1/1000 to only about 1/100," and indeed "we actually find that all the major religions ... still share between them practically all of humanity." As for the future, "in 100,000 years the fraction y ... will have increased to only about 2/3." Do we have all the information we need to verify these calculations with our formula for $y(t)$? [/yes/no]

3. (1 point) Consider the system $dr/dt = -j$, $dj/dt = r$, where $r(t)$ represents Romeo's love (positive values) or hate (negative values) for Juliet at time t, and $j(t)$ similarly represents Juliet's feelings toward Romeo. [/Romeo/Juliet] "loves to be loved", while [/Romeo/Juliet] is intrigued by rejection. Romeo's and Juliet's families are enemies. This can be expressed in the initial condition $(r,j) =$ ([/1/0/-1], [/1/0/-1]) at time $t = 0$. What happens in the long run?

- They are mutually in love a quarter of the time.
- They are mutually in love half the time.
- They are sometimes in love with the other but never at the same time.
- They end up being in love.
- They end up not being in love.
- None of the above

”In the Spring a young man’s fancy lightly turns to thoughts of love,” says Tennyson. In mathematical terms this could be reflected in the system having:

- a forcing term
- an unstable equilibrium
- a nonlinear function for t
- none of the above

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4. (1 point) With air resistance, four qualitatively different types of pendulum behaviours can arise:

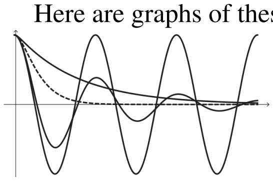
The ”undamped” pendulum keeps on swinging forever. This is the idealised case where there is no air resistance. It has solutions such as $x = \cos t$.

The ”damped” pendulum gradually swings shorter and shorter arcs. This is the realistic case of an ordinary pendulum with air resistance. It has solutions such as $x = e^{-t}\cos \alpha t$.

The ”overdamped” pendulum goes slowly to its lowest point without oscillating. This could be for example a pendulum submerged in thick syrup. It has solutions such as $x = e^{-t}$.

The ”critically damped” pendulum is right on the boundary between damped and overdamped. It has solutions such as $x = e^{-t} + te^{-t}$.

Here are graphs of these four cases:



As you can see, the critically damped case (dashed) gives a quick yet soft return to equilibrium. Therefore it is often desirable for applications such as shock-absorbing suspensions or the hydraulic arm of a door.

$x'' + bx' + 6x = 0$ is critically damped when $b =$ _____

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5. (1 point) If we ignore air resistance the only force acting on a falling object is the constant gravitational acceleration ($g \approx 10$), which gives the differential equation $v' = -10$. Suppose you fire a gun straight up into the air. The initial velocity of the bullet is 1000 meters/second.
Solve the differential equation for v as a function of t with the given initial condition. Find the height of the bullet as a function of time. With what velocity does the bullet strike the ground when it lands? _____ meters per second. (Mind the sign of your answer.)
Of course this is unrealistic. The resistance of the air is considerable. Experience shows that it is roughly proportional to the square of the velocity. It also needs to have the right sign both when the bullet is going up and when it is coming down. Which differential equation accomplishes this?

- $v' = -10 - 0.004v|v|$
- $v' = -10 - 0.004v^2$
- $v' = -10 + 0.004v^2$
- $v' = -10 + 0.004v|v|$

With what velocity does the bullet strike the ground according to this model? _____ meters per second. (Mind the sign of your answer.)
Hint: Instead of solving the differential equation to find this out, use the reasonable assumption that the descending bullet will reach terminal velocity (i.e., a velocity at which it is no longer accelerating) before hitting the ground.

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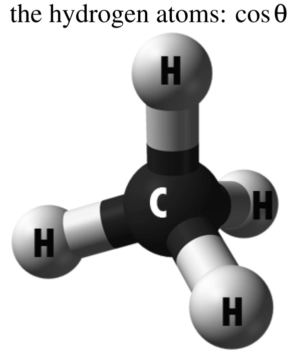
6. (1 point) A molecule of methane, CH_4, forms a regular tetrahedron with the four hydrogen atoms at the vertices and the carbon atom at the centroid. Let the vertices of the first three hydrogen atoms be $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Argue that any point of the form (a,a,a) is equidistant from each of these three points. Use the condition that all hydrogen atoms must be equidistant to determine the coordinates of the fourth hydrogen atom: (___,___,___) Determine the coordinates of the carbon atom by computing the ”average” position of the four hydrogen atoms (i.e., add the hydrogen position vectors and divide by 4). Then use scalar products to compute the bond angle θ, i.e., the angle between the vectors that join the carbon atom to two of the hydrogen atoms: $\cos\theta =$ _____



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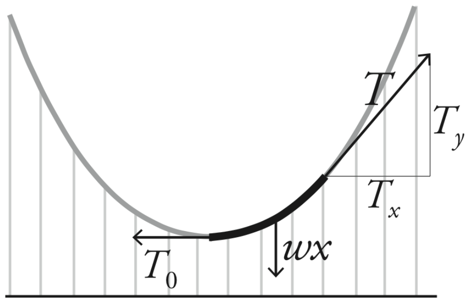
7. (1 point) In this problem we shall find the shape of a suspension bridge cable. We assume that the weight of the roadway and cars is uniformly distributed with density w, so that a segment of the bridge with length x has weight wx. This weight translates into tension in the cable. In the figure we have indicated the tension forces T_0 and T. If the cable was cut at one of these points, then T_0 or T indicates the force we would need to apply at that point to keep the shape of the cable intact.



Thus there are three forces acting on the cable segment: the tensions T_0 and T, and the weight wx. Since the cable is in equilibrium these forces must cancel out. Therefore the horizontal component T_x of T must equal T_0, and the vertical component T_y of T much equal wx. Argue that T_y/T_x is proportional to dy/dx. Use this to express the condition of equilibrium as a differential equation. Then solve it and write the solution in the form $y = f(x)$. The highest power of a term involving x is ___ Which of the following types of expressions also occur in the solution?

- A. arc-trigonometric
- B. trigonometric
- C. exponential
- D. logarithmic
- E. None of the above

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8. (1 point) Newton's law of cooling says that the temperature, H, of a hot object decreases at a rate proportional to the difference between its temperature and that of its surroundings, S:

$$\frac{dH}{dt} = -k(H - S)$$

The body of a murder victim is found at noon in a room with a constant temperature of 20°C. At noon the temperature of the body is 35°C; two hours later the temperature of the body is 32 °C.

Find the temperature of the body as a function of t, the time since it was found. Note that you can check that your answer seems reasonable by observing that the temperature of the body should approach ___°C as $t \to \infty$.

Assume that the victim had the normal body temperature 37°C at the time of the murder. When did the murder occur? __________ hours before noon.