HISTORY OF MATHEMATICS READER

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R1.6. Is Sagan right about the reasons for astrology's prominence?

B. L. VAN DER WAERDEN, *Science Awakening II*, Springer, 1973.

Early Egyptian astronomy seems to have been of a practical origin. It was noticed that Sirius was the "herald of the flood" (8) "The flooding of the Nile over its banks is the most important event in the Egyptian agricultural year. It gives new life to the parched land. This event is heralded some weeks beforehand by a striking event in the firmament, namely the first visibility of Sirius in the morning sky." (9) Other practical advice based on stars include: "when strong Orion begins to set, then remember to plough"; and "fifty days after the solstice is the right time for men to go sailing" (12) The stars were also used to tell time at night. "In the course of the centuries these Stars of Time became Gods of Time and Destiny." (14) "'From their might derives everything that humanity encouters in the way of disasters,' says the revelation of Hermes Trismegistos." (29) "According to Hermes Trismetgistos the decans can also be called 'horoskopoi'—hour indicators. The decan that rises in the hour of the birth of a child determines the nature of the child." (32)

Babylonian astronomy, on the other hand, seems to be linked to, and largely dictated by, astrology as far back as the record goes. "The oldest cuneiform texts giving the positions of the planets in the zodiac date from the second half of the fifth century B.C. To just this period, and to Babylon too, belongs the oldest horoscope that has been preserved." (2) Of course Babylonian astronomy is much older than this, but precise knowledge of planetary positions were not important as long as astrology was impersonal, perhaps for the reasons given below. Indeed, "Old-Babylonian astrology was not interested, or at least not in the first place, in the fate of the individual. Its principal interest was the well-being of the country. Its predictions concern the weather and the harvest, drought and famine, war or peace and of course also the fate of the Kings." (48-49)

The rationale for impersonal astrology may have included the following. "Just as the great Gods Sin (the moon) and Shamash (the sun) are obviously responsible for the regular procession of months, days and years, and thus influence our entire life, so it was thought that the Goddess Ishtar [Venus] communicates important things to us by her appearances and disappearances." (57) Above we saw some examples of apparently important influences of the stars, in the spirit of which one will say things like "O Ursa major … Put truth for me" (58), as one prayer reads. A further consideration is the plausibility of the idea of a strictly periodic universe (of course the world would be periodic if it was determined by the heavens, which are paradigmatically periodic). As Eudemos was later to relate, "If we are to believe the Pythagoreans, I shall in the future, even as everything recurs according to the Number, again tell you tales here, holding this little stick in my hand, while you will sit before me as you do now; and likewise everything else will be the same." (114) The periodicity at which the world repeats is presumably a common multiple of all planetary periods.

The rationale for individual astrology seems to have included the following. The idea that the souls of the dead rise to the heavens is an old one. Not the first example is that "the inscription for the fallen at the battle of Potidea (-431) says: 'The aether will receive their souls, as the earth receives their bodies'" (146) From here it is a rather short step to the idea that, as expressed for example "in Servius' commentary on Aeneid VI 714, the souls before birth go down through the planetary spheres, acquiring thereby from Saturn inertia, from Mars wrath, from Venus lust, from Mercury avarice, from Jupiter ambition" (144) Another argument in support of this view is that the heavens are the paradigm of self-motion, which is not displayed by soulless objects. As Plato puts it: "the soul which has lost its wings is borne along until it gets hold of something solid, … taking upon itself an earthly body, which seems to be selfmoving, because of the power of the soul within it." (147, Phaedrus 246b-c)

VETTIUS VALENS, *Anthologies*, c. 150. Quoted from Roger Beck, *A Brief History of Ancient Astrology*, Blackwell, 2007, 74–76.

"The Sun is the overseer of all; he is fiery, he is the light of the intellect and the instrument of the soul's perception. In a horoscope he indicates kingship, leadership, … the father, the master …

The Moon … indicates human life at birth, the body, the mother …, nurture … housekeeping, the queen, the mistress …

Saturn makes those born under him … solitary, … robed in black, importunate, miserable … He causes … laziness, inactivity, hindrances, long drawn out litigation, reversals, secrets, oppression, fetters, griefs, accusations, tears, loss of parents, captivity, banishment. He makes … tax collectors … He brings things to completion … He makes people single or widowed, orphaned or childless.

Jupiter indicates … abundance, salaries, large gifts, good crop yields, justice, rulership, … release from chains, freedom …

Mars indicates violence, wars, plundering, uproar, excess, adultery …; masculinity, perjury, error, negotiations on bad terms; those who work with fire or iron, artisans, masons. He makes military commanders …

Venus is desire and erotic love. She indicates the mother and the nurturer. She causes … reconciliations for good ends, marriages, refined arts and crafts, good singing voices, music, sweetness of melody, beauty of form, painting …

Mercury indicates education, letters, argumentation, logic, brotherhood, … calculations, geometry, commerce, youth, play, theft, … discoveries … He is the giver of discernment and judgment. He is in charge of brothers, younger children, …" (I.1)

CARL SAGAN, *Cosmos*, TV show, PBS, 1980.

"Our language preserves an astrological consciousness … The word 'disaster' comes from the Greek for 'bad star'. The Italians once believed that disease was caused by the influence of the stars. It's the origin of our word 'influenza'. …

Astrology developed into a strange discipline, a mixture of careful observations, mathematics and record keeping with fuzzy thinking and pious fraud. Nevertheless, astrology survived and flourished. Why? Because it seems to lend a cosmic significance to the routine of our daily lives. It pretends to satisfy our longing to feel personally connected to the universe. Astrology suggests a dangerous fatalism. If our lives are controlled by a set of traffic signals in the sky, why try to change anything? …

Astrology can be tested by the lives of twins. There are many real cases like this: One twin is killed in childhood in, say, a riding accident, or is struck by lightning but the other lives to a prosperous old age. Suppose that happened to me. My twin and I would be born in precisely the same place and within minutes of each other. Exactly the same planets would be rising at our births. If astrology were valid how could we have such profoundly different fates? …

Also, how could it possibly work? How could the rising of Mars at the moment of my birth affect me then or now? I was born in a closed room. Light from Mars couldn't get in. The only influence of Mars which could affect me was its gravity. But the gravitational influence of the obstetrician was much larger than the gravitational influence of Mars. Mars is a lot more massive but the obstetrician was a lot closer."

AUGUSTINE, *The Confessions*, c. 400. Translated by Henry Chadwick, Oxford University Press, 1991.

"I now wished to attack and with ridicule to refute … one of those charlatans who make money out of astrology. … I therefore gave attention to those who are born twins. Most of them emerge from the womb in succession at a brief interval of time. They may contend that in the realm of nature this interval has considerable consequences. But it cannot be recorded by human observation and noted in the tables that the astrologer will inspect to give a true forecast. … Someone inspecting the identical tables ought to have been able to say that Esau and Jacob would have the same destiny. Yet things turned out differently in each case." (VII.vi.10)

§ R2. Early numeration

R2.1. What general theses can be supported by evidence from both this section and §R4?

R2.2. Why is it interesting that some number words used to be conjugated like adjectives?

R2.3. Name the words for 9 and "new" in each language you know.

GRAHAM FLEGG, *Numbers Through the Ages*, Sheridan House, 1989.

"There is evidence of tallying (carving notches on bone or wood) going back more than thirty thousand years" (37)—the oldest signs of mathematical activity in the historical record.

Tallying remained a widespread record of transactions well into modern times, as witnessed for instance in Shakespeare (48): "There shall be no money, all shall eat and drink on my score." (Henry VI) "He parted well and paid his score." (Macbeth)

"Tally sticks were admitted as legal documents even as late as the eighteenth and nineteenth centuries." (43) For instance, "Napoleon's book of civil law ... of 1804" states: "The tally sticks which match their stocks have the force of contracts between persons who are accustomed to declare in this manner the deliveries they have made or received." (44)

"The word for 'contract' in Chinese is written by means of two characters at the top, one for a notched stick and one for a knife, and another at the bottom which means 'large'. Thus a contract in Chinese is symbolised as 'a large tally stick'." (44)

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As for verbal representation of numbers, "Over most of aboriginal Australia one finds essentially only two number words, 'one' and 'two': many of the tribes are reported definitively as not counting beyond 2, and as indicating higher multiplicities by the word 'many', while others go a little further by compounding these words—for example, expressing 3 as 'two-one', 4 as 'two-two' ... A similar method of counting, the so-called 2-system, is found in New Guinea, in South America, and in South Africa." (7, quoting Seidenberg)

"At a very early stage of counting, the 'number' of something that was counted was felt to be one of its attributes." (57) Therefore, "In Gothic as well as High and Middle German the number words for 2 and 3 do appear as adjectives; they have separate masculine, feminine and neuter forms, and they are inflected like adjectives. The same is true for Latin and Greek." (58)

However, "In all known Indo-European languages, numbers beyond 4 are not treated as adjectives." (59) "It is quite possible that in the original Indo-European language 'one', 'two', 'three', and 'four' were earlier number words, which were more closely connected with the counted objects and treated like adjectives, whereas 'five', 'six', etc. were later number words, perhaps taken over from a foreign language, which were no longer considered as adjectives and hence remained unchanged in all cases." (60)

"Another argument in favour of [this interpretation] is given by the form of the number words for 8 in Latin and Greek. In Latin the word is octo, in Greek oktô; the ending -ô in Greek is a dual ending. In the Gothic and Sanskrit words for 8, ahtaú and astau, the ending -au is also a dual ending. So it seems that 8 was originally conceived as a dual, i.e. as two fours ... In many primitive languages, the number word for 8 is formed as 'twice 4'. So it is quite possible that in the mother language of the Indo-European family the independent number words originally ended at 4 and that 8 was expressed as twice 4." (60)

"It is also possible that at one time counting stopped at 8. This would occur naturally when two handsbreadths had been used up in measurement. After the doubling of 'four' to give 'eight', there would be a need for a 'new' number [or hand] before counting could continue. It is a striking fact that there is a similarity in most Indo-European languages between the word for 9 and the word for 'new'." (60)

EDWARD SAPIR, Notes on the Takelma Indians of Southwestern Oregon, *American Anthropologist*, New Series, 9(2), 1907, 251-275.

Number words of "the Takelma numeral system" have these literal meanings: "Four is evidently nothing but 'two two'; ... six, seven, eight, and nine are respectively equivalent to 'one finger in,' 'two fingers in,' 'three fingers in,' and 'four fingers in' ...; ten is 'two hands' ... twenty is quite transparently 'one person' ..., i.e., 'two hands and two feet'." (266)

§ R3. Beginnings of geometry

R3.1. Herodotus, Strabo, and Proclus give subtly different accounts of the origins of geometry in Egypt. Based on this information alone, are there grounds for judging their relative credibility?

HERODOTUS, *The Histories*, c. -440, translation by A. D. Godley.

"This king [Sesostris] also (they said) divided the country among all the Egyptians by giving each an equal parcel of land, and made this his source of revenue, assessing the payment of a yearly tax. And any man who was robbed by the river of part of his land could come to Sesostris and declare what had happened; then the king would send men to look into it and calculate the part by which the land was diminished, so that thereafter it should pay in proportion to the tax originally imposed. From this, in my opinion, the Greeks learned the art of measuring land." (2.109)

STRABO, *Geographica*, c. -10, quoted from *The Geography of Strabo*, translated by H.C. Hamilton & W. Falconer, London, 1903.

"An exact and minute division of the country was required by the frequent confusion of boundaries occasioned at the time of the rise of the Nile, which takes away, adds, and alters the various shapes of the bounds, and obliterates other marks by which the property of one person is distinguished from that of another. It was consequently necessary to measure the land repeatedly. Hence it is said geometry originated here, as the art of keeping accounts and arithmetic originated with the Phoenicians, in consequence of their commerce." (17.1.3)

PROCLUS, *A Commentary on the First Book of Euclid's Elements*, c. 450, translated by Glenn R. Morrow, Princeton University Press, 1992.

"Geometry was first discovered by the Egyptians and originated in the remeasuring of their lands. This was necessary for them because the Nile overflows and obliterates the boundary lines between their properties. It is not surprising that the discovery of this and the other sciences had its origin in necessity, since everything in the world of generation proceeds from imperfection to perfection. Thus they would naturally pass from sense-perception to calculation and from calculation to reason. Just as among the Phoenicians the necessities of trade gave the impetus to the accurate study of number, so also among the Egyptians the invention of geometry came about from the cause mentioned." (52)

LEONARD MLODINOW, *Euclid's window: The Story of Geometry from Parallel Lines to Hyperspace*, Touchstone, 2002.

