Spherical geometry: isometries

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Isometries of S^2

An $\mathbb{E}^3$ isometry T can be restricted to a function $\underline{T}$ acting on S^2.
An S^2 isometry T can be extended to a function

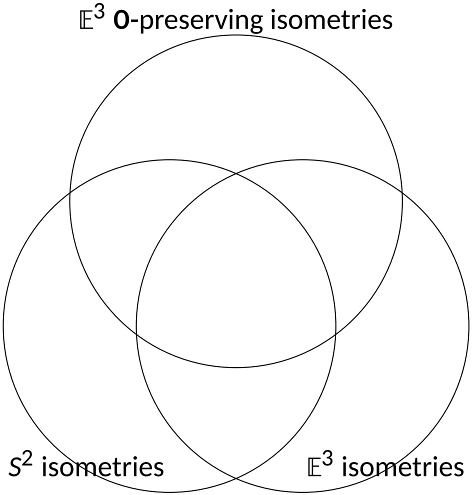
$$\overline{T}(\mathbf{x}) := \begin{cases} |\mathbf{x}|\, T(\mathbf{x}/|\mathbf{x}|) \\ \mathbf{0} \text{ if } \mathbf{x} = \mathbf{0} \end{cases}$$

acting on $\mathbb{E}^3$.

With this correspondence, what can we say about the Venn diagram?

3.4.1. T is an S^2 isometry $\implies$ $\overline{T}$ is an $\mathbb{E}^3$ isometry
3.4.2. T is an $\mathbb{E}^3$ isometry $\implies$

$$\underline{T} \text{ is an } S^2 \text{ isometry} \iff T(\mathbf{0}) = \mathbf{0}$$



Proof of 3.4.1

Given that T is an S^2 isometry, prove that $\overline{T}(\mathbf{x}) := \begin{cases} |\mathbf{x}|\, T(\mathbf{x}/|\mathbf{x}|) \\ \mathbf{0} \text{ if } \mathbf{x} = \mathbf{0} \end{cases}$ is an $\mathbb{E}^3$ isometry.

- $\overline{T}$ is distance-preserving: $d(\overline{T}(\mathbf{x}), \overline{T}(\mathbf{y})) = d(\mathbf{x}, \mathbf{y})$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{E}^3$.
  Proof idea: re-express $d(\mathbf{x}, \mathbf{y})$ in terms of things centered at the origin, since that fits well with T. In fact, by SAS triangle congruence, $d(\mathbf{x}, \mathbf{y})$ is preserved if $\angle(\mathbf{x}, \mathbf{y})$ and $d(0, \mathbf{x})$ and $d(0, \mathbf{y})$ are preserved. This can be verified from the definition of $\overline{T}$:
  - Angles at origin preserved:

$$
\begin{aligned}
\angle(\overline{T}(\mathbf{x}), \overline{T}(\mathbf{y})) &\overset{*}{=} \angle(|\mathbf{x}|\, T(\mathbf{x}/|\mathbf{x}|), |\mathbf{y}|\, T(\mathbf{y}/|\mathbf{y}|)) \\
&= \angle(T(\mathbf{x}/|\mathbf{x}|), T(\mathbf{y}/|\mathbf{y}|)) \\
&= \angle(\mathbf{x}/|\mathbf{x}|, \mathbf{y}/|\mathbf{y}|) \\
&= \angle(\mathbf{x}, \mathbf{y})
\end{aligned}
$$

    * Unless $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$: verify these cases separately.

  - Lengths from origin preserved:

$$
\begin{aligned}
d(\mathbf{0}, \overline{T}(\mathbf{x})) &\overset{*}{=} d(\mathbf{0}, |\mathbf{x}|\, T(\mathbf{x}/|\mathbf{x}|)) \\
&= \big||\mathbf{x}|\, T(\mathbf{x}/|\mathbf{x}|)\big| \\
&= |\mathbf{x}| \, \big|T(\mathbf{x}/|\mathbf{x}|)\big| \\
&= |\mathbf{x}| \cdot 1 \\
&= |\mathbf{x}| \\
&= d(\mathbf{0}, \mathbf{x})
\end{aligned}
$$

- Left to prove: $\overline{T}$ is surjective.

Proof of 3.4.1 continued

- Given: T is an S^2 isometry.
- Already proved: Its extension $\overline{T}(\mathbf{x}) := \begin{cases} |\mathbf{x}|\, T(\mathbf{x}/|\mathbf{x}|) \\ \mathbf{0} \text{ if } \mathbf{x} = \mathbf{0} \end{cases}$ is a distance-preserving function $\mathbb{E}^3 \to \mathbb{E}^3$.
- Left to prove: $\overline{T}$ is surjective.
- In fact, any distance-preserving $T : \mathbb{E}^n \to \mathbb{E}^n$ is automatically (surjective and hence) an isometry, since d-preserving $\implies T = M\mathbf{x} + \mathbf{v} \implies \exists T^{-1}$. $\square$

❓ What happens if we try to apply this logic to

$$T : \mathbb{E}^2 \to \mathbb{E}^3 \qquad (x, y) \mapsto (x, y, 0)?$$

❓ d-preserving? ❓ of form $T = M\mathbf{x} + \mathbf{v}$? ❓ $\exists T^{-1}$?

