Spherical geometry: unique properties

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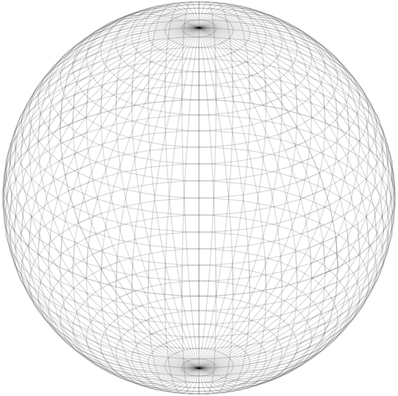
❓ True or false on a sphere?

- ❓ Two points determine a unique line (= great circle).
- ❓ The set of all points equidistant to a line is a line. (Or two lines: one on either side.)
- ❓ Some circles are lines. (A circle can be defined as $\{\mathbf{x} \in S^2 : d(\mathbf{m}, \mathbf{x}) = d(\mathbf{m}, \mathbf{r})\}$ for a given midpoint $\mathbf{m} \in S^2$ and circumference point $\mathbf{r} \in S^2$ ($\mathbf{r} \neq \mathbf{m}$).)
- ❓ Some lines are circles.
- ❓ Some circles are points.
- ❓ The set of all points equidistant to a line is a circle. (Or two circles: one on either side.)
- ❓ Squares exist. (Figure with four equal straight sides and four right angles.)

- ❓ Equilateral triangles exist.
- ❓ Superright triangles exist. (Triangles all of whose angles are right).



Is Wikipedia right? (Hint: no.)

Spherical geometry

From Wikipedia, the free encyclopedia

Relation to Euclid's postulates [edit]

If "line" is taken to mean great circle, spherical geometry obeys two of Euclid's postulates: the second postulate ("to produce [extend] a finite straight line continuously in a straight line") and the fourth postulate ("that all right angles are equal to one another"). However, it violates the other three. Contrary to the first postulate ("that between any two points, there is a unique line segment joining them"), there is not a unique shortest route between any two points (antipodal points such as the north and south poles on a spherical globe are counterexamples); contrary to the third postulate, a sphere does not contain circles of arbitrarily great radius; and contrary to the fifth (parallel) postulate, there is no point through which a line can be drawn that never intersects a given line.[8]

Postulates

1. Let it have been postulated to draw a straight line from any point to any point.

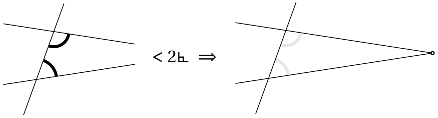
2. And to produce a finite straight line continuously in a straight line.

3. And to draw a circle with any center and radius.

4. And that all right angles are equal to one another.

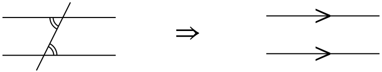
5. And that if a straight line falling across two other straight lines makes internal angles on the same side of itself whose sum is less than two right angles, then the two other straight lines, being produced to infinity, meet on that side of the original straight line that the sum of the internal angles is less than two right angles.

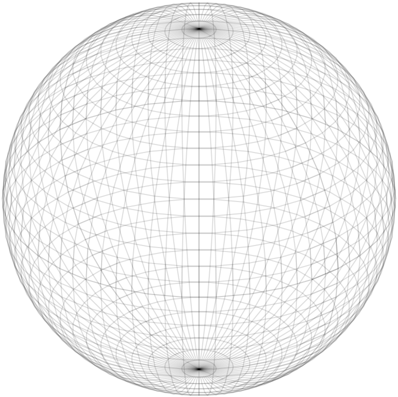
Postulate 5: condition for crossing.



Attempting $\mathbb{E}^2$ proofs in S^2 can be instructive

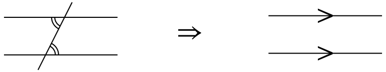
Proposition 27: alternate angles equal $\Rightarrow$ parallel.



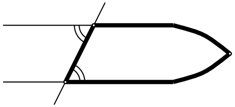


❓ Is this theorem (from Euclid's *Elements*) true on a sphere?