"Picture a windswept, desolate desert, the date, 2580 B.C. The architect had laid out a papyrus with the plans for your structure. His job was easy—square base, triangular faces—and, oh yeah, it has to be 480 feet high and made of solid stone blocks weighing over 2 tons each. You were charged with overseeing completion of structure. Sorry, no laser sight, no fancy surveyor's instruments at your disposal, just some wood and rope. As many homeowners know, marking the foundation of a building or the perimeter of even a simple patio using only a carpenter's square and measuring tape is a difficult task. In building this pyramid, just a degree off from true, and thousands of tons of rocks, thousands of person-years later, hundreds of feet in the air, the triangular faces of your pyramid miss, forming not an apex but a sloppy four-pointed spike. The Pharaohs, worshipped as gods, with armies who cut the phalluses off enemy dead just to help them keep count, were not the kind of all-powerful deities you would want to present with a crooked pyramid. Applied Egyptian geometry became a well-developed subject." (§2)

§ R4. Babylonia

R4.1. The two readings embody internalistic (focussed on analysis of mathematical details) and externalistic (focussed on societal context) approaches respectively. What are strengths and weaknesses of each of these modes of doing history?

R4.2. How would an ancient Babylonian mathematician fare on a modern high school quiz on quadratic equations?

R4.3. Was the development of Babylonian mathematics driven by practical need?

JENS HØYRUP, *Lengths, Widths, Surfaces: A Portrait of Old Babylonian Algebra and Its Kin*, Springer, 2002.

The quadratic equation example in the lecture is taken from Høyrup's translation, page 50. The translation of such texts is far from straightforward. In fact, Høyrup's translation is quite radically different from previous ones. Babylonian mathematical sources are extremely sparse in words, as is understandable since they are meticulously inscribed on small clay tablets. This extremely condensed, telegraphic style, allows for a considerable scope of interpretation. A typical phrase such as "30 a-na 7 ta-na-sima 210" (5) can be interpreted alternately in purely arithmetical ("multiply 30 by 7; the result is 210") or geometrical terms ("form a rectangle with sides 30 and 7; its area is 210"). The former option is the traditional one adopted by Neugebauer et al. The primary argument for this interpretation is the fact that the sources often add lengths and areas together, which is geometrically nonsensical. On the other hand the terms used for multiplication seem to point, linguistically speaking, to the geometrical interpretation ("to raise", "to hold", etc.), and indeed certain words for "multiplication" are only used for multiplying two lengths, never areas. Furthermore, certain words that can be read as "protrude", "break", etc., can be understood quite literally in the geometrical reading, whereas they are basically ignored in the arithmetical readings:

"We may say that the received interpretation made sense of the numbers occurring in the text. But it obliterated the distinction made in the texts which after all need not be synonymous unless the arithmetical interpretation is taken for granted; … and it had to dismiss some phrases as irrelevant … or to explain them by gratuitous ad-hoc hypotheses." (13)

The difference between the two readings is quite important since only according to Høyrup's reading does it follow that "The procedure is … algorithm and proof in one. … [It] performs all steps in such a way that their correctness is obvious." (98)

But the geometrical reading does not mean that the procedure is applicable to geometrical problems only. On the contrary, the procedure is "functionally abstract": there are examples where a segment represents a number, an area, a volume, or a commercial rate (280). Virtually all texts "use the Sumerograms *us* and *sag* unerringly for the lengths and widths of the standard representation; 'real' linear dimensions (the length of a wall, the distance bricks are to be carried, the width of a canal), in contrast, may as well be written in syllabic Akkadian … This suggests strongly that the Old Babylonian authors were explicitly aware of the functionally abstract character of their standard representation." (280–281) "In this sense, Neugebauer was right in considering the *us* and *sag* as equivalents of the symbols of modern algebra." (10)

ELEANOR ROBSON, *Mathematics in Ancient Iraq: A Social History*, Princeton University Press, 2008.

"From about 6000 BCE, long before writing, Neolithic villagers used simple geometric counters in clay and stone to record exchange transactions, funded by agricultural surpluses. As societies and economies grew in size and complexity, ever more strain was placed on trust and memory. By the late fourth millennium the intricacies of institutional management necessitated both an increasing numerical sophistication and the invention of written signs for the commodities, agents, and actions involved in controlling them." (27) "Mesopotamian city states had implemented an extensible and powerful literate technology for the quantitative control and management of their assets and labour force. In doing so, they had created in parallel a new social class—in Uruk called the umbisag 'accountant/scribe'—who was neither economically productive nor politically powerful, but whose role was to manage the primary producers on the elite's behalf." (40)

"It was the … state bureaucracy in which the scribes were embedded that … drove the need for … the sexagesimal place value system … by imposing increasingly high calculational standards on its functionaries through the demand for complex annual balanced accounts." (83) This went hand in hand with "centrally imposed reforms of weights and measures throughout the third millennium" (84). "None of these newly invented units of measure was recorded with compound metrological numerals, but always written as numbers recorded according to the discrete notation system followed by a separate sign for the metrological unit," (76) unlike the earliest sources, where, "while there was a single word for 'ten'," "there was no single numeral but different signs for 'ten-discrete-objects', 'ten-units-of-grain', and 'ten-units-of-land'" (33).

"In the early Old Babylonian period [c. 1850 BCE], elementary scribal training underwent a revolution … in which emphasis was more on the ability to manipulate imaginary lines and areas in almost algebraic ways than on the ability to count livestock or calculate work rates." (86) "Topics range from apparently abstract 'naive-geometrical algebra', via plane geometry, to practical pretexts for setting a problem—whether agricultural labour, land inheritance, or metrological conversions. Even the most abstract problems may be dressed up with 'practical' scenarios." (89)

This was "a style of mathematics that encapsulated the principles of … justice" on which the society was based; "in solving abstruse puzzles about measured space, the true scribe demonstrated his or her technical capability … for upholding justice and maintaining social and political stability on behalf of king and god" (266).

As one scribe put it: "When I go to divide a plot, I can divide it; when I go to apportion a field, I can apportion the pieces, so that when wronged men have a quarrel I soothe their hearts … Brother will be at peace with brother." (122)

"The Sumerian word for justice was nig-si-sa, literally 'straightness, equality, squareness', Akkadian misarum 'means of making straight'. The royal regalia of justice were the measuring rod and rope … In [this] light … Old Babylonian mathematics, with its twin preoccupation of land and labour management on the one hand and cut-and-paste geometrical algebra on the other, becomes truly comprehensible." (123-124)

This tradition effectively came to an end with "the collapse of the Old Babylonian kingdom in c. 1600 BCE" (151), though "traces of Old Babylonian mathematical learning lingered on long after the political ideology that it supported had disappeared" (181). "Evidence suggests that mathematics … was still a vital component of Babylonian intellectual life" (151) for a while, but ultimately a massive decline followed. "In the first half of the first millennium we find a low level of mathematical sophistication in school, consumer, and professional contexts." (212)

"Mathematics and mathematical astronomy were central components of the last flowering of cuneiform culture." (261) "From the mid-seventh century [BCE] onwards, … compilers of eclipse records and astronomical diaries had begun to think in terms of divine quantification. … Apparently random events of great ominous significance were observed, quantified, and recorded in the hope that numerical patterns could be detected amongst them. The ultimate aim was to understand the will of the gods, to ensure that they were propitiated and would act benignly to the king and humanity. Thus in later Babylonia mathematics became a priestly concern." (268)

These priests "comprised a tiny number of individuals from a restricted social circle, intermarrying, working closely together to train each successive generation, and highly valuing privacy and secrecy" (261-262). "Their sole aim was to uphold the belief systems and religious practices of ancient times." "In this context [they] developed increasingly mathematically sophisticated means to ensure the calendrical accuracy of their rituals." (262)

§ R5. Greek geometry and philosophy: beginnings

R5.1. Why did the conception of mathematics as a field of knowledge characterised first and foremost by rigorous deductive reasoning arise in ancient Greece?

R5.2. What does early Greek philosophy have in common with mathematics? What are some differences between the two?

R5.3. Why did the mathematicians care what the philosophers were doing?

R5.4. In what ways can Zeno be taken as a case in point illustrating the main theses of Lloyd and Szabó?

R5.5. Are Zeno's two arguments just different literary elaborations of the same idea, or is there something substantially different between the two cases?

G. E. R. LLOYD, *Magic, Reason and Experience: Studies in the Origin and Development of Greek Science*, Cambridge University Press, 1979.

"Public debates between contending speakers in front of a lay audience" was a prominent part of ancient Greek culture (93). Science and philosophy were born on this stage. Many otherwise peculiar characteristics of Greek thought are explained by this format.

The stage debate requires the speakers to proclaim bold and provocative theses, and to strive to avoid reconciliation with other viewpoints at all costs. This is why early Greek thought is rife with crackpot claims such as that motion is impossible (74) or "that man is all air, or fire, or water, or earth" (92). Indeed, the format demands a multiplicity of such viewpoints in competition with one another, whence "the remarkable proliferation of theories dealing with the same central issues" that "may well be considered one of the great strengths ... of Presocratic natural philosophy" (97).

The stage debate also explains why these theses were invariably defended by abstract deductive reasoning. "Given an interested but inexpert audience, technical detail, and even careful marshalling of data, might well be quite inappropriate, and would, in any event, be likely to be less telling than the well-chosen plausible—or would-be demonstrative—argument." (98) Hence we understand why "with the Eleatics logos—reasoned argument—comes to be recognised explicitly as *the* method of philosophical inquiry." (78) This "notion of the supremacy of pure reason may ... be said to have promoted some of the triumphs of Greek science." (121)

However, these triumphs of reason "were sometimes bought at the price of ... a certain impoverishment of the empirical content of the inquiry" (121). In early Greek science, "observations are cited to illustrate and support particular doctrines, almost, we might say, as one of the dialectical devices available to the advocates of the thesis in question." (221) Also, "observations and tests ... could be deployed ... destructively ... [to disprove an opponent's thesis], as they were by Aristotle especially, with great effect." (224) These uses of observation fit well within the stage debate format. However, "theories were not put at risk by being checked against further observations carried out open-endedly and without prejudice as regards the outcome." (222) We can understand why since "The speaker's role was to advocate his own cause, to present his own thesis in as favourable a light as possible. It was not his responsibility to scrutinise ... the weaknesses of his own case with the same keenness with which he probed those of his opponent." (98)

Of course, everyone was well aware of the deceptive potential of sly rhetoric for "making the worse argument appear the better" (99)—so much so that "early on it became a commonplace to insist on your own lack of skill in speaking" (100). But the Greeks did not see this as a reason to abandon the stage disputation format—instead they focussed on explicating "the correct rules of procedure for conducting a dialectical inquiry" (101).

Why did these things arise in Greece and not elsewhere? "Aristotle associated the development of speculative thought with the leisure produced by wealth," (236) and indeed "the level of technology and that of economic development [in ancient Greece] were far in advance of those of many modern non-industrialised societies" (266). However, "Egypt and Babylonia ... were, economically, incomparably more powerful than any of the Greek city-states" (236), so the explanation for the "additional distinctively Greek factor" of "generalised scepticism" and "critical inquiry directed at fundamental issues" (264) must be sought elsewhere.

The answer may lie in "a particular social and political situation in ancient Greece, especially the experience of radical political debate and confrontation in small-scale, face-to-face societies. The institutions of the city-state ... put a premium on skill in speaking and produced a public who appreciated and the exercise of that skill. Claims to particular wisdom and knowledge in other fields besides the political were similarly liable to scrutiny, and in the competition between many and varied new claimants to such knowledge those who deployed evidence and argument were at an advantage compared with those who did not." (266)

G. E. R. LLOYD, *Adversaries and Authorities: Investigations into Ancient Greek and Chinese Science*, Cambridge University Press, 1996.

"Any acquaintance with early Greek natural philosophy immediately brings to light a very large number of instances of philosophers criticising other thinkers." (21) Being a philosopher means being "subjected to blistering attack" (22). "From the list of occasions when philosophers are attacked by name ..., one could pretty well reconstruct the main lines of the development of Hellenistic philosophy itself." (22-23) "Nor is this just a matter of 'philosophy', howsoever understood." (23) There is "hard-hitting polemic" (23) in mathematics, medicine, and art as well. There is a "lack of great authority figures"; even Homer "is attacked more often than revered" (24).