❓ How do we know that the case we need is not like that?

Proof of 3.4.2 ($\Rightarrow$)

Given that T is an $\mathbb{E}^3$ isometry and its restriction $\underline{T}$ is an S^2 isometry, show that $T(\mathbf{0}) = \mathbf{0}$.

- We will prove this by contradiction. Suppose T sends the origin somewhere else, $T(\mathbf{0}) = \mathbf{m} \neq \mathbf{0}$.
- Then $\mathbf{0}, \mathbf{m}$ define a line $L := \text{line}_{\mathbb{E}^3}(\mathbf{0}, \mathbf{m})$.
- Lines through $\mathbf{0}$ cut S^2 in two points: $L \cap S^2 = \{\mathbf{a}, \mathbf{b}\}$.
- Since T is an isometry, it has an inverse T^{-1} that is also an isometry.
- Since isometries of $\mathbb{E}^3$ send lines to lines, the inverse image $L^{-1} := T^{-1}(L)$ of L must be a line as well.
- Furthermore, this line L^{-1} must go through $\mathbf{0}$ as well, since it contains $T^{-1}(\mathbf{l})$ for any $\mathbf{l} \in L$, and hence in particular $T^{-1}(\mathbf{m}) = T^{-1}(T(\mathbf{0})) = \mathbf{0}$.
- Hence L^{-1} too cuts S^2 in two points: $L^{-1} \cap S^2 = \{\mathbf{a}', \mathbf{b}'\}$.
- By definition, T sends L^{-1} to L, so it must send its sphere points $\mathbf{a}', \mathbf{b}'$ somewhere on L. And since $\underline{T}$ is an S^2 isometry, T must map S^2 to itself. Therefore T must send $\mathbf{a}', \mathbf{b}' \mapsto \mathbf{a}, \mathbf{b}$.
- Thus T is an isometry that sends the line segment $\mathbf{a}'\mathbf{b}'$ to the line segment $\mathbf{ab}$. Hence it must also send the midpoint of $\mathbf{a}'\mathbf{b}'$ to the midpoint of $\mathbf{ab}$, that is to say, $\mathbf{0}$ to $\mathbf{0}$.
- Which contradicts the assumption $T(\mathbf{0}) = \mathbf{m} \neq \mathbf{0}$. $\square$

Proof of 3.4.2 ($\Leftarrow$)

Given that T is an $\mathbb{E}^3$ isometry and $T(\mathbf{0}) = \mathbf{0}$, show that its restriction $\underline{T}$ is an S^2 isometry.

Need to show: $\underline{T}$ is a surjective, distance-preserving function $S^2 \to S^2$.

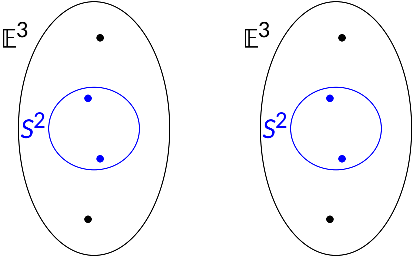
We will show:

- $\underline{T}$ is a function $S^2 \to S^2$.
- $T^{-1}(S^2) \subseteq S^2$.
- Hence $\underline{T}$ is a surjective function $S^2 \to S^2$.

Illustrate the meaning of these steps in terms of the diagram below.

Why can we not argue as follows:

$$\begin{aligned} T \text{ isometry} &\implies T \text{ surjective} \\ &\implies \underline{T} \text{ surjective} \end{aligned}$$



Proof of 3.4.2 ($\Leftarrow$) continued

Given that T is an $\mathbb{E}^3$ isometry and $T(\mathbf{0}) = \mathbf{0}$, show that its restriction $\underline{T}$ is an S^2 isometry.

- $\underline{T}$ is a function $S^2 \to S^2$:

$$\begin{aligned}
\mathbf{s} \in S^2 &\implies d_{\mathbb{E}^3}(\mathbf{0}, \mathbf{s}) = 1 \\
&\implies d_{\mathbb{E}^3}(T(\mathbf{0}), T(\mathbf{s})) = 1 \\
&\implies d_{\mathbb{E}^3}(\mathbf{0}, T(\mathbf{s})) = 1 \\
&\implies T(\mathbf{s}) \in S^2
\end{aligned}$$

- T isometry of $\mathbb{E}^3 \implies$ T^{-1} isometry of $\mathbb{E}^3$. This inverse preserves S^2 ($T^{-1}(S^2) \subseteq S^2$):

$$\begin{aligned}
\mathbf{s} \in S^2 &\implies d_{\mathbb{E}^3}(\mathbf{0}, \mathbf{s}) = 1 \\
&\implies d_{\mathbb{E}^3}(T^{-1}(\mathbf{0}), T^{-1}(\mathbf{s})) = 1 \\
&\implies d_{\mathbb{E}^3}(\mathbf{0}, T^{-1}(\mathbf{s})) = 1 \\
&\implies T^{-1}(\mathbf{s}) \in S^2
\end{aligned}$$

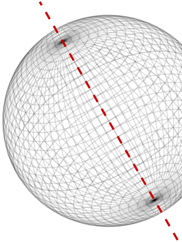
- Hence $\underline{T}$ is a surjective function: For any $\mathbf{s} \in S^2$ there is a $T^{-1}(\mathbf{s}) \in S^2$ such that $T(T^{-1}(\mathbf{s})) = \mathbf{s}$.
- $\underline{T}$ is distance-preserving.

$$\begin{aligned}
d_{S^2}(T(\mathbf{x}), T(\mathbf{y})) &= \angle T(\mathbf{x})\mathbf{0}T(\mathbf{y}) \\
&= \angle \mathbf{x}\mathbf{0}\mathbf{y} \\
&= d_{S^2}(\mathbf{x}, \mathbf{y}) \quad \square
\end{aligned}$$

Classification of isometries of S^2

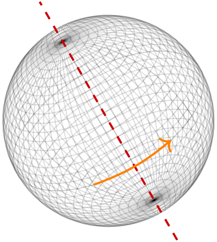
$$
\begin{aligned}
\text{isometries of } S^2 \leftrightarrow \mathbf{0}\text{-preserving isometries of } \mathbb{E}^3 \\
= \text{orthogonal } 3 \times 3 \text{ matrices } M \\
= \begin{pmatrix} \pm 1 & 0 \\ 0 & R_\varphi \end{pmatrix}
\end{aligned}
$$

identity



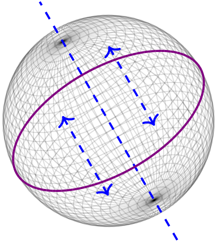
$\begin{pmatrix} 1 & 0 \\ 0 & R_0 \end{pmatrix}$

rotation



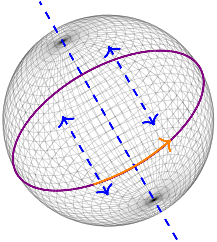
$\begin{pmatrix} 1 & 0 \\ 0 & R_\varphi \end{pmatrix}$

reflection



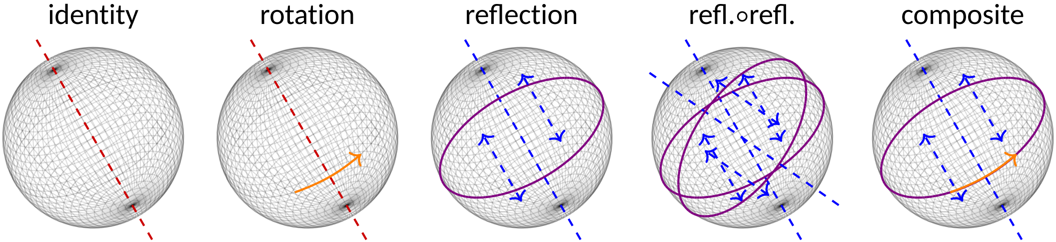
$\begin{pmatrix} -1 & 0 \\ 0 & R_0 \end{pmatrix}$

composite



$\begin{pmatrix} -1 & 0 \\ 0 & R_\varphi \end{pmatrix}$

❓ Not every isometry of S^2 can be expressed as a composition of one or two reflections



This is best proved using:

- ❓ direct/indirect
- ❓ fixed points
- ❓ three-reflections theorem of $\mathbb{E}^2$
- ❓ three-reflections theorem of $\mathbb{E}^3$

❓ Isometry definition of lines

A “curve” can be defined as the image of $\mathbb{R}$ under a continuous function: $\{\mathbf{r}(t) : t \in \mathbb{R}\}$ (a parametrisation).

Playfair’s definition of straight lines: A curve L is called a line if and only if

$$\left|L \cap T(L)\right| \geq 2 \implies L = T(L)$$

for any isometry T.

- ❓ This definition correctly identifies straight lines and only straight lines in

  - ❓ $\mathbb{E}^2$?
  - ❓ $\mathbb{E}^3$?
  - ❓ S^2?