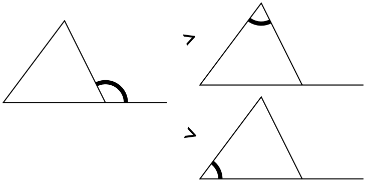
Proposition 27: alternate angles equal ⇒ parallel.



Euclid proves this by contradiction. If they meet, that contradicts Prop. 16.



Proposition 16: △ external angle > each opposite internal angle.



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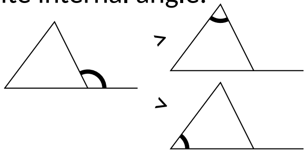
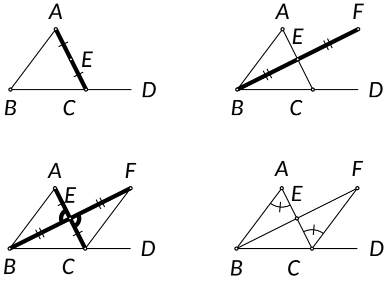
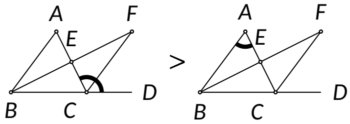
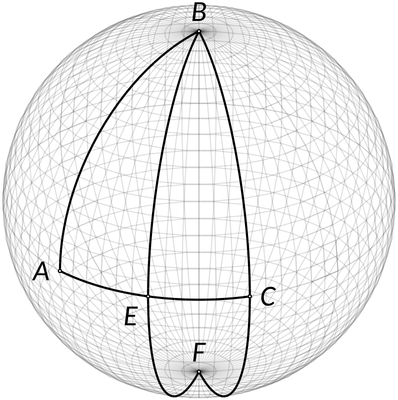


Euclid’s proof:





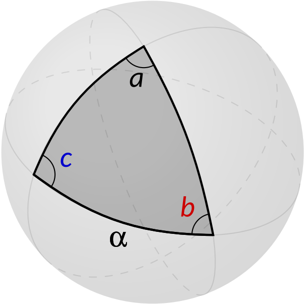
❓ Which step goes wrong on the sphere?

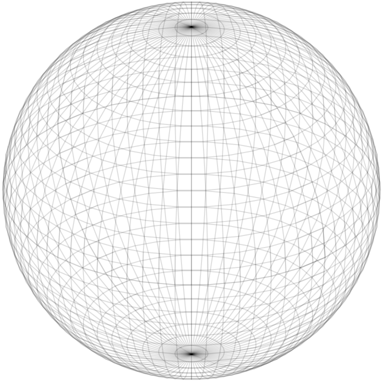


❓ Is AAS a triangle congruence case?

Alternative cosine rule:

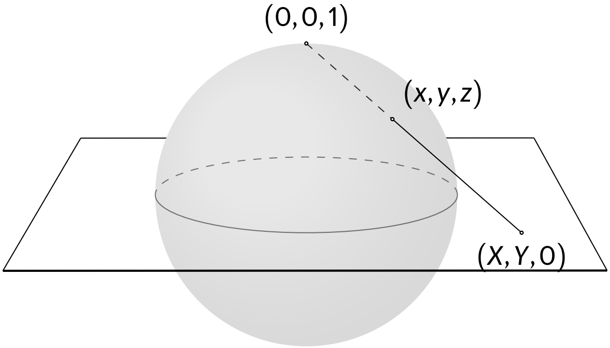
$$\cos a = \cos\alpha \sin b \sin c - \cos b \cos c$$





Hint: What happens if you put *b* at the north pole and *a* and *c* at the equator?