The Greek style of philosophy is connected to its social context. “Greek pupils could and did pick and choose between teachers. … Direct criticism of teachers is possible, and even quite common. … Argument and debate … are one of the means of attracting and holding students, and secondly they serve to mark the … boundaries of groups.” (35) “The Greek schools were there not just, and not even primarily, to hand on a body of learned texts, but to attract pupils and to win arguments with their rivals. They may even be said to have needed their rivals, the better to define their own positions by contrast with theirs.” (38) “Dialectical debate, on which the reputations of philosophers and scientists alike so often depended, stimulated, when it did not dictate, confrontation. … The recurrent confrontations between rival masters of truth left little room for the development of a consensus, let alone an orthodoxy, little sense of the need or desirability of a common intellectual programme.” (137)

“It was the rivalry between competing claimants to intellectual leadership and prestige in Greece, that stimulated the analysis of proving and of proof.” (57) “Many have assume that the internal dynamic of the development of mathematics itself would, somehow inevitably, eventually lead to a demand for strict axiomatic-deductive demonstration, and that there is accordingly no need to pustulate any external stimulus such as I have conjectured … Yet the difficulty for that view is … [that] other, non-Greek, ancient mathematical traditions — Babylonian, Egyptian, Hindu, Chinese — … all got along perfectly well without any notion corresponding to axioms and … the particular notion of strict demonstration that went with it.” (58)

The underlying cause is perhaps captured by an admittedly “simplistic” dichotomy between “adversarial Greeks and irenic, authority-bound, Chinese” (44).

The different philosophical styles of ancient Greece and China reflect differences in their political systems. “Extensive political and legal debates, in the assemblies, councils and law-courts, were … a prominent feature of the life of Greek citizens.” (74) “Greek philosophical and medical schools used, as the chief means for the expression of their own ideas and theories, both lectures … and open, often public debates, sometimes modelled directly on the adversarial exchanges so familiar in Greek law-courts and political assemblies.” (39)

“So far from positively delighting in litigation, as many Greeks seem to have done, so far from developing a taste for confrontational argument in that context and becoming quite expert in its evaluations, the Chinese avoided any brush with the law as far as they could. Disputes that could not be resolved by arbitration were felt to be a breakdown of due order and as such reflect unfavourably on both parties, whoever was in the right.” (220)

“The … typical target audience envisaged in Greek rhetoric is some group of fellow citizens” (79), just as “In Greek law-courts the decisions rested with the ‘dicasts’ … [who] were chosen by lot … [and] combined the roles of both judge and jury.” (79) “In China, the [intended] audience for much philosophical and scientific work was very different: the ruler or emperor himself.” (39) “The Chinese were never in any doubt that the wise and benevolent rule of a monarch is the ideal.” (43)

“We often find Greek philosophers adopting a stance of fierce independence vis-a-vis rulers. … With this independence came … a disadvantage. Compared with their Chinese counterparts, Greek philosophers and scientists had appreciably less chance of having their ideas put into practice. … Autocrats … — as … in China — … could and did move swiftly from theoretical approval to practical implementation.” (43) “The superiority of theory to practice is a theme repeatedly taken up by scientists as well as philosophers in Greece: but that was sometimes to make a virtue out of necessity.” (44)

“Unlike in classical Greece, … the bid to consolidate a comprehensive unified world-view was largely successful in China.” (116) “The prime duty of members of a Chinese Jia was the preservation and transmission of a received body of texts. In that context, pupils did not criticise teachers, and any given Jia did not see it as a primary task to take on and defeat other Jia in argument.” (44-45) While the Greeks “adopted a stance of aggressive egotism in debate, the tactics of Chinese advisers was rather to build on what could be taken as common ground, [and] certainly on what could be represented as sanctioned by tradition.” (221-222) “The emphasis is not on points at which [earlier philosophers] disagreed, … but rather on what each of them had positively to contribute, how each succeeded, at least in part, in grasping some part of the Dao.” (25)

ÁRPÁD SZABÓ, *The Beginnings of Greek Mathematics*, Springer, 1978.

“Mathematics itself grew out of the more ancient subject of dialectic” (245). “Quite obviously it was dialectic which came first. The mere fact that all those terms which relate to the foundations of mathematics are of dialectical origin should have led us to this conclusion. Dialectic did not borrow any of its vocabulary from mathematics; instead, perfectly ordinary expressions from dialectic were transformed into technical mathematical terms. Hence early Greek mathematics, at least when it is viewed as an elaborately constructed system of knowledge, can properly be called a branch of dialectic.” (253-254)

The terms referred to here are those for axiom, postulate, hypothesis, etc., all of which, in dialectic, essentially stood for some variant of “concessions which the participants in a discussion have agreed to make” (238). Even in the best sources the uses of such terms are not very systematic. For example, a clear distinction between definition and axiom is often lacking. On this basis “we can … conjecture that the earliest foundations were composed only of definitions. After all, we know that the word hypotheseis (as a mathematical term) had two meanings; it could denote either a fundamental principle or just a definition. This in itself seems to indicate that fundamental principles were at one time identified with definitions. Furthermore, it is obvious that the very first hypotheseis on which the partners in a dialectical debate have to agree take the form of definitions." (255)

"We know that the term aitema [=postulate] came from dialectic where it was used to denote a 'demand' about which the second partner in a dialogue had reservations. Let us see whether there is any connection between this early meaning of the word and Euclid's postulates. At first glance, Postulates 1-3 appear to be such simple, self-evident and easily fulfilled 'demands' that one is tempted to disregard the literal meaning of their name." (276) But they involve movement (of compasses etc.), which is a problematic notion especially in the view of Zeno and the Eleatics. "If we bear this in mind, it is easy to understand why Euclid's first three postulates had to be laid down. … They really are demands (aitemata) and not agreements (homologema); for they postulate motion, and anyone who adhered consistently to Eleatic teaching would not have been able to accept statements of this kind as a basis for further discussion." (279)

"Our text of Euclid" has a separate heading called common notions, but this was not a well-entrenched term and these principles "obviously bore the name axioma in pre-Euclidean times" (281), and "the noun axioma, when used as a dialectical term, was originally synonymous with aitema and just meant a 'demand' or 'request'" (286). Indeed, Euclid's common notions "all assert properties of a relation (equality) which must have been regarded (by the Eleatics at least) as 'self-contradictory'. It is by no means evident that two distinct things (i.e. two things which are not the same) can ever be 'equal to one another'; furthermore, it makes no sense to speak of two things unless they can be distinguished from one another in some way." (290)

The common notions are also dubious in that they "are assertions which are justified by practical experience and, in some cases, directly by sense-perception. Axiom 7, for example, states that 'things which coincide with one another are equal to one another'. It can literally be seen that plane figures which coincide are actually equal; hence this axiom is verified by sensory experience." (290) Therefore the common notions "could not have been accepted by the Eleatics, who required that all knowledge be obtained by purely intellectual means and without appealing to the senses. These principles were originally called demands (axiomata) because the other party in a dialectical debate had reservations about accepting them as a basis for further inquiry or, in other words, because their acceptance could only be demanded." (301)

"After Plato's time, however, the essentials of Eleatic dialectic were no longer very well understood; hence the ancient term axioma acquired a new meaning. Since it had always been used to refer to a group of principles which, from the viewpoint of common sense, were evidently valid, it came now to denote those statements whose truth was 'accepted as a matter of course'." (301)

Early geometry may well have been based on visual reasoning. For example, the original meaning of the word for "prove" (deiknymi) was "show" or "point out" in a visual sense (although "we also find deiknymi used with the meaning 'to make known by words' in texts as early as the Odyssey" (188)). "Mathematics was at one time a practical science whose subject matter was empirical and whose principles were established by empirical methods"; indeed "the sudden turn away from empiricism [in mature Greek mathematics] seems somewhat surprising" (216): one must "explain why a predominantly empirical mathematical tradition suddenly and for no apparent reason became anti-empirical and anti-visual" (217).

This radical break was perhaps due to "the decisive influence of the Eleatic school of philosophy" (217). The notion of proof by contradiction provides the link between mathematics and the Eleatics. "It is apparent that indirect arguments played a very important part in Eleatic philosophy. Without them it would not have been possible to establish such central doctrines as that there is no motion, no change, no becoming, no perishing, no space and no time. Of course, these doctrines contradict the evidence of our senses and are incompatible with empiricism, nonetheless the Eleatics, bolstered by their belief that reason was the only guide to truth, accepted them." (218)

Mathematicians were drawn into this school of philosophy since "the existence of incommensurability could not be conclusively proved by practical or empirical methods. Hence a complex of problems associated with incommensurability made it necessary to adopt Eleatic techniques of proof in geometry." (316-317) Thus the earlier sense of proof, with its visual connotations, was replaced by axiomatic foundations in the sense of dialectic, and anti-empiricism came with the bargain.

KEN ALBALA, *Food: A Cultural Culinary History*, The Great Courses Plus, 2017.

Islands $\implies$ good conditions for sea trade; independent city states. Hilly landscapes $\implies$ olives, grapes $\implies$ oil, wine = expensive non-perishables $\implies$ trade wealth $\implies$ large middle class with leisure time.

H. D. LEE (ED.), *Zeno of Elea: A Text, with Translation and Notes*, Cambridge University Press, 1936.

Zeno (c. -450) argued that motion is impossible. His "dichotomy" argument says:

"An object in motion must move through a certain distance; but since every distance is infinitely divisible the moving object must first traverse half the distance through which it is moving, and then the whole distance; but before it traverses the whole of the half distance, it must traverse half of the half, and again the half of this half. If then these halves are infinite in number, because it is always possible to halve any given length, and if it is impossible to traverse an infinite number of positions in a finite time … [then] therefore it is impossible to traverse any magnitude in a finite time." (45; Simplicius 1013.4)

There is also the "Achilles" form of the argument:

"The argument is called the Achilles because of the introduction into it of Achilles, who, the argument says, cannot possibly overtake the tortoise he is pursuing. For the overtaker must, before he overtakes the pursued, first come to the point from which the pursued started. But during the time taken by the pursuer to reach this point, the pursued always advances a certain distance; even if this distance is less than that covered by the pursuer, because the pursued is the slower of the two, yet none the less it does advance, for it is not at rest. And again during the time which the pursuer takes to clever this distance which the pursued has advanced, the pursued again covers a certain distance ... And so, during every period of time in which the pursuer is covering the distance which the pursued ... has already advanced, the pursued advances a yet further distance; for even though this distance decreases at each step, yet, since the pursued is also definitely in motion, it does advance some positive distance. And so ... we arrive at the conclusion that not only will Hector never be overcome by Achilles, but not even the tortoise." (51; Simplicius 1014.9)

§ R6. Textual aspects of Greek mathematics

R6.1. Is the written record a good representation of Greek geometrical thought?

R6.2. Were Greek ways of recording mathematics constrained by technology? By tradition?

R6.3. Was Euclid a Platonist who reasoned about eternal, abstract objects, or did he think of geometry as something physically produced by ruler and compass? What can we conclude in this regard from his definitions of point and line, and the passive formulations of construction steps in his proofs?

REVIEL NETZ, *The Shaping of Deduction in Greek Mathematics*, Cambridge University Press, 2003.

"THEAANDTHEBTAKENTOGETHERAREEQUALTOTHECANDTHED This is how the Greeks would write A+B=C+D, had they written in English. And it becomes clear that only by going beyond the written form can the reader realise the structural core of the expressions. Script must be transformed into prewritten language, and then be interpreted through the natural capacity for seeing form in language. Greek mathematical formulae are post-oral, but pre-written. They no longer rely on the aural; they do not yet rely on the layout." (163)

"The lettered diagram is a distinctive mark of Greek mathematics. ... No other culture developed it independently." (58) "The overwhelming rule in Greek mathematics is that propositions are individuated by their diagrams" (38), contrary to the economy of using the same diagram for several propositions, and contrary even to plain sense, it would seem, in the use of completely functionless diagrams for number-theoretic propositions (41).

