History and Philosophy of Geometry

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Contents

§ 0. Intro

This is a set of discussion questions to accompany my History and Philosophy of Geometry podcast episodes.

§ 1. Why the Greeks?

1.1. Why did the conception of mathematics as a field of knowledge characterised first and foremost by rigorous deductive reasoning arise in ancient Greece?

1.2. What does early Greek philosophy have in common with mathematics? What aspects of mathematics fits well with this view, and what aspects less well?

1.3. Which of the factors that led to mathematics in ancient Greece were or were not present in other cultures, such as for example Renaissance Europe (where mathematics was embraced) or the Roman Empire (where it was not)?

§ 2. Societal role of geometry in early civilisations

2.1. What aspects of mathematics were especially crucial for the societal function mathematics served in early civilisations, and why?

2.2. Mathematics textbooks often contain "pseudo-applications": problems that are supposedly about the real-world scenarios, but that are hopelessly unrealistic and artificial. Argue that history vindicates such problems and suggests that they serve some rational purposes.

§ 3. First proofs: Thales and the beginnings of geometry

philosophy	both	mathematics
disagreement	hyper-critical dialectic	consensus
rival schools	reason > experience	unity
"subtractive"	rationalism > empiricism	"additive"
(critiques, refutations)	deduction > induction	(cumulative knowledge)
	definitions of terms crucial in theory-building	

Table 1: Similarities and differences between early Greek philosophy and mathematics.

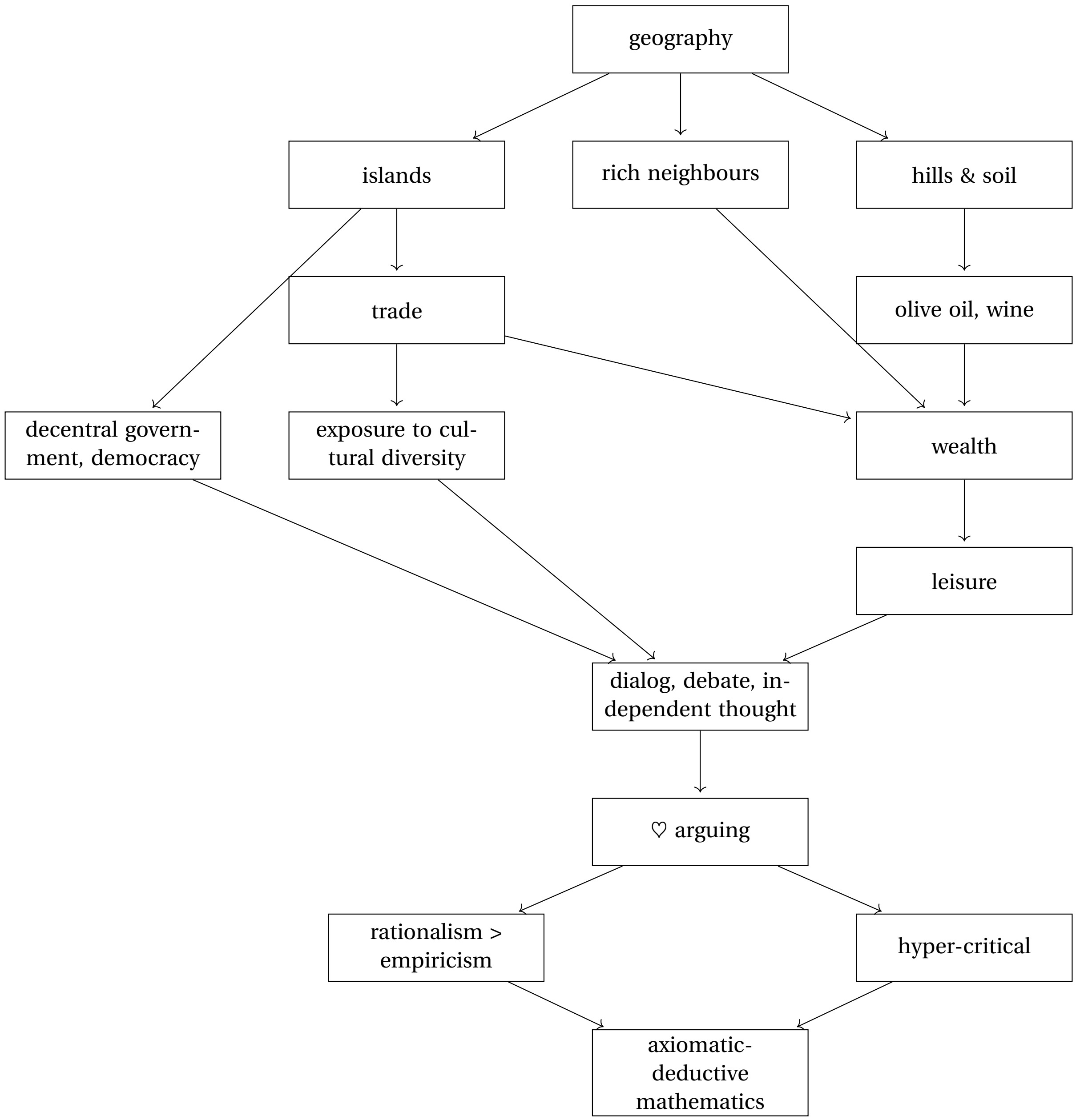
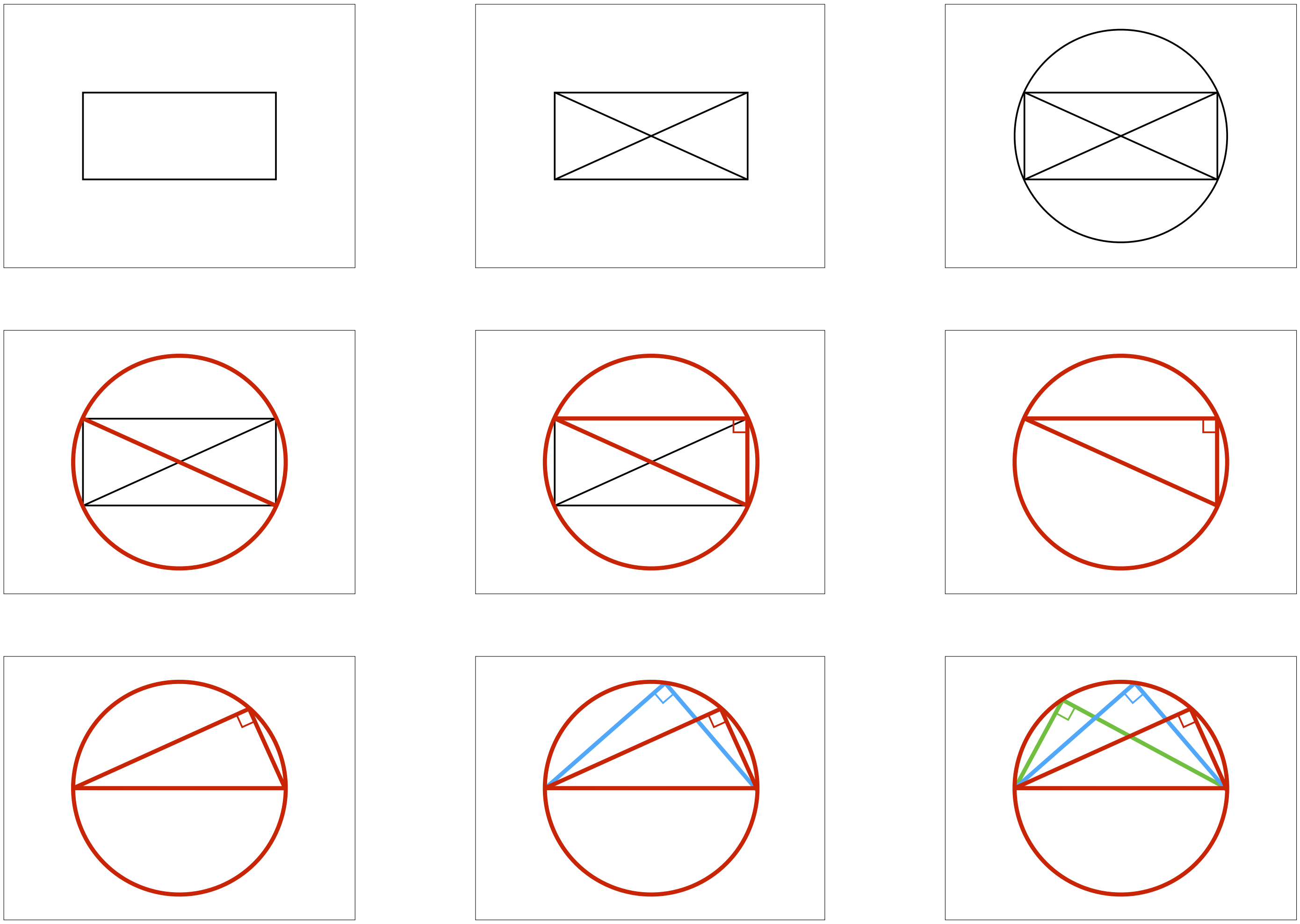


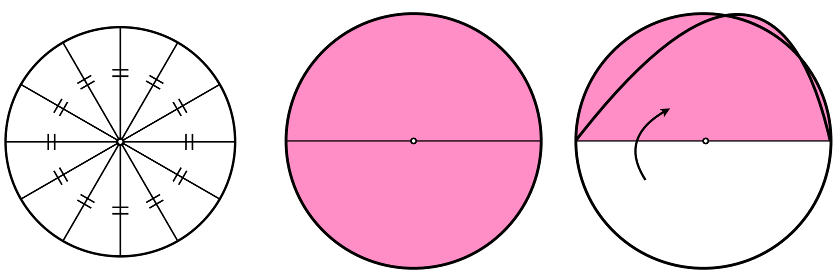
Figure 1: Factors influencing the emergence of rigorous mathematics in ancient Greece.

3.1. In what sense can the theorems attributed to Thales have been the beginning of deductive mathematics?

"Thales' Theorem" about triangles inscribed in circles:



Proof by contradiction that diameter cuts circle in half:



Jakob Steiner's proof of the isoperimetric property of the circle:

§ 4. Singing Euclid: the oral character of Greek geometry

4.1. Is the written record a good representation of Greek geometrical thought?

4.2. Were Greek ways of recording mathematics constrained by technology? By tradition?

4.3. The way the Elements is written is in some ways comparable to songs, poems, tape recordings, or old movies. Discuss.

§ 5. Read Euclid backwards: history and purpose of Pythagorean Theorem

5.1. A mathematical proof can serve many purposes: to explain (create an "aha" moment for the reader), the carefully verify (a result that may be doubted), to exhibit logical relations within a formal system, to reduce complex statements to more basic fundamental principles, etc. Discuss which of these goals of proofs seem to be primary in Euclid's proof of the Pythagorean Theorem (and perhaps the Elements generally).

§ 6. Consequentia mirabilis: the dream of reduction to logic

6.1. What is consequentia mirabilis? Illustrate with examples.

6.2. What makes mathematical knowledge special? What sets it apart from other fields?

§ 7. What makes a good axiom?

7.1. In the Platonic worldview, what is the relation between mathematics and physical reality? What are strengths and weaknesses of this view?

7.2. What aspects of Euclid's *Elements* or other technical mathematics fit especially well with empirical, rationalistic, or logical interpretations of mathematics respectively?

7.3. In Rafael's famous fresco "The School of Athens," Plato is pointing toward the sky and Aristotle is pointing straight ahead. Why?

7.4. Are there examples of theories in which the principles are not primitives, or the primitives are not principles, in Aristotle's sense? (Cf. §14.)

<table>
<tr><th></th><th colspan="3">Axioms of mathematics are based on:</th></tr>
<tr><th></th><th>the physical world, generalised experience</th><th>the mind, intuition</th><th>pure logic</th></tr>
<tr><td>Examples:</td><td colspan="2">Euclid's Postulates</td><td></td></tr>
<tr><td></td><td colspan="3">Euclid's Definitions, Common Notions</td></tr>
<tr><td></td><td>Newton's law of gravity</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td>consequentia mirabilis, e.g.:</td></tr>
<tr><td></td><td></td><td colspan="2">"I think therefore I am."
"we ought to philosophise"</td></tr>
<tr><td>View of:</td><td>Aristotle, Newton</td><td>Plato, Proclus</td><td>modern mathematics</td></tr>
<tr><td>Associated philosophy:</td><td>empiricism</td><td colspan="2">rationalism</td></tr>
<tr><td>Reading Euclid:</td><td>backwards</td><td colspan="2">forwards</td></tr>
<tr><td>Problems:</td><td>uniqueness of math.
(≠ botany, anthropology)</td><td></td><td></td></tr>
<tr><td></td><td></td><td colspan="2">"unreasonable effectiveness" of applied math.</td></tr>
<tr><td></td><td></td><td></td><td>consistency
(e.g. Russell's Paradox)</td></tr>
</table>

§ 8. That which has no part: Euclid's definitions

8.1. What do Euclid's definitions of point and line say about the nature of geometry, in particular with respect to Platonist versus physicalist interpretations? Use ancient sources to argue for both sides.

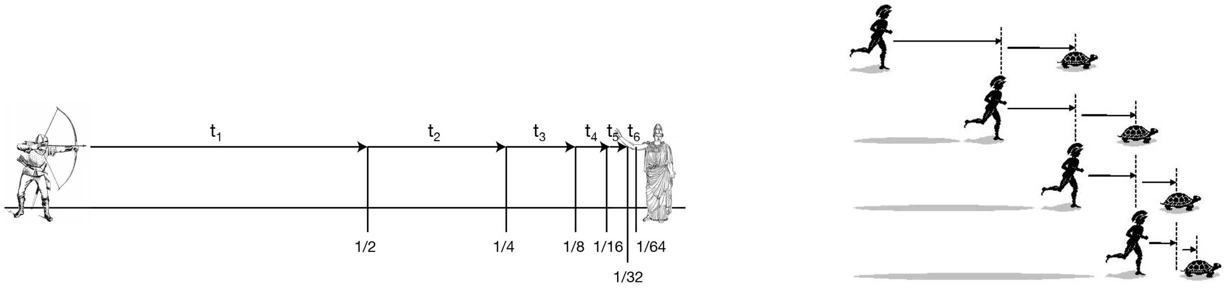


Figure 2: Zeno's paradoxes.

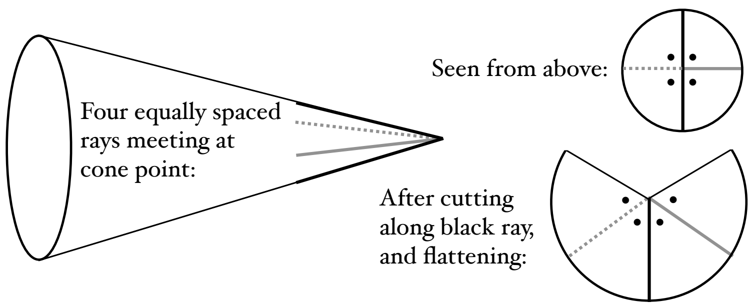


Figure 3: Right angles at a cone point. On this 240 degree cone, the right angles at the cone point are 60 degrees.

§ 9. Created equal: Euclid's Postulates 1-4

9.1. Why did Euclid feel the need to postulate that "all right angles are equal"?

9.2. In what ways can Zeno be taken as a case in point illustrating the thesis (advanced by Lloyd and Szabó) of the dialectical origins of Greek geometry?

9.3. Are Zeno's dichotomy and Achilles arguments just different literary elaborations of the same idea, or is there something substantially different between the two cases?

§ 10. Why construct?

10.1. Why did constructivist philosophies of the foundations of mathematics and science gain popularity in the 20th century?

10.2. In what ways can 20th-century constructivist philosophy be considered similar to what Euclid was doing?

10.3. Suppose I add to Euclid's *Elements* the definition: "A superright triangle is a triangle each of whose angles is a right angle." So its angle sum is three right angles. But we also know from Proposition 32 that the angle sum of a triangle is two right angles. Therefore 3 = 2. How can we avoid this paradox? How can guarantee that mathematics is free of such contradictions?

10.4. How do we know that mathematics does not contain statements that can be proven both true and false?

§ 11. Let it have been drawn: the role of diagrams in geometry

11.1. (Returning question:) Is the written record a good representation of Greek geometrical thought?

11.2. (Returning question:) Were Greek ways of recording mathematics constrained by technology? By tradition?

↪	"I think therefore I am"	(by "consequentia mirabilis")
$\Longrightarrow$	my thoughts are actual	
	my thoughts include the idea of perfection	
$\Longrightarrow$	something perfect (= God) must exist	(something perfect cannot be caused by something less perfect)
$\Longrightarrow$	God is not a deceiver	(because a deceiver is not perfect)
$\Longrightarrow$	intuitions (innate ideas) are reliable	(since implanted by the creator = God = not a deceiver)
$\Longrightarrow$	Euclid's axioms are reliable	(since they are intuitive)

Table 2: Descartes, *Principles of Philosophy*, 1644, I.7–30.

11.3. 💬 Was Euclid a Platonist who reasoned about eternal, abstract objects, or did he think of geometry as something physically produced by ruler and compass? What can we conclude in this regard from the passive formulations of construction steps in his proofs?

§ 12. Maker's knowledge: early modern philosophical interpretations of geometry

🎤 ↗

12.1. 💬 How did 17th-century philosophers interpret the purpose of constructions in Euclidean geometry?

12.2. 💬 What implications did they draw from this for philosophy and other branches of knowledge?

12.3. 💬 What did Descartes see as the key aspects of geometrical reasoning that were to be generalised to other domains of thought?

12.4. 💬 What is the relation between God and geometry, according to Descartes?

12.5. 💬 What aspects of mathematics can be construed as showing that its proofs merely demonstrate propositions logically without explaining why they are true?

12.6. 💬 How can mathematics be defended against the charge that it is inferior to other sciences because its proofs are not explanatory and causal?

§ 13. Cultural reception of geometry in early modern Europe

🎤 ↗

13.1. 💬 What is the mathematical justification for the Renaissance recipe for drawing a tiled floor (given by Alberti, *De pictura*, 1435)?

13.2. 💬 In what way has perspective painting played a role in broader philosophical and scientific developments?

13.3. 💬 Do a Google Images search for Galileo's moon drawings. Discuss.

13.4. 💬 In the early modern period, who had respect for mathematics, and why? Who were critical of mathematics, and why?

13.5. 💬 Which themes in these debates are timeless, and which were only relevant in their particular context?

§ 14. Rationalism versus empiricism

🎤 ↗

14.1. 💬 How do rationalists and empiricists differ in how they interpret the Euclidean method?

14.2. 💬 Newton and Leibniz disagreed on what it means to treat gravity scientifically. In what way does their disagreement parallel their views on the nature of geometry?

14.3. 💬 Argue that rationalism and synthesis forms a natural pair, as does empiricism and analysis.

14.4. 💬 Do a Google Images search for "birth of Athena." Argue that it is a metaphor for one of the isms.

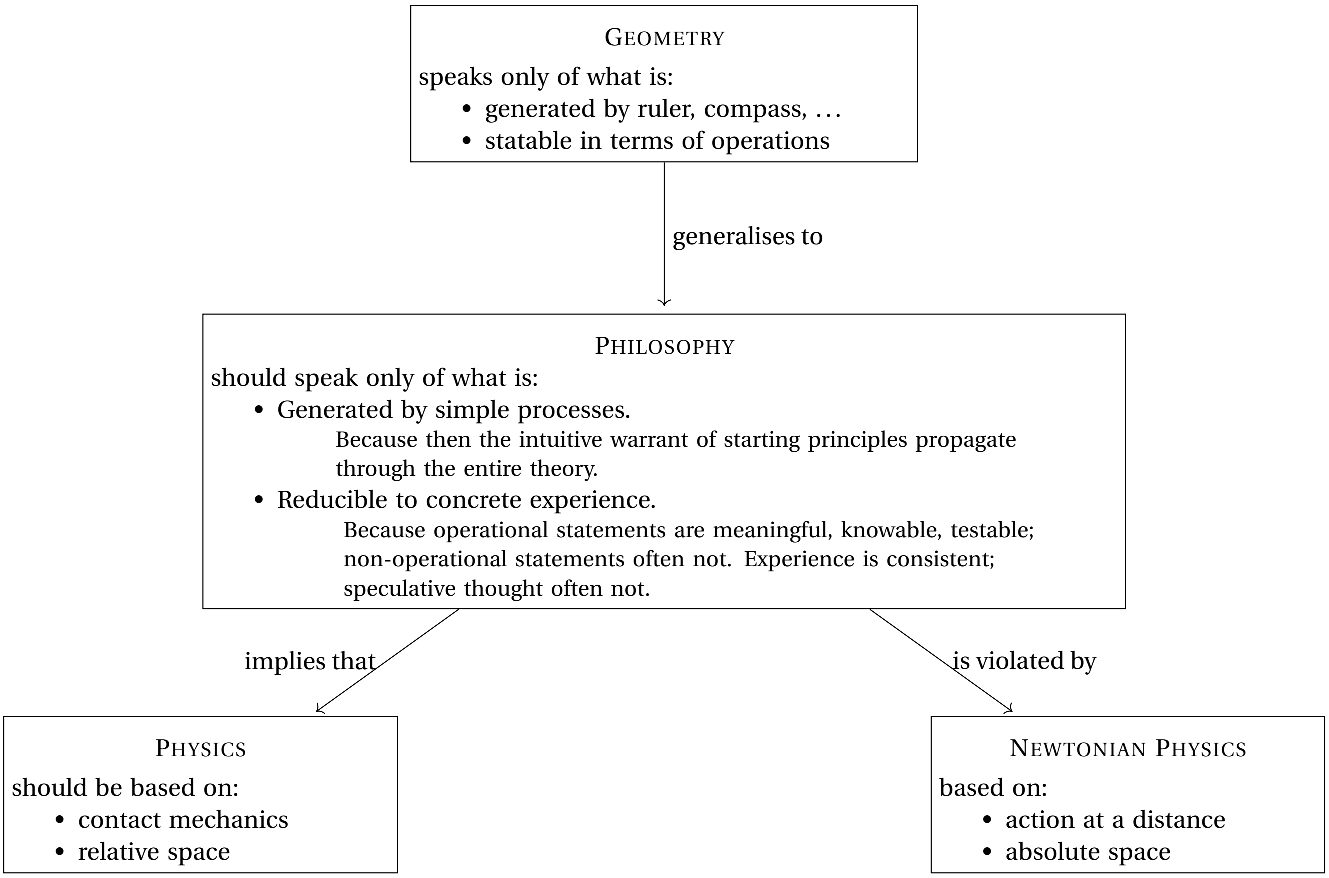


Figure 4: 17th-century operationalist view of scientific method.

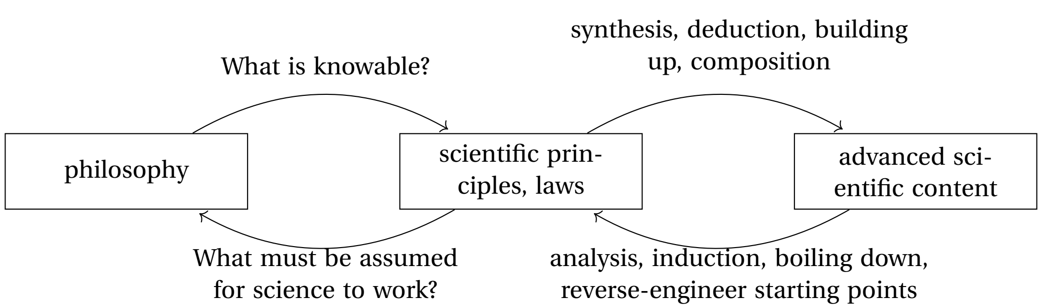


Figure 5: Opposite views on the relation between science and philosophy. Top: The view of Descartes, Leibniz, et al., according to which science must flow from philosophically justified starting principles. Bottom: The view of Newton, according to which philosophical and scientific principles are subordinated to science and retrofitted to agree with science after the fact.

	Descartes, Leibniz Continental rationalism	Newton British empiricism
The search for knowledge starts with …	intuitively clear primitive notions	the rich diversity of phenomena
… and consists in …	deducing the diversity of phenomena from them.	reducing them to a few simple principles.
Intrinsic justification of the axiomatic principles is …	immediate by their intuitive nature	external to the matter at hand
… and is therefore …	the crucial epistemological cornerstone of the entire enterprise.	of secondary importance at best.
In the case of physics, the axiomatic principles are …	the laws of contact mechanics	Newton's three force laws and the law of gravity
… which are established by means of …	their intuitively immediate nature.	induction from the phenomena.
In the case of geometry, the study of curves starts with …	the primitive intuition of local motion	the diversity of curves conceived in any exact manner whatever
… and consists in …	constructively building up a theory of all knowable curves on this basis.	investigating their properties in a systematic fashion.
Geometrical axioms are thus …	the intuitively immediate principles that define and generate the entire subject.	the outcomes of the reductive study of curves, which it was found convenient and illuminating to take as assumptions when the time came to write a systematic account.
The certainty of geometrical reasoning …	stems directly from the axioms' intuitive warrant and the constructive manner in which the rest is built up from them.	stems not from the axioms as such, but from the general method and exactitude of geometrical reasoning.

Table 3: Overview of the two competing interpretations of the Euclidean method in the 17th century.

§ 15. Rationalism 2.0: Kant's philosophy of geometry

15.1. What is the relation between geometry and empirical data regarding spatial relations, according to Kant?

15.2. Kant reconciled elements of rationalism and empiricism that seemed irreconcilable. How?

15.3. Is geometrical knowledge in some sense specifically human or tied to human nature? Compare Kant with Plato and Descartes in this regard.

15.4. What led Descartes and Leibniz to insist on a relativistic notion of space? And Newton on an absolutist one? How does Kant reconcile the two?

15.5. Why was philosophy "dead" before Kant, and how did Kant reverse this?

§ 16. Repugnant to the nature of a straight line: Non-Euclidean geometry

16.1. What is the standard modern view of the relation between geometry and empirical data (exemplified by mathematicians such as Hilbert and physicists such as Einstein)? What are its strengths and weaknesses compared to older conceptions?

16.2. Has the history of the epistemology of geometry been one of constant retreats? That is, has the perceived status and claim to truth of geometrical knowledge been shrinking over time?

16.3. Should all of Euclid's theorems be seen as essentially conditional statements—"*if* you accept the postulates, *then* logic compels you to also accept this theorem"—as opposed to truths in and of themselves?

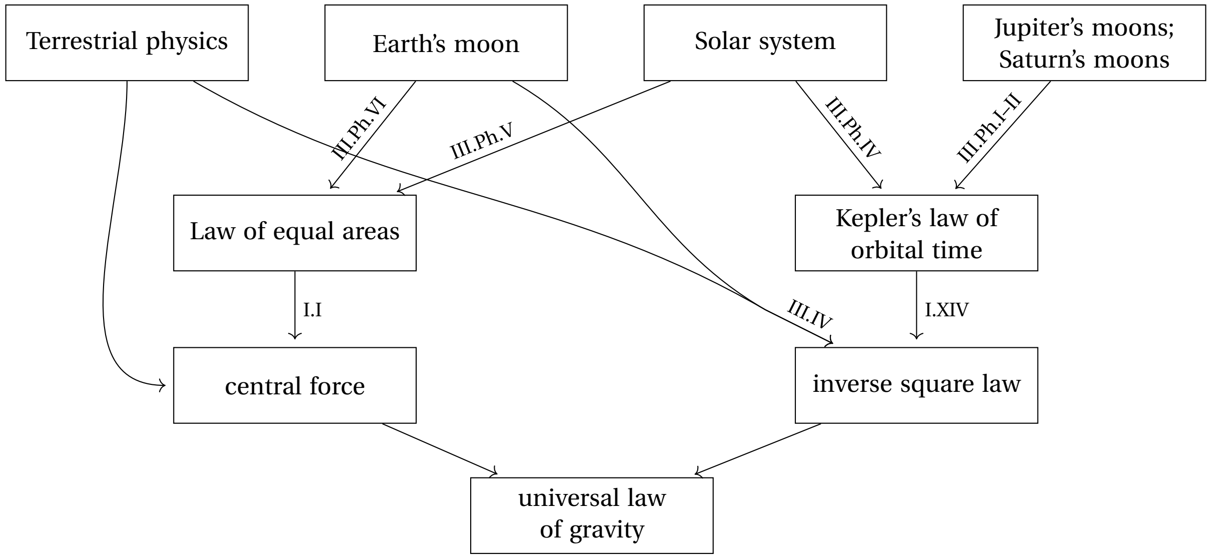


Figure 6: Flowchart illustrating Newton's method of reducing "phenomena" to universal scientific principles in the *Principia*.

rationalism — … since both made by God.

The architecture of our mind contains seeds of truths about data from the outside world …

Kant — … since data biased by the mind.

… and depend on subjective presuppositions necessary for scientific thought to be possible.

Laws/patterns in observed data are building blocks of scientific thought …

empiricism — … and non-subjective, mind-independent.

Table 4: Kant as middle position between rationalism and empiricism: he is able to maintain bits of each by "contaminating" each with the other.

From the point of view of	rationalism	empiricism
absolute space is unknowable since it	cannot be proven by pure thought	is not susceptible to measurement
Kantian solution: yes, but it is knowable that	absolute space is a necessary precondition of our thought	sensory data will conform to this preconception of our minds
so absolute space may not be	logically necessary truth	objective fact about the world
but it is inextricably baked into	human thought	sensory data
it is delusional to hope for anything better since there is no such thing as	"pure thought" ("Critique of Pure Reason")	"objective data"
in fact, such a thing would be undesirable and amount to	a bird trying to fly in a vacuum	kaleidoscopic chaos of impressions

Table 5: Kantian outlook illustrated by the example of absolute space.

Euclidean geometry is:	Plato	Kant	modern
true objectively/absolutely/unconditionally	■		
the constitution of physical space	■		
the constitution of perceived space	■	■	
innately pre-programmed into humans	■	■	
the only coherently thinkable geometry fitting Euclid's Definitions and Postulates 1–4	■	■	
axiomatic-deductive: the theorems follow logically from the axioms	■	■	■

Table 6: Evolution of the epistemology of geometry across time.

§ 17. The universal grammar of space: what geometry is innate?

17.1. Geometry may be shaped by sensory perception in a way similar to how one's native language is formed by one's linguistic environment. What speaks in favour of this view?

17.2. How, and under what conditions, can a mind without specific geometrical preconceptions be led to impose a geometrical structure on sensory perceptions?

17.3. What does the experience of blind people becoming sighted tell us about the relation between space and perception?

language

geometry

- When we know only one language/geometry, it seems an intuitive necessity of thought.
- But the existence of other languages/geometries shows that we "over-universalised" our subjective intuitions.
- Not all intuitive content is innate or necessary; rather, it is shaped by environmental input.
- Nevertheless, this is only made possible by pre-fixed innate concepts that:
  - Underlie and structure all possible languages/geometries. ("universal grammar"; group-transformational concepts)
  - Enable the selection of the relevant (linguistic/spatial) data from the environment.

Table 7: Similarities between language and geometry.

§ 18. Operational Einstein: constructivist principles of special relativity

18.1. What is "relative" in relativity theory?

18.2. Relativity theory vindicates 17th-century opposition to the notion of absolute space. Discuss.

18.3. Relativity theory is an example of a scientific advance driven by philosophical, rather than internal scientific, considerations. Discuss.

18.4. "I had studied [the work of David Hume] avidly and with admiration shortly before discovering the theory of relativity," Einstein said. Discuss and contextualise what philosophical views Einstein and Hume had in common.

18.5. The thought experiment of a person chained to a wall can be used to illustrate both Einsteinian (space and time in special relativity) and pre-Einsteinian (Poincaré's philosophy of space) ideas. Explain.

	innate ("universal grammar")	intuitive ("native" lang./geom.)
necessary for linguistic/geometrical thought	■	■
essence of language/geometry-ness	■	
synthetic a priori	■	
particular contingent instantiation		■
. . . but doesn't limit the scope of conceivability		■
conflated in single notion pre non-Euclidean geometry	■	■
still credibly universal post non-Euclidean geometry	■	

Table 8: Intuition versus innateness.

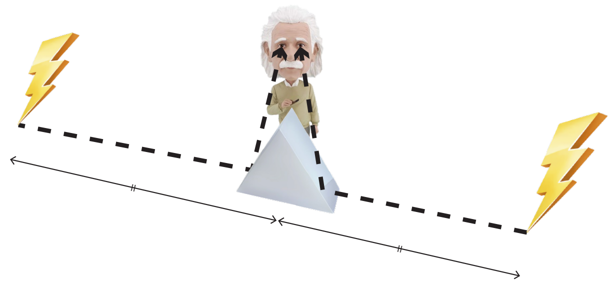


Figure 7: Einstein's definition of simultaneity. Two events occurred simultaneously if the light from them reach the eyes of an observer positioned halfway between them at the same time.

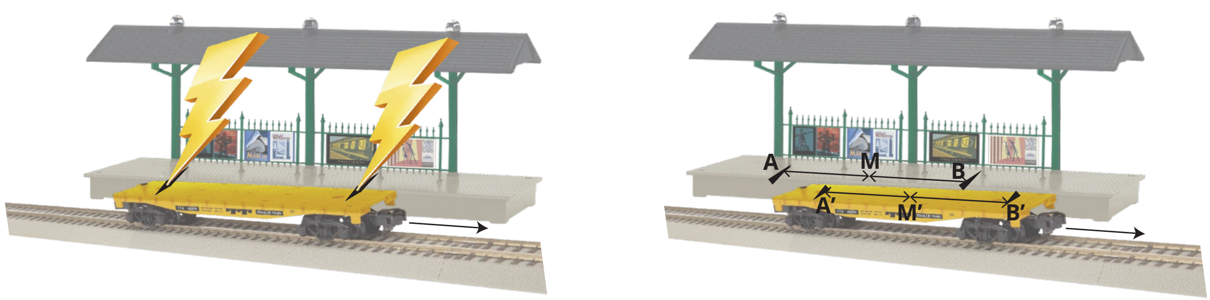


Figure 8: Relativity of simultaneity. Two lightning bolts strike a train car travelling at a certain velocity. Burn marks A, A', B, B' indicate the positions where the lighting struck. Using Einstein's definition of simultaneity, the observer on the stationary platform will say that the lighting strikes were simultaneous if the light rays from them coincide at M. However, if the light rays coincide at M, then the light from B has already passed M' while the light from A has not passed M'. Thus the observer on the train does not agree that the lighting strikes were simultaneous, and instead feels that the event at the front of the train (B, B') occurred earlier. Conversely, if the lighting strikes are simultaneous according to the observer on the train (light rays coincide at M'), then the observer on the platform (at M) will feel that the event at the back of the train (A, A') occurred first (since the light from this event reaches M before it reaches M').

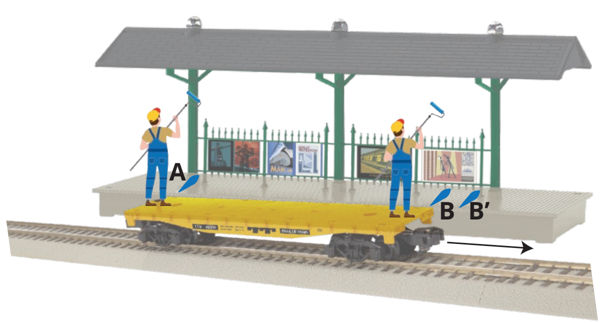


Figure 9: Relativity of length. The length of the train is transferred onto the platform by two paint marks drawn simultaneously. Because of the relativity of simultaneity, this gives different results for different observers. Marks A, B were made simultaneously according to an observer on the platform. Marks A, B' were made simultaneously according to an observer on the train.