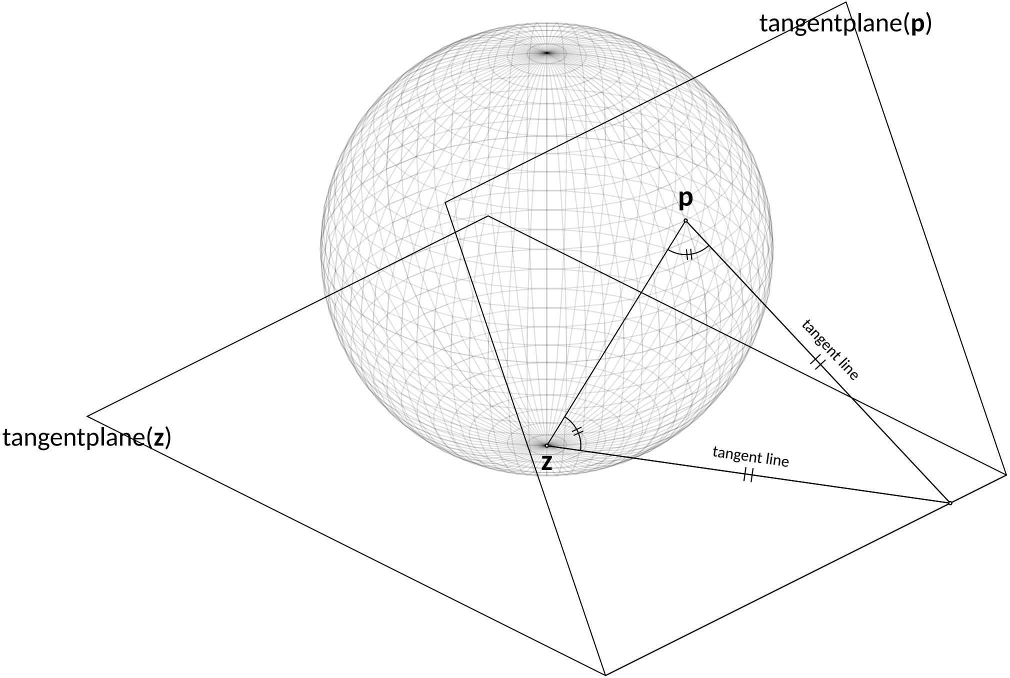
Stereographic projection $S^2 \to \mathbb{R}^2$



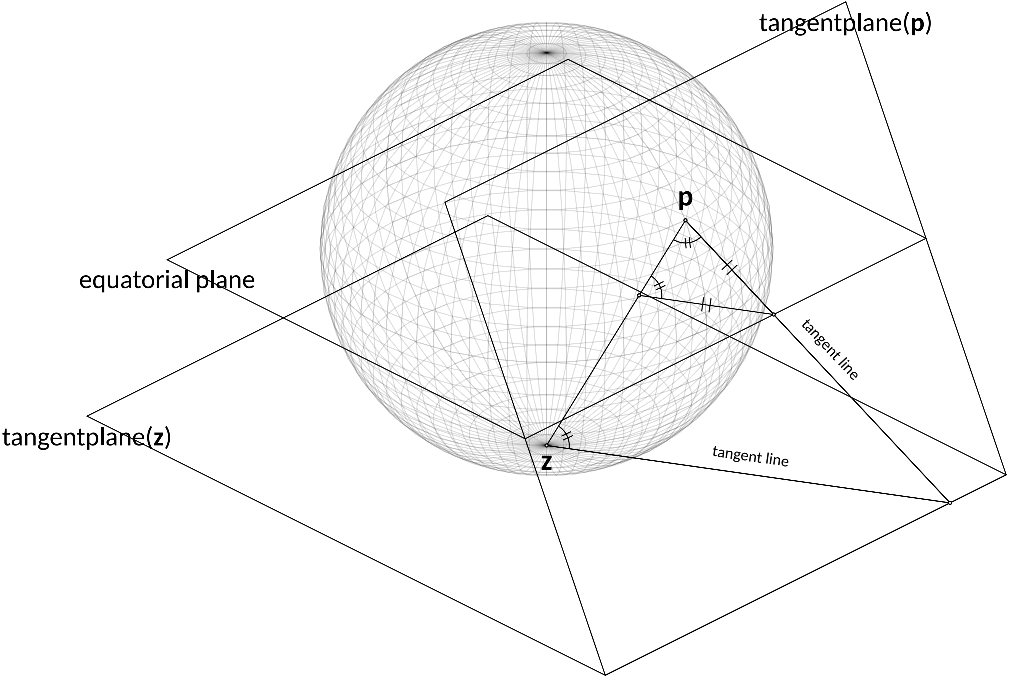
$$(x,y,z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)$$
$$(X,Y,0) \leftarrow \left(\frac{2X}{X^2+Y^2+1}, \frac{2Y}{X^2+Y^2+1}, \frac{X^2+Y^2-1}{X^2+Y^2+1}\right)$$

Astrolabes are based on stereographic projection

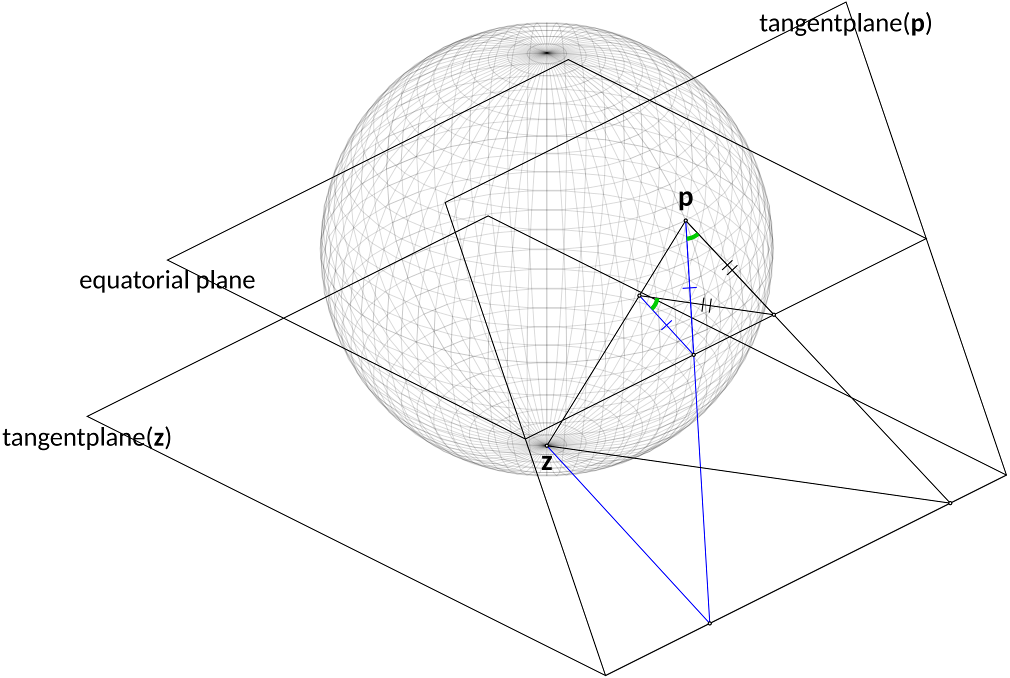
Symmetry of tangents to sphere



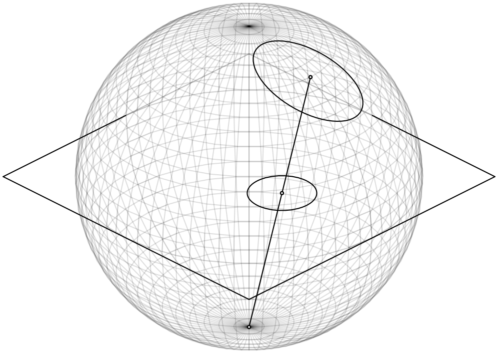
Symmetry transferred to projection plane



Symmetry of two tangents at $\mathbf{p} \implies$ stereogr. proj. conformal



Circles $\mapsto$ circles



- Projection of circle = closed-curve intersection of cone with plane = circle or ellipse.

- Consider a cone tangent to the circle on the sphere. All lines from the cone vertex cut the circle at right angles.
- Those lines map to lines (plane$\cap$plane).
- By angle preservation, the images of those lines cut the image of the circle at right angles.
- Hence there is a point (the image of the cone point) from which all lines cut the image of the circle at right angles.
- Hence the image is a circle.