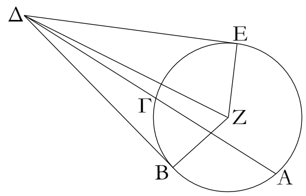
But the diagrams were schematic only, with for example conic sections being crudely represented by circular arcs (34). The diagrams were also static since "of the media available to the Greeks ... none had ease of writing and rewriting" (14). Standard media were papyri and wax tablets, and, for larger audiences, such as Aristotle's lectures, "the only practical option was wood ... painted white" (16). "None of these [ways of representing figures] is essentially different from a diagram as it appears in a book. ... The limitations of the media available suggest ... the preparation of the diagram prior to the communicative act—a consequence of the inability to erase." (16) "This, in fact, is the simple explanation for the use of perfect imperatives in the references to the setting out—'let the point A have been taken'. It reflects nothing more than the fact that, by the time one comes to discuss the diagram, it has already been drawn." (25)

KEN SAITO, Diagrams and traces of oral teaching in Euclid's Elements: labels and references, *ZDM Mathematics Education*, 50(5), 2018, 921–936.

"The teaching of mathematics in ancient times was prevalently oral." (§2.3) "The written demonstration is rather an exceptional form of transmitting mathematical ideas to a person living far away, and written text was probably not the main task of a mathematician." (§7) Many aspects of Greek texts are explained by this circumstance, for example why propositions are never referred to by number (§3.1) or why points are labelled in alphabetical order as they are named in the proof, without any regard for continuity of notation across sequences of very closely related propositions (§3.2).

"A proposition of Euclid's Elements begins with so-called 'protasis', or general enunciation, where the proposition is stated in a general way, without a diagram. ... For example, the protasis of proposition III.13 goes: 'A circle does not touch a circle at more than one point, whether it touches it internally or externally.' This is fine. One understands what it purports. However, a protasis can become so long and complicated, that it is difficult to understand it. The proposition III.37 ... offers a good example. 'If a point be taken outside a circle and from the point two straight lines fall on the circle, and if one of them cut the circle, and the other fall on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference be equal to the square on the straight line which falls on the circle, the straight line which falls on it will touch the circle.' Probably few people are endowed with such intelligence as to understand without difficulty what this proposition means on reading it for the first time. So when we read Euclid's Elements, we often skip the protasis and begin with the ekthesis, or setting out, which follows the protasis and explains the premises of the protasis by assigning names to the points. 'For let a point Δ be taken outside the circle $AB\Gamma$; from Δ let the two straight lines $\Delta\Gamma A$, ΔB fall on the circle $A\Gamma B$; let $\Delta\Gamma A$ cut the circle and ΔB fall on it; and let the rectangle $A\Delta$, $\Delta\Gamma$ be equal to the square on ΔB.' Then

Euclid's restates the conclusion with the names of points. This part is called diorismos (specification). 'I say that ΔB touches the circle $AB\Gamma$.'" (§3.4)



"Now, if the ekthesis, setting out, with diagram and names of points, is much easier to understand, why does the text of the Elements always preserve the protatis, which is often skipped by modern readers? Moreover, in many books of the Elements, the protasis is almost literally repeated at the end of each proposition as sumperasma (conclusion), adding only one word "ara" (therefore) to the protasis. Taking into account the difficulty and high cost of copying and preserving long text in antiquity, the generosity of the author (and/or ancient editors) of the Elements for protasis and sumperasma is quite impressive. Why did it not occur to them to reduce the length of propositions by suppressing the protasis? The answer should be that the protasis (repeated in sumperasma) deserved the space it occupied. But what was its value or function for ancient mathematicians and teachers?" (§3.4)

Since, in oral teaching, "the diagram of a proposition was probably erased when the next proposition had to be treated, … there must have been some way of memorizing and referring to the propositions. I believe that this was exactly the role of the protatis. Indeed, although a protasis such as that of III.37 is long and incomprehensible to someone who reads it for the first time, it is not so hard to memorize after one has learned the proposition and one has understood what it means. … Another important function of the protasis is that a protasis is useful when you want to apply the proposition you know in the demonstration of a later proposition. The protasis is a quotable format. … If you want to apply a proposition whose protasis you have in memory in a demonstration you are working on, you recite the protasis, replacing the indication of geometrical objects by the expression with labels in the diagram, and you have the argument you need." (§4.4)

LUCIO RUSSO, The Definitions of Fundamental Geometric Entities Contained in Book I of Euclid's Elements, *Archive for History of Exact Science*, 52 (1998), 195–219.

Definition 4 of Euclid's Elements reads: "a straight line is [a line] which lies uniformly in respect to [all] its points." "Definitions like these are today considered useless and their inclusion in the Elements is usually seen as a serious flaw in the Elements." But "the presence of the above definitions in our manuscripts of the Elements is … far from warranting their authenticity, in view of the scant reliability of the textual tradition." (196) Manuscript evidence suggests that "the original text of Book I of Euclid's Elements did contain some of the definitions," but that "Euclid did not hesitate in using geometrical terms he had not defined in advance" (such as "circumference")—a practice "avoided in the Imperial age" when "more definitions were again included in textbooks." (198) In the works of Archimedes and Apollonius (who "belong to the same scientific tradition" as Euclid) "there is nothing analogous to the pseudo-definitions of fundamental geometrical entities contained in the Elements. The introduction of terms implicitly defined through postulates is instead frequent." (209–210)

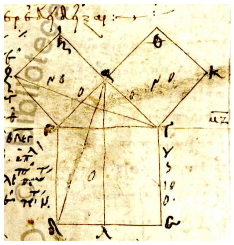
Heron of Alexandria (c. 50) wrote a work devoted to, as he says, "describing and sketching for you as briefly as possible … the technical terms premised in the elements of geometry" (213). Heron's description of a straight line begins as follows: "a straight line is [a line] which, uniformly in respect to [all] its points, lies upright and stretched to the utmost towards the ends, such that, given two points, it is the shortest of the lines having them as ends." "Archimedes … had in fact assumed that among all lines with the same ends the straight line has the minimum length. It is worth noting that Archimedes' statement was not a 'definition', but [a postulate]. In order to draw a 'definition' from Archimedes' postulate, Heron, however, could not restrict his statement to only one couple of points; he had to require that Archimedes' property should be verified uniformly in respect to all its points … Heron's sentence is therefore completely clear." (215)

"We know that the obscure scholar who compiled the list of definitions in the form in which they now appear in Book I of the Elements was not a mathematician of any value. We have supposed that he had decided to use as definitions of elementary geometrical entities some excerpts from Heron's long illustrations. In our case he might have truncated Heron's first sentence as soon as he could get a syntactically correct sentence, even if empty of mathematical meaning." (215) The goal in doing so may have been "to get a set of short 'definitions' suitable to be learnt by heart in the schools … If such a list was usually premised to the Elements, it could hardly avoid being eventually confused with Euclid's text." (203)

CHRISTIÁN C. CARMAN, Accounting for overspecification and indifference to visual accuracy in manuscript diagrams: A tentative explanation based on transmission, *Historia Mathematica*, 45, 2018, 217–236.

"The first time you encounter a medieval manuscript of a Greek mathematical or astronomical work, like those of Archimedes, Euclid, or Aristarchus, the most impressive feature is the odd configuration many diagrams show. There is a tendency to represent more regularity among the geometric objects than what the argument demands and usually they are not accurate graphical depictions of the mathematical object discussed in the text." (217)

Here for example is a typical manuscript depiction of the figure for the Pythagorean Theorem (*Elements* I.47):



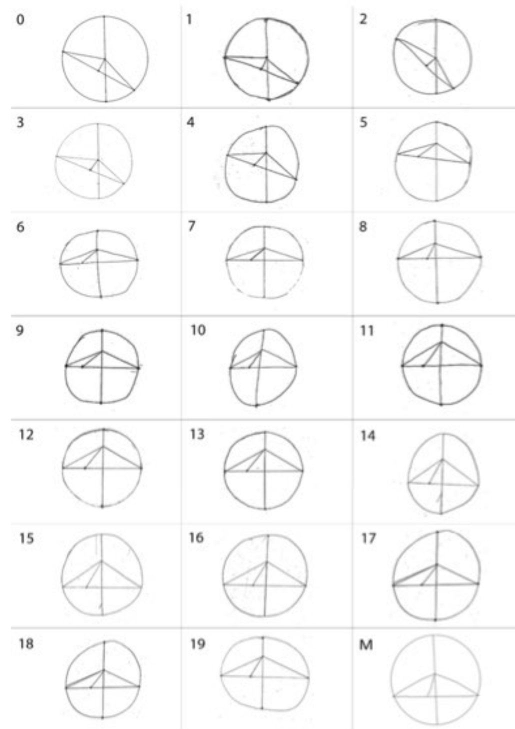
Even though the theorem holds for any right-angle triangle, the manuscript has drawn the special case where the two legs are equal, thereby giving the misleading impression that the theorem is less general than it really is.

"Considering that, except for very few testimonies extant in papyrus (from which not much can be inferred), the most ancient witnesses of Greek diagrams are medieval, we cannot conclusively decide whether these characteristics go back to the Greek authors or they are the result of medieval copyists." (222)

One hypothesis is that, "starting from the supposed original diagram (that is, the diagram well done), a series of copies will generally transform it into a diagram similar to that of the manuscripts." (222) It is indeed plausible that repeated copying converges to specificity, assuming copyists largely ignorant of mathematical content. For example, in the case of the Pythagorean Theorem, a scribe might get a version where the legs look similar and mistakenly assume that exact equality was intended. He then copies it this way, and specificity is introduced. No one will restore more generality in the diagram, because that would require mathematical understanding.

"To test this hypothesis I asked many groups of university students to play the role of copyists. … I gave a folder to one of them, containing a drawing of the diagram as I presume the Greek author drew it. I asked the student to copy the diagram and to place her copy in the folder, over the starting diagram she had copied before hiding it, and to pass the folder to the next student, who would copy the drawing made by the first student, and so on." (222–223)

Below is an example of the successive copies of the original figure (0), which do indeed approach the surviving manuscript figure (M). Note for example that the short segment is supposed to be perpendicular to the long one.



§ R7. Greek geometry and philosophy: classical age

R7.1. What makes mathematical knowledge special? What sets it apart from other fields?

R7.2. In the Platonic worldview (shared by Proclus), what is the relation between mathematics and physical reality? What are strengths and weaknesses of this view?

R7.3. What aspects of technical mathematics do the authors below seem to have had especially in mind?

R7.4. In Rafael's famous fresco "The School of Athens," Plato is pointing toward the sky and Aristotle is pointing straight ahead. Why?

R7.5. Are there examples of theories in which the principles are not primitives, or the primitives are not principles, in Aristotle's sense? (Cf. §R19.)

PROCLUS, *A Commentary on the First Book of Euclid's Elements*, c. 450, translated by Glenn R. Morrow, Princeton University Press, 1992.

"Proclus' commentary on book I of Euclid's Elements is almost certainly a written version of lectures which he presented to students and associates in Athens in the mid-fifth century. … Readers of the commentary should always bear in mind that, although it is the work of Proclus, it is also a record of an educational and intellectual tradition." (ix)

Mathematics stems from the soul, not sense experience. "Should we admit that [the objects of mathematics] are derived from sense objects, either by abstraction, as is commonly said, or by collection from particulars to one common definition?" (10) No, because "The unchangeable, stable, and incontrovertible character of the propositions [of mathematics] shows that it is superior to the kinds of things that move about in matter." (3) "And how can we get the exactness of our precise and irrefutable concepts from things that are not precise? … We must therefore posit the soul as the generatrix of mathematical forms and ideas." (11)

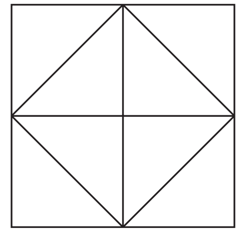
Nevertheless mathematics has many uses. For although we just argued that it is "immaterial and theoretical; when it touches on the material world it delivers out of itself a variety of sciences—such as geodesy, mechanics, and optics—by which it benefits the life of mortals. … and many things incredible to men it has made credible to all. Recall what Hieron of Syracuse is said to have remarked about Archimedes, who had built a three masted vessel … When all the Syracusans together were unable to launch it and Archimedes made it possible for Hieron alone to move it down to the shore [by a system of pulleys], he exclaimed, in his amazement: 'From this day forth we must believe everything that Archimedes says.' … Many of our predecessors have recorded such things in praise of mathematics." (50-51)

"There are nevertheless contentious persons who endeavor to detract from the worth of this science, … declaring that the empirical sciences concerned with sense objects are more useful than the general theorems of mathematics. Mensuration, they say, is more useful than geometry, popular arithmetic than the theory of numbers, and navigation than general astronomy. For we do not become rich by knowing what wealth is but by using it, nor happy by knowing what happiness is but by living happily. Hence we shall agree, they say, that the empirical sciences, not the theories of the mathematicians, contribute most to human life and conduct. Those who are ignorant of principles but practiced in dealing with particular problems are far and away superior in meeting human needs to those who have spent their time in the schools pursuing theory alone." (22)

Reply to this objection. "We do not think it proper … to measure its utility by looking to human needs and making necessity our chief concern. … We must therefore posit mathematical knowledge and the vision that results from it as being worthy of choice for their own sakes, and not because they satisfy human needs. And if we must relate their usefulness to something outside them, it is to intellectual insight that they must be said to be contributory. For to that they lead the way and prepare us by purifying the eye of the soul and removing the hindrances that the senses present to our knowing the whole of things. … Consequently instead of crying down mathematics for the reason that it contributes nothing to human needs … we should, on the contrary, esteem it highly because it is above material needs and has its good in itself alone." (23-24)

PLATO, *Meno*, translated by W. K. C. Guthrie, Penguin, 1956.

Socrates wants to illustrate the nature of mathematical knowledge by leading an uneducated slave boy to discover how to double a given square of area four square feet. In the course of the dialog below he draws this figure in the sand, starting with one of the four small squares:



"SOCRATES: Tell me, boy, is not this our square of four feet? You understand? BOY: Yes. SOCRATES: Now we can add another equal to it like this? BOY: Yes. SOCRATES: And a third here, equal to each of the others? BOY: Yes. SOCRATES: And then we can fill in this one in the corner? BOY: Yes. SOCRATES: Then here we have four equal squares? BOY: Yes. SOCRATES: And how many times the size of the first square is the whole? BOY: Four times. SOCRATES: And we want one double the size. You remember? BOY: Yes. SOCRATES: Now does this line going from corner to corner cut each of these squares in half? BOY: Yes. SOCRATES: And these are four equal lines enclosing this area? BOY: They are. SOCRATES: Now think. How big is this area? BOY: I don't understand. SOCRATES: Here are four squares. Has not each line cut off the inner half of each of them? BOY: Yes. SOCRATES: And how many such halves are there in this figure? BOY: Four. SOCRATES: And how many in this one? BOY: Two. SOCRATES: And what is the relation of four to two? BOY: Double. SOCRATES: How big is this figure then? BOY: Eight feet. SOCRATES: On what base? BOY: This one. SOCRATES: The line which goes from corner to corner of the square of four feet? BOY: Yes. SOCRATES: The technical name for it is 'diagonal'; so if we use that name, it is your personal opinion that the square on the diagonal of the original square is double its area. BOY: That is so, Socrates. SOCRATES: What do you think, Meno? Has he answered with any opinions that were not his own? MENO: No, they were all his. SOCRATES: Yet he did not know, as we agreed a few minutes ago. MENO: True. SOCRATES: But these opinions were somewhere in him, were they not? MENO: Yes. SOCRATES: So a man who does not know has in himself true opinions on a subject without having knowledge. … This knowledge will not come from teaching but from questioning. He will recover it for himself." (84d-85d)

PLATO, *Republic*, c. −380, quoted from *Complete Works*, ed. John M. Cooper, Hackett, 1997.

"We must require those in your fine city not to neglect geometry in any way, for even its byproducts are not insignificant. … When it comes to better understanding any subject, there is a world of difference between someone who has grasped geometry and someone who hasn't."

But: "No one with even a little experience of geometry will dispute that this science is entirely the opposite of what is said about it in the accounts of its practitioners. ... They give ridiculous accounts of it, ... for they speak like practical men, and all their accounts refer to doing things. They talk of 'squaring,' 'applying,' 'adding,' and the like, whereas the entire subject is pursued for the sake of knowledge ... [and] for the sake of knowing what always is, not what comes into being and passes away." (VII, 527)

PLATO, *Timaeus*, translated by Donald J. Zeyl, Hackett, 2000.

Polyhedral theory of the elements. "Let us now assign to fire, earth, water, and air the [regular polyhedra]. To earth let us give the cube, because of the four kinds of bodies earth is the most immobile and the most pliable ... And of the solid figures that are left, we shall next assign the least mobile of them to water, to fire the most mobile, and to air the one in between" (55d-56a). The dodecahedron "still remained, and this one the god used for the whole universe" (55c).

Applications of the polyhedral theory ("a moderate and sensible diversion," 59d). Water=icosahedron has 20 equilateral triangles as its sides, while fire=tetrahedron has 4 and air=octahedron 8, so "when water is broken up into parts by fire or even by air, it could happen that the parts recombine to form one corpuscle of fire and two of air" (56d), i.e., steam is two parts air and one part fire. A second example may illustrate how the relative sizes of the polyhedra matter (61a). Fire is of course the smallest, followed by air. Thus, for example, water can normally be dissolved by air (evaporation) by air octahedra slipping in between the water icosahedra. But since the fire tetrahedra are smaller they dissolve water much more efficiently. And if the water is sufficiently packed (ice) then air cannot dissolve it at all since only fire can get through the cracks.

Human anatomy is an appendix to the soul. "The entire body" was created "as its vehicle" (69c), and its properties were designed to serve the soul, e.g., "They wound the intestines round in coils to prevent the nourishment from passing through so quickly that the body would of necessity require fresh nourishment just as quickly, there by rendering it insatiable. Such gluttony would make our whole race incapable of philosophy and the arts, and incapable of heeding the most divine part within us." (73a). Even eyesight was created not for worldly purposes but primarily to give us the mind the idea of number and time:

Origins of human understanding. "Our ability to see the periods of day-and-night, of months and of years, of equinoxes and solstices, has led to the invention of number and has given us the idea of time and opened the path to inquiry into the nature of the universe. These pursuits have given us philosophy, a gift from the gods to the mortal race whose value neither has been nor ever will be surpassed. I'm quite prepared to declare this to be the supreme good our eyesight offers us." (47a-b).

Generation of animals. "[Birds] descended from ... simple-minded men, men who studied the heavenly bodies but in their naiveté believed that the most reliable proofs concerning them could be based upon visual observation. Land animals ... came from men who had no tincture of philosophy and who made no study of the heavens whatsoever ... As a consequence ... they carried their forelimbs and their heads dragging toward the ground." (91d-92a).

ARISTOTLE, *Posterior Analytics*, translated by Jonathan Barnes, Oxford University Press, 1994.

The essence Aristotle's view of the axiomatic-deductive method is summed up in the following sentence: "Demonstrative understanding ... must proceed from items which are true and primitive and immediate and more familiar than and prior to and explanatory of the conclusions." (71b)

Three notable consequences of this thesis are:

The axiomatic-deductive method is much more than mere logic. "There can be a deduction even if these conditions are not met, but there cannot be a demonstration—for it will not bring about understanding." (71b)

There is a fundamental distinction between "demonstrations which are said to demonstrate and those which lead to the impossible" (85a), i.e., proofs by contradiction, which must be seen as intrinsically inferior (87a).

Axioms stem from perception. "I call prior and more familiar in relation to us items which are nearer perception" (72a), so immediate perception must be the ultimate foundations of "demonstrative understanding." "We must get to know the primitives [i.e., axioms] by induction; for this is the way in which perception instills universals." (100b) However, "for the principles [i.e., axioms] a geometer as geometer should not supply arguments" (77b). Note the two coextensive words for "axiom"—indeed, "I call the same things principles and primitives" (72a), since logical starting points of a deductive system and immediately given truths should be the same thing.

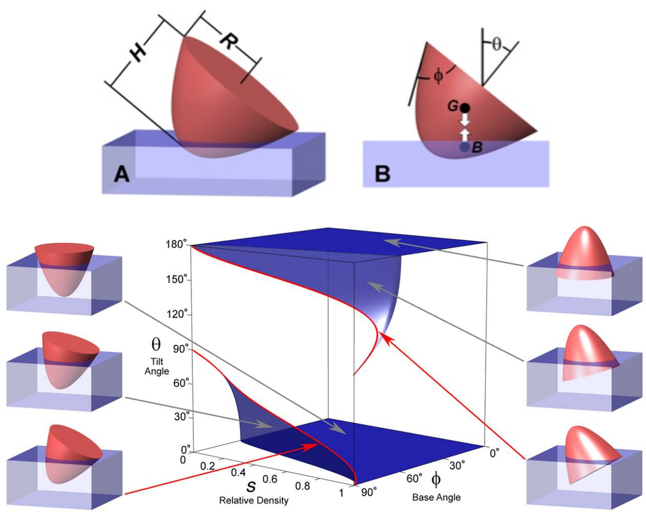
§ R8. Greek science

R8.1. What aspects of modern scientific thinking were lacking in Greek works? Use of mathematics in the study of nature? Experimental and empirical method? Unity of abstract analytic thought and hands-on practice? Unified treatment of terrestrial and celestial phenomena?

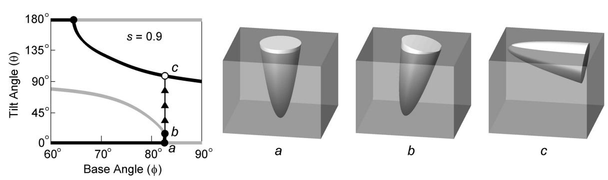
R8.2. How is this related to Plato's philosophy?

CHRIS RORRES, Completing Book II of Archimedes's On Floating Bodies, *Mathematical Intelligencer*, 26(3), 2004, 32–42.

Archimedes's work of floating bodies "ranks with Newton's *Principia Mathematica* as a work in which basic physical laws are both formulated and accompanied by superb applications," namely a detailed investigation of the floatation behaviour of paraboloids that was "the standard starting point for scientists and naval architects examining the stability of ships" still thousands of years later, and that can also be used to explain phenomena such as "the sudden tumbling of a melting iceberg or the toppling of a tall structure due to liquefaction of the ground beneath it." (32–33). Archimedes did not discuss these practical applications, but he did derive exact, detailed, quantitative, highly non-trivial, empirically verifiable results regarding the floatation behaviour of paraboloids submerged in water and stability conditions of such paraboloids in terms of their shape, tilt, and density.



Iceberg toppling as ϕ passes crucial threshold due to melting:



LUCIO RUSSO, *The Forgotten Revolution: How Science Was Born in 300 BC and Why it Had to Be Reborn*, Springer, 2004.

The "Scientific Revolution" took place not in the 17th century but in Hellenistic times. Its root was the marriage of philosophy and technology:

"Despite all the achievements of their culture, the Greeks of the classical age were still behind the Egyptians and Mesopotamians from the technological point of view." (28) "The technological development of all three cultures – classical Greece, Egypt and Mesopotamia – having proceeded by a gradual accumulation and transmission of empirical knowledge, it is natural that the extra millennia would give the two older civilizations a technological advantage." (29) "The Greeks who moved to the new kingdoms that arose from Alexander's conquests had to administer and control these more advanced economies and technologies with which they were not familiar; their one crucial advantage and guide consisted in the sophisticated methods of rational analysis developed by the Greek cultural tradition during the preceding centuries. It is in this situation that science is born." (29)

The practical and technological aspect of Hellenistic thought and science is often not fully appreciated because: "Lack of interest in applied science is of course documented among many classical-era Greek thinkers (who lived before the full blossoming of the scientific method) and among imperial-era Roman intellectuals (to whom the scientific method remained alien). … If one believes in a homogeneous attitude of the 'Ancients' regarding science, one can be tempted to reconstruct it by dismissing as unrepresentative all the true scientists … This misunderstanding is further compounded by the fact that most of what we know about Hellenistic scientists comes to us through the sieve of imperial-era writers." (198)

—Physical astronomy.

Archimedes is the quintessential Hellenistic scientist: a first-rate mathematician who is also deeply immersed in practical engineering. That this combination gave rise to brilliant science is proved perhaps most clearly by Archimedes's hydrostatics of paraboloids (74).

You might object that the hydrostatics of paraboloids is a very narrow topic, and try to write it off on those grounds. But you would be wrong. In the same work, "Archimedes showed that simple postulates on gravity (essentially that gravity is a spherically symmetric pull toward the center of the earth, as Aristotle thought), together with simple postulates about fluids, necessarily imply the spherical shape of the oceans (in rest conditions). … There is no doubt that Archimedes' demonstration was also used to explain the form of the earth as a whole … Indeed, the idea that the earth was originally fluid is reported in several sources, and in particular by Diodorus Siculus, who explicitly relates the earth's shape to gravity. … This is a good example of how exact science can connect apparently distant subjects through logical ties: Archimedes' theorem not only cast light on the earth's geological past, but also had important astronomical and cosmological consequences[, for] Once gravity is used to explain the roundness of the earth, the next step is inevitable, namely explaining in the same way the obvious spherical shape of the sun and of the moon." (303)

Thus we are led naturally to the idea of each heavenly body having its own gravitational attraction. "Plutarch says explicitly: 'Just as the sun attracts to itself the parts of which it consists, so does the earth.'" (304) From here it is a short step to the idea that the sun has a pull not only on its own parts but also on other bodies such as the earth. Indeed the Greeks were well aware that tides can be explained this way: ancient sources "characterize without doubt the lunisolar theory" (308) of tides—that is to say, the correct explanation of tides—postulating the causal role of the sun and moon, and describing the effects in extensive and accurate detail.

Now, if every heavenly body pulls on every other, this leads to the idea of a dynamic theory of planetary motions, seeing planetary motions as composed of rectilinear inertia and gravitational pull. Indeed we read in Vitruvius: "the sun's powerful force attracts to itself the planets by means of rays projected in the shape of triangles; as if braking their forward movement or holding them back, the sun does not allow them to go forth but [forces them] to return to it" (297). Pliny likewise has it that planets are "prevented by a triangular solar ray from following a straight path" (298). The reference to triangles suggests an underlying mathematical treatment, and indeed there are further traces of this (298-302). Furthermore, "the technical tool of vector addition for displacements is present in Heron and in the pseudo-Aristotelian Mechanics, and indeed it is used in this latter work to explain how a uniform circular motion can be regarded as a continuous superposition of a displacement 'according to nature', along the tangent, with one 'contrary to nature', directed toward the center." (301-302)

Moreover, "Simplicius … tells us that Hipparchus wrote a work on gravity titled On bodies thrust down because of gravity. This is the same terminology used several times by Plutarch" (291) in describing the question posed by "the folks who introduced the thrust toward the center" as to whether "boulders thrust through [a tunnel into] the depths of the earth, upon reaching the center, should stay still with nothing touching or supporting them; [or whether] if thrust down with impetus they should overshoot the center and turn back again and keep bobbing back and forth" (287). An indication that this was a serious mathematical question is Simplicius's statement that "Hipparchus contradicts Aristotle regarding weight, as he says that the further something is, the heavier it is." (292) "The only way to make [this statement] comprehensible is to suppose that Hipparchus meant the weight of bodies inside the earth, recognizing that it decreases as the body nears the center." (293)

Another link between celestial and everyday mechanics is this. Plutarch: "To help the moon, that it may not fall, there is its motion itself and the whizzing nature of its rotation, just as objects placed in a sling are prevented from falling by the circular motion. … For this reason the moon does not follow its weight, which is cancelled by the counterweight of the rotation." (286) This metaphor has a suggestive consequence, for we may observe that "Anyone who has tried to spin around a weight at the end of a string has noticed that it is impossible to do this while keeping perfectly still; likewise a hammer thrower never remains immobile, but swings his own body in a small circle as he spins the hammer in a larger one." (314) Perhaps this is what Seleucus has in mind when, according to Aetius: "Seleucus the mathematician (also one of those who think the earth moves) says that the moon's revolution counteracts the whirlpool motion of the earth." (315) For perhaps "what is meant is the earth's revolution along a very small circle (that is, around the earth-moon barycenter), a wobbling very much like that of a largish object caught near the center of a whirlpool. So the moon's revolution counteracts, or is counterposed to, the earth's wobbling: as the moon describes a large orbit, the earth describes a small one, both remaining always opposed in relation to the center of the orbit, just as the hammer thrower moves in a small circle, keeping diametrically opposite the projectile in its circular trajectory." (315)

This dynamical perspective gives a further argument for heliocentrism. Seneca: "You are mistaken in thinking that any star [=planet] stops on its track or turns backward. Heavenly bodies cannot be detained or turned back; they forever move forth; as they once were sent on their way, so they continue; … if ever [these bodies] stop, they will fall upon one another." (294) Just as a rock in a sling would fall to the ground if not held in orbit by its rotational speed, so the planetary system, if robbed of speed, would collapse into a point under mutual gravitational attraction. "Heliocentrism is able to solve the dynamical problem mentioned by Seneca: the sling argument can be applied to planetary motion exactly as to lunar motion, by making the sun, rather than the earth, be the center." (295) For on this account the planets are never actually stationary but only appear to be so when the earth is overtaking them in its orbit. This goes hand in hand with "Seneca's statement that planetary stations are just an illusion and the ship analogy [illustrating relativity of motion, which Seneca discusses]" (295).

It is well-known that Hellenistic astronomers advocated heliocentrism (80-82, 294-295, 297). Less clear is how far they worked out a complete quantitative theory of the planets using a heliocentric model. If anyone did it would most likely have been Hipparchus, and indeed there are indications that he did (285-286, 293-294). Also "several technical elements of Ptolemaic astronomy can only be explained as derivatives of an earlier heliocentric model" (317).

It is in any case known for a fact that important mathematical works from this era, and Hipparchus in particular, are completely lost almost without a trace. One clear example is this correct solution to an advanced combinatorial problem, incidentally transmitted in complete isolation: "in Plutarch's dialogues we find this remark: 'Chrysippus said that the number of intertwinings obtainable from ten simple statements is over one million. Hipparchus contradicted him, showing that affirmatively there are 103,049 intertwinings.'" (281)

—Unity of mathematics and applied science.

Philo of Byzantium says on artillery design that "in this techne many calculations are needed, and someone who makes a small departure in the individual parts causes a large error in the result." (280) At the same time, however, "everything cannot be accomplished through pure thought and the methods of mechanics, but much is found also by experiment" (111). "Thus Hellenistic scientists had already enunciated explicitly the relationship between mathematics and experiments that is usually considered typical of the Galilean method. Soon after this passage Philo gives the formula for the diameter of the opening that the spring (tension rope) goes through, and hence the diameter of the spring itself, as a function of the weight of the projectile that one wishes to throw a given distance; the diameter is proportional to the cube root of the weight, the proportionality constants being given by Philo." (111)

We also see in this example how this practical engineering problem connects directly to the most abstract mathematics of the day: "The famous problem of the doubling of the cube (extraction of cube roots) thus reveals its practical interest in the task of 'calibrating' catapults. An ingenious instrument, the mesolabe, was designed by Eratosthenes to perform the extraction. … Eratosthenes mentions the usefulness of his instrument in designing catapults." (111)

More generally, "The modern distinction between physical and mathematical sciences was alien to Hellenistic science, which was unitary. … Just as works on statics and optics bear a clear relation to concrete activities such as the use of balances and optical instruments …, the exact same relation … obtains between Euclidean geometry and drawing with ruler and compass." (189) Indeed, the other classical construction problems of geometry likewise "had some practical interest in antiquity": "the trisection of the angle … to draw divisions corresponding to the hours in sundials" and "the quadrature of the circle … to compute trigonometric functions, essential in topography and astronomy" (201).

"Consider the following two propositions: Construct an equilateral triangle on a given segment. With a given force move a given weight by means of gears. The first is taken from the Elements, the second from Heron's Mechanics. From the point of view of Hellenistic science these two statements (or 'problems') are strictly analogous: both are followed by an exposition of the necessary construction and then the demonstration that, based on propositions already known, the construction does satisfy the statement's conditions." (185-186) Thus for example when "Archimedes … solved the problem of lifting a given weight with a given force" (71) this must have meant giving a concrete construction recipe, and indeed Heron describes a machine to this end in detail (99).

In Europe, "Renaissance intellectuals were not in a position to understand Hellenistic scientific theories, but, like bright children whose lively curiosity is set astir by a first visit to the library, they found in the manuscripts many captivating topics, especially those that came with illustrations. … The most famous intellectual attracted by all these 'novelties' was Leonardo da Vinci." (335) "Leonardo's 'futuristic' technical drawings … was not a science-fiction voyage into the future so much as a plunge into a distant past. Leonardo's drawings often show objects that could not have been built in his time because the relevant technology did not exist. This is not due to a special genius for divining the future, but to the mundane fact that behind those drawings … there were older drawings from a time when technology was far more advanced." (336)

§ R9. Origins of conic sections

R9.1. What are the strengths as weaknesses of Zeuthen's and Neugebauer's theories regarding the origins of conic sections?

R9.2. Describe step by step how to set up Neugebauer's sundial. What knowledge is required to operate it? Is it likely that early sundials were of this type?

H. G. ZEUTHEN, *Die Lehre von den Kegelschnitten im Altertum*, A. F. Höst & Sohn, 1886.

What led the Greeks to take an interest in conic sections? Very little is known about the early history of conic sections, but one striking fact is that in the early period conic sections were defined as the intersection of a cone with a plane *perpendicular* to its side. So instead of obtaining all possible types of conic sections by slicing a single cone by various planes, as we do today, the Greeks used only perpendicular planes and hence through of the various types of conic sections as arising not from using different cutting planes but different cones (with varying cone angles).

"[The restriction to perpendicular cutting planes in the early theory of conics] calls out for an explanation, in my opinion. I find that the explanation is that the task was not to seek plane sections of a cone, but rather, conversely, to find a representation of curves already known. This goal was best achieved by means of a completely determined and limited form of representation. This explanation agrees well with what is known about the relation between the discovery of conic sections and the duplication [of the cube]. … Hippocrates [reduced this problem to]

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{b} \tag{$*$}$$

[which, if we take $a = 1$, $b = 2$ implies $x = \sqrt[3]{2}$ = the side of a cube with twice the volume of a unit cube]. Once this is found, the goal becomes to find two geometrical curves whose intersection makes it possible to carry out this construction [i.e., construct the line segment x]. Surely this started with efforts to reformulate [($*$)] in terms that could be represented by line and circle, that is, which would led to ordinary geometrical constructions. … [When this failed,] one fell back on [($*$)] as the simplest relations. These were then subject to increasingly detail study. The goal thereof mush have been to bring out … a geometrical definition. … [This led to] a geometric determination of the curves in question, namely as sections of right cones cut in a certain way. … Whether these could be brought about in other ways as sections of cones was irrelevant." (458–460)

OTTO NEUGEBAUER, The Astronomical Origin of the Theory of Conic Sections, *Proceedings of the American Philosophical Society*, 92(3) (1948), 136–138.

"The strange condition of perpendicularity of the intersecting plane always seemed to me to point to only one explanation, the theory of sundials. The generating line must be the 'gnomon' … adjusted in such a way that it always points to the sun when it culminates. The plane onto which the shadow is cast is perpendicular to the gnomon." (136) In other words, the gnomon is pointing towards the highest (i.e., noon) position of the sun in any given day, and the plane recording its shadow is perpendicular to this gnomon. This arrangement, sure enough, produces conic sections consistent with the perpendicularity condition, for in this case the gnomon is contained within the surface of the cone defined by the circular path of the sun and the tip of the gnomon, whence the plane perpendicular to the gnomon is also perpendicular to the side of the cone, as required.

"The adjustment towards the culminating point is very easy to control: one must merely prevent the noon shadow from becoming visibly different from zero. Thus the whole construction is very simple in practical execution." (136) "Though I feel confident that the above explanation gives the real motivation for the early Greek theory of conic sections, I must admit that I do not know of the existence of sundials of this type." (138)

§ R10. Perspective art

R10.1. What is the mathematical justification for Alberti's recipe for drawing a tiled floor?

R10.2. In what way has perspective painting played a role in broader philosophical and scientific developments?

> VITRUVIUS, *De Architectura*, 1st century BC, VII.11, quoted from Vitruvius, *The Ten Books on Architecture*, translated by Morris Hicky Morgan, Harvard University Press, 1914, 198.

Geometrical principles of perspective can be used to deceive the eye. "By this deception a faithful representation of the appearance of buildings might be given in painted scenery, … so that, though all is drawn on a vertical flat facade, some parts may seem to be withdrawing into the background, and others to be standing out in front."

> PLATO, *Republic*, c. −380, quoted from *Complete Works*, ed. John M. Cooper, Hackett, 1997.

"*Trompe l'œil* painting" has "powers that are little short of magical," "because they exploit this weakness in our nature," bypassing "the rational part of the soul." (X.602d) The solution to this problem, as Plato saw it, was a solid mathematical education. Since "sense perception seems to produce no sound result" with these "*trompe l'œil* paintings" (VII.523b), "it makes all the difference whether someone is a geometer or not" (VII.526d). "There's no knowledge of … sensible things, whether by gaping upward or squinting downward." (VII.529b) For example, "let's study astronomy by means of problems, as we do geometry, and leave the things in the sky alone." (VII.530b)

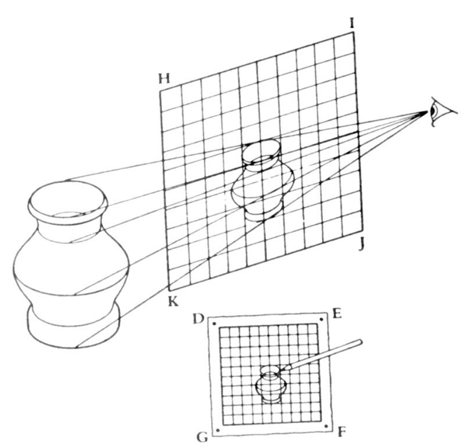
> PLATO, *Protagoras*, c. −380, quoted from *Complete Works*, ed. John M. Cooper, Hackett, 1997.

"The power of appearance often makes us wander all over the place in confusion, often changing our minds about the same thing and regretting our actions and choices with respect to things large and small." "The art of measurement," by contrast, "would make the appearances lose their power" and "give us peace of mind firmly rooted in the truth." (356d)

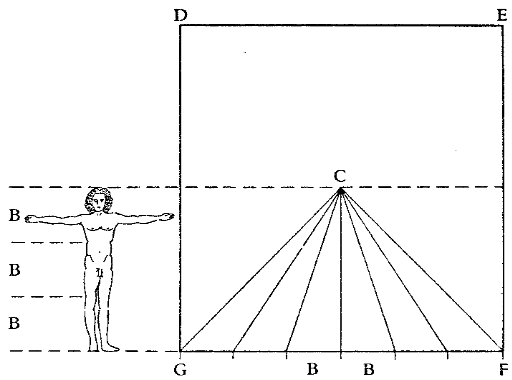
> LEON BATTISTA ALBERTI, *De pictura*, 1435. Quoted from the Penguin Classics edition, translated by Martin Kemp, 1991.

"Let us … start from the opinion of philosophers who say that surfaces are measured by certain rays, ministers of vision as it were, which they therefore call visual rays, since by their agency the images of things are impressed on the senses. … Let us imagine the rays, like extended very fine threads gathered tightly in a bunch at one end, going back together inside the eye where lies the sense of sight." (§5) "The eye measures [the size of objects] with the extrinsic rays rather like a pair of dividers." (§6) "The extrinsic rays, which hold on like teeth to the whole of the outline [of a body], form an enclosure around the entire surface like a cage. This is why they say that vision takes place by means of a pyramid of rays." (§7)

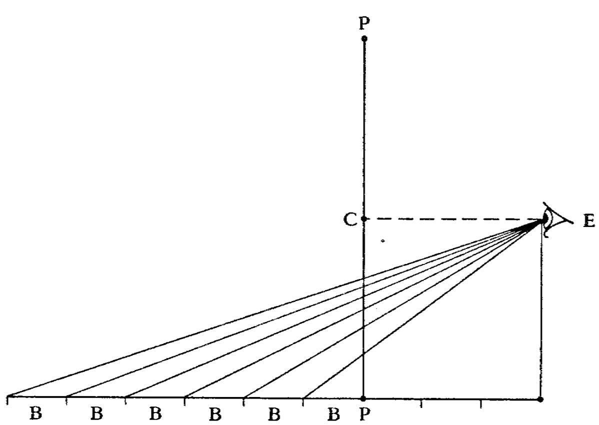
"A painting will be the intersection of a visual pyramid …, represented by art with lines and colours on a given surface." (§12) "To do this well, I believe nothing more convenient can be found than the veil, … whose usage I was the first to discover. It is like this: a veil loosely woven of fine thread … divided up by thicker threads into as many parallel square sections as you like, and stretched on a frame. I set this up between the eye and the object to be represented, so that the visual pyramid passes through the loose weave of the veil. … [In this way] the boundaries of the surfaces can easily be established accurately on the painting panel; for just as you see the forehead in one parallel, the nose in the next, … and everything else in its particular place, so you can situate precisely all the features on the panel or wall which you have similarly divides into appropriate parallels." (§31)



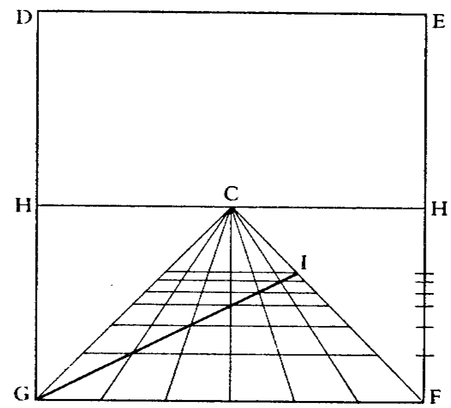
"Let me tell you what I do when I am painting. … I decide how large I wish the human figures in the painting to be. I divide the height of this man into three parts. … With this measure I divide the bottom line … into as many parts as it will hold. Then I establish a point … wherever I wish; and as it occupied the place where the centric ray strikes, I shall call this the centric point. The suitable position for this centric point is no higher from the base line than the height of the man … for in this way both the viewers and the objects in the painting will seem to be on the same plane. Having placed the centric point, I draw lines from it to each of the divisions on the base line." (§19)



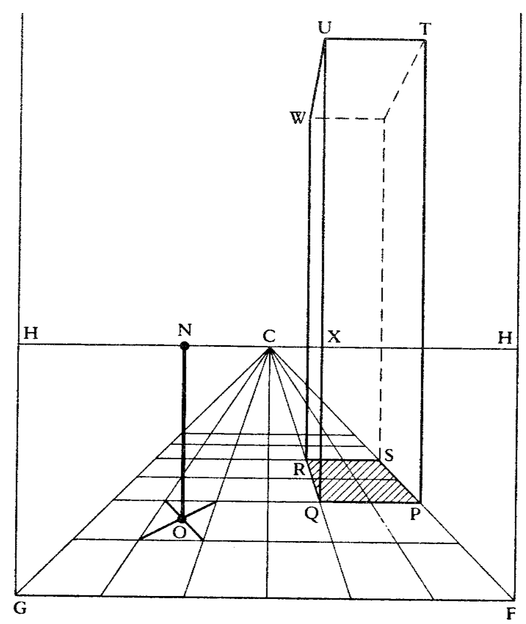
"As regards the successive [parallels of a tiled floor] I observe the following method. I have a drawing surface [on the canvas margin] on which I describe a single straight line, and this I divide in parts like those into which the base line of the rectangle is divided. Then I place a point above this line, directly over one end of it, at the same height as the centric point … and from this point I draw lines to each of the divisions of the line. Then I determine the distance I want between the eye of the spectator and the painting, and, having established the position of the intersection at this distance, I effect the intersection with … a perpendicular. … This perpendicular will give me, at the places it cuts the other lines, the measure of what the distance should be in each case between the transverse equidistant lines of the pavement." (§20)



"A proof of whether they are correctly drawn will be if a single straight line forms the diagonal of connected quadrangles in the pavement. … When I have carefully done these things, I draw a line across, … which … passes through the centric point. This line is for me a limit or boundary, which no quantity exceeds that is not higher than the eye of the spectator … This is why men depicted standing in the parallel [to the horizon] furthest away are a great deal smaller than those in the nearer ones—a phenomenon which is clearly demonstrated by nature herself, for in churches we see the heads of men walking about, moving at more or less the same height, while the feet of those further away may correspond to the knee-level of those in front." (§20)



"On the pavement that is divided up into parallels, you have to construct the sides of walls and other similar surfaces. … I will explain briefly how I proceed in this construction. I begin first from the foundations. I draw the breadth and length of the walls on the pavement. … I determine what I wish their length and breadth to be by the parallels traced on the pavement, for I take up as many parallels as I want them to be braccia. … So, from the scale of the parallels I easily draw the width and length of walls that rise from the ground. Then I go from there without any difficulty to do the heights of the surfaces. … [For example,] if you want this [height] from the ground to the top to be [three] times the height of a man …, you must continue it upwards [two] times again the distance from the centric line to the foot of the [wall]." (§33)



Samuel Y. Edgerton, *Renaissance Rediscovery of Linear Perspective*, Icon, 1976.

"Perspective rules were accepted by … early artists because it gave their depicted scenes a sense of harmony with natural law, thereby underscoring man's moral responsibility within God's geometrically ordered universe." A passage in Dante "upholds the wisdom of looking someone 'straight in the eye', with all its overtones of detecting another's truthfulness and communicating one's own sincerity"—perhaps this is why Renaissance artists preferred the central viewpoint where walls and floor tiles tend towards the centric point in the middle of the horizon. "During the fourteenth century there had been a temporary favoring of charming oblique views. Thereafter it was as if painters wanted to recapture the solemn spirit of the old traditional Christian messages" and "came more and more to believe that things planned or seen from a central viewpoint had greater monumentality and moral authority."

Clifford D. Conner, *A People's History of Science: Miners, Midwives, and Low Mechanicks*, Nation Books, 2005.

"The invention of perspective by the Renaissance artists, … by demonstrating that mathematics could be usefully applied to physical space itself, [constituted] a momentous step … toward the general representation of physical phenomena in mathematical terms." (270)

David Wootton, *The Invention of Science: A New History of the Scientific Revolution*, Allen Lane, Penguin Random House, 2015.

"The mathematization of the sublunary world begins not with Galileo but with Alberti." (§5.8)

H. Floris Cohen, *The Scientific Revolution: A Historiographical Inquiry*, University of Chicago Press, 1994.

"Galileo may fruitfully be seen as the culmination point of a tradition in Archimedean thought which, by itself, had run into a dead end. What enabled Galileo to overcome its limitations … seems easily explicable upon considering Galileo's background in the arts and crafts." (349)

§ R11. Mathematics and society in early modern Europe

R11.1. In the early modern period, who had respect for mathematics, and why? Who were critical of mathematics, and why?

R11.2. What was the relation between mathematics and religion?

R11.3. What was the relation between mathematics and science?

R11.4. Which themes in these debates are timeless, and which were only relevant in their particular context?

Otto Georg von Simson, *The Gothic Cathedral*, Princeton University Press, 1988.

"At least one literary document survives that explains the use of geometry in Gothic architecture: the minutes of the architectural conferences held during 1391 and the following years in Milan. … The question debated at Milan is not whether the cathedral is to be built according to a geometrical formula, but merely whether the figure to be used is to be the square … or the equilateral triangle. … The minutes of one particularly stormy session relate an angry dispute between the French expert, Jean Mignot, and the Italians. Overruled by them on a technical issue, Mignot remarks bitterly that his opponents have set aside the rules of geometry by alleging science to be one thing and art another. Art, however, he concludes, is nothing without science, *ars sine scientia nihil est.* … This argument was considered unassailable even by Mignot's opponents. They hasten to affirm that they are in complete agreement as regards this theoretical point and have nothing but contempt for an architect who presumes to ignore the dictates of geometry."

Roger Ascham, *The Schoolmaster*, 1570.

"Some wits, moderate enough by nature, be many times marred by over much study and use of some sciences, namely, music, arithmetic, and geometry. These sciences, as they sharpen men's wits over much, so they charge men's manners over sore, if they be not moderately mingled, and wisely applied to some good use of life. Mark all mathematical heads, which be wholly and only bent to those sciences, how solitary they be themselves, how unapt to serve in the world."

William Kempe, translator's dedication in Ramus, *The Art of Arithmeticke*, 1592.

"Take away arithmetic, ye take away the merchant's eye, whereby he seeth his direction in buying and selling; ye take the goldsmith's discretion, whereby he mixeth his metals in due quantities; ye take away the captain's dexteritie, whereby he embattaileth his army in convenient order; finally ye take from all sorts of men, the faculty of executing their functions aright. Arithmetic then teacheth unto us matters in divinity, judgeth civil causes uprightly, cureth diseases, searcheth out the nature of things created, singeth sweetly, buyeth, selleth, maketh accompts, weigheth metals and worketh them, skirmisheth with the enemy, goeth on warfare, and setteth her hand almost to every good work, so profitable is she to mankind."

TOBIAS DANTZIG, *Number: The Language of Science*, Masterpiece Science edition, Pi Press, 2005.

"There is a story of a German merchant of the fifteenth century, which I have not succeeded in authenticating, but it is so characteristic of the situation then existing that I cannot resist the temptation of telling it. It appears that the merchant had a son whom he desired to give an advanced commercial education. He appealed to a prominent professor of a university for advice as to where he should send his son. The reply was that if the mathematical curriculum of the young man was to be confined to adding and subtracting, he perhaps could obtain the instruction in a German university; but the art of multiplying and dividing, he continued, had been greatly developed in Italy, which in his opinion was the only country where such advanced instruction could be obtained." (26)

CHRISTOPH J. SCRIBA, The Autobiography of John Wallis, F.R.S., *Notes and Records of the Royal Society of London*, 25(1), 1970, 17–46.

When Wallis went to Oxford in 1632 there was no one at the university who could teach him mathematics. "For Mathematicks, (at that time, with us) were scarce looked upon as Accademical studies, but rather Mechanical; as the business of Traders, Merchants, Seamen, Carpenters, Surveyors of Lands, or the like." (27)

CHRISTOPHER CLAVIUS, *In disciplinas mathematicas prolegomena*, 1574. Translation quoted from J. M. Lattis, *Between Copernicus and Galileo*, University of Chicago Press, 1994, 35–36.

"The theorems of Euclid and the rest of the mathematicians, still today many years past, retain … their true purity, their real certitude, and their strong and firm demonstrations. … And thus so much do the mathematical disciplines desire, esteem, and foster truth that they reject not only whatever is false, but even anything merely probable … So there can be no doubt but that the first place among the other sciences should be conceded to mathematics." Mathematics is "not only useful, but in fact necessary" in many fields, but "of all these benefits [of mathematical studies], perhaps the greatest is the entertainment and pleasure that fills the soul as a result of the cultivation and exercise of those arts."

THOMAS HOBBES, *De Cive*, 1642. Quoted from *The English Works of Thomas Hobbes of Malmesbury*, Volume II.

"And truly the geometricians have very admirably performed their part. For whatsoever assistance doth accrue to the life of man, whether from the observation of the heavens, or from the description of the earth, from the notation of times, or from the remotest experiments of navigation; finally, whatsoever things they are in which this present age doth differ from the rude simpleness of antiquity, we must acknowledge to be a debt which we owe merely to geometry. If the moral philosophers had as happily discharged their duty, I know not what could have been added by humane Industry to the completion of that happiness, which is consistent with humane life." (iv)

AMIR ALEXANDER, *Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World*, Scientific American / Farrar, Straus and Giroux, 2014.

—Jesuit opposition to infinitesimals.

The Jesuits were the intellectual leaders of the Catholic world in the 17th century. They ran hundreds of colleges across Europe, notable as much for their "sheer educational quality" (46) as for their doctrinal role "in the fight to defeat Protestantism" (41).

The Jesuit colleges placed great emphasis on Euclidean mathematics. "It was a logical sequence of studies, but for [the Jesuits] it also represented a deeper ideological commitment. Geometry, being rigorous and hierarchical, was, to the Jesuit, the ideal science. The mathematical sciences that followed—astronomy, geography, perspective, music—were all derived from the truths of geometry … Consequently, Clavius's mathematical curriculum … demonstrated how absolute eternal truths shape the world and govern it" (72), thereby serving as a model for their religious doctrine and worldview. "Euclidean geometry thus came to be associated with a particular form of social and political organization, which … the Jesuits strived for: rigid, unchanging, hierarchical, and encompassing all aspects of life." (218)

For this reason, "the Jesuits reacted with … fury to the rise of infinitesimal methods. For the mathematics of the infinitely small was everything that Euclidean geometry was not. Where geometry began with clear universal principles, the new methods began with a vague and unreliable intuition that objects were made of a multitude of minuscule parts. Most devastatingly, whereas the truths of geometry were incontestable, the results of the method of indivisibles were anything but" (175), thereby undermining "the Jesuit quest for a single, authortized, and universally accepted truth" (176).

Thus infinitesimal mathematics was dangerous to the Jesuits not for intrinsic mathematical reasons but because it was associated with diversity of thought unchecked by authority. "'Unless mind are contained within certain limits', warned Father Leone Santi, prefect of studies at the Collegio Romano …, 'their excursions into exotic and new doctrines will be infinite', leading to 'great confusion and perturbation to the Church'." (122)

Consequently, "In a fierce decades-long campaign, the Jesuits worked relentlessly to discredit the doctrine of the infinitely small and deprive its adherents of standing and voice in the mathematical community. Their efforts were not in vain: as 1647 was drawing to a close, the brilliant tradition of Italian mathematics was coming to an end as well." (117)

According to Alexander the Jesuit campaign against infinitesimals was extremely successful and influential. As he puts it, "champions of the infinitely small (Galileo, Cavalieri, and Torricelli) pioneered new techniques that would transform the very foundations of mathematical inquiry and practice. But when the Jesuits triumphed over the advocates of the infinitely small, this brilliant tradition died a quick death," (178), leaving "no one left in Italy to carry the torch of the infinitely small" (165). "In Italy, the stage was set for centuries of backwardness and stagnation." (180) "No city or prince wished to risk the wrath of the Jesuits, and as a result, no university chairs or positions of honor at princely courts were in the offing for supporters of the infinitely small." (165)

It should be noted, however, that Cavalieri's generation did in fact have a number of direct followers in prominent university positions. Cavalieri's student "Pietro Mengoli (1626-84) succeeded Cavalieri to the mathematics chair at Bologna," (164) a position he held for the remaining 39 years of his life. Angeli was another student of Cavalieri's, and a prima facie counterexample to Alexander's thesis. "In 1662, he was appointed to the chair of mathematics at the University of Padua, a position once held by Galileo. The Jesuits, so powerful elsewhere in Italy, could only fume as the upstart … was raised to one of the most prestigious mathematical posts in all Europe." (170-171) "Angeli … took on the Jesuits like no one had dared since the days of Galileo himself. He called them names [and] ridiculed [them]," (170) and "published no fewer than nine books promoting and using the method of indivisibles" (174), and held his chair for the rest of his life.

—Mathematics and politics in Britain.

Wallis's work on infinite series was based on daring, unrigorous extrapolations and generalisations, which he considered "'a very good Method of Investigation … which doth very often lead us to the early discovery of a General Rule'. Most important, 'it need not … any further Demonstration'." (270)

Hobbes, by contrast, appealed to the authority and rigour of Euclidean geometry as a model for reasoning as well as political organisation.

"Wallis and Hobbes both believed that mathematical order was the foundation of the social and political order, but beyond this common assumption, they could agree on practically nothing else. Hobbes advocated a strict and rigorous deductive mathematical method, which was his model for an absolutist, rigid, and hierarchical state. Wallis advocated a modest, tolerant, and consensus-driven mathematics, which was designed to encourage the same qualities in the body politic as a whole." (256)

Wallis's vision of mathematics was very agreeable to the experimental scientists of the Royal Society. "Experimentalism is a humbling pursuit, very different from the brilliance and dash of systematic philosophers such as Descartes and Hobbes. It is, wrote Sprat, 'a laborious philosophy … that teaches men humility and acquaints them with their own errors'. And that is precisely what the founders of the Royal Society liked about it. Experimentalism, as Sprat noted, 'removes all haughtiness of mind and swelling imaginations', teaching men to work hard, to acknowledge their own failures, and to recognize the contributions of others." (253)

"Mathematics, [the Royal Society founders] believed, was the ally and the tool of the dogmatic philosopher. It was the model for the elaborate systems of the rationalists, and the pride of the mathematicians was the foundation of the pride of Descartes and Hobbes. And just as the dogmatism of those rationalists would lead to intolerance, confrontation, and even civil war, so it was with mathematics. Mathematical results, after all, left no room for competing opinions, discussions, or compromise of the kind cherished by the Royal Society. Mathematical results were produced in private, not in a public demonstration, by a tiny priesthood of professionals who spoke their own language; used their own methods, and accepted no input from laymen. Once introduced, mathematical results imposed themselves with tyrannical, power, demanding perfect assent and no opposition. This, of course, was precisely what Hobbes so admired about mathematics, but it was also what Boyle and his fellows feared: mathematics, by its very nature, they believed, leads to claims of absolute truth, dogmatism, threats of tyranny." (256)

JED Z. BUCHWALD & I. BERNARD COHEN (EDS.), *Isaac Newton's Natural Philosophy*, MIT Press, 2004.

The Royal Society in London was long fraught with tension between mathematicians and naturalists. "There has been much canvassing and intrigue made use of, as if the fate of the Kingdome depended on it" (77). "On the eve of Newton's election as president, matters had deteriorated to such an extent that various fellows could be restrained only with difficulty from a public exchange of blows (or, in one case, the drawing of swords)" (93).

So what was this conflict on which "the fate of the Kingdome" depended? The "philomats" identifying with Newton attacked the naturalists thus: "That Great Man [Newton] was sensible, that something more than knowing the Name, the Shape and obvious Qualities of an Insect, a Pebble, a Plant, or a Shell, was requisite to form a Philosopher, even of the lowest rank, much more to qualifie one to sit at the Head of so great and learned a Body." (77)

The naturalists, for their part, identified with Bacon, who had complained about “the daintiness and pride of mathematicians, who will needs have this science almost domineer over Physic. For it has come to pass, I know not how, that Mathematics and Logic, which ought to be but the handmaids of Physic, nevertheless presume on the strength of the certainty which they possess to exercise dominion over it.” (80)

Similar points were raised many times, as here in 1700 by a minor figure: “Mathematical Arguments, of which the World is become most immoderately fond, looking upon every thing as trivial, that bears no relation to the Compasse, and establishing the most distant parts of Humane Knowledge; all Speculations, whether Physical, Logical, Ethical, Political, or any other upon the particular results of number and Magnitude. … In any other commonwealth but that of Learning such attempts towards an absolute monarchy would quickly meet with opposition. It may be a kind of treason, perhaps, to intimate thus much; but who can any longer forbear, when he sees the most noble, and most usefull portions of Philosophy lie fallow and deserted for opportunities of learning how to prove the Whole bigger than the Part, etc.” (90)

HELENA PYCIOR, *Symbols, Impossible Numbers, and Geometric Entanglements*, Cambridge University Press, 1997.

In the 17$^{\text{th}}$ century, some took “the symbolical style [of algebra] as a model for terse, scientific expression.” Indeed, the Royal Society “exacted from all their members a close, naked, natural way of speaking … bringing all things as near the Mathematicall plainness as they can.” (Sprat, 46) Others disagreed. “Symbols are poor unhandsome, though necessary, scaffolds of demonstration; and ought no more to appear in public, than the most deformed necessary business which you do in your chambers.” (Hobbes, 145) Pages of algebra often look “as if a hen had been scraping there.” (Hobbes, 147)

Negative numbers. “That which most perplexes narrow minds in this way of thinking, is, that in common life, most quantities lose their names when they cease to be affirmative, and acquire new ones so soon as they begin to be negative: thus we call negative goods, debts; negative gain, loss; negative heat, cold; negative descent, ascent, &c.: and in this sense indeed, it may not be so easy to conceive, how a quantity can be less than nothing, that is, how a quantity under any particular denomination, can be said to be less than nothing, so long as it retains that denomination.” (Saunderson, 287)

Technology in teaching. “That the true way of Art is not by Instruments, but by demonstration: and that it is a preposterous course of vulgar Teachers, to beginne with Instruments, and not with the Sciences, and so in stead of Artists, to make their Schollers onely doers of tricks, and as it were jugglers.” (Oughtred, 68)

§ R12. Do mathematical proofs explain?

R12.1. What aspects of mathematics can be construed as showing that its proofs merely demonstrate propositions logically without explaining why they are true?

R12.2. How can mathematics be defended against the charge that it is inferior to other sciences because its proofs are not explanatory and causal? (Answer this after reading §§R16, R17, R18.)

PAOLO MANCOSU, *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*, Oxford University Press, 1996.

“The classical texts of Euclid, Archimedes, Apollonius, Pappus, … constituted an extremely stable body of results whose absolute certainty had seldom been put into doubt. In the scholastic Aristotelian tradition a long list of commentators argued for this certainty by remarking that mathematics conforms to the specifications for a perfect science set down by Aristotle in his Posterior Analytics. However, during the middle of the sixteenth century several objections were raised against such justifications of the certainty of mathematics. The debate that ensued from such criticisms is known as the Quaestio de Certitudine Mathematicarum. The main issues raised by this debate were (a) Does mathematics fit the definition of Aristotelian science or does it fall short of it? This problem led in turn to a careful analysis of mathematical demonstrations. And (b) If the certainty of mathematics cannot be argued by appealing to its logical structure on what other grounds can we justify it?” (10)

“My opinion is that the mathematical disciplines are not proper sciences. … To have science is to acquire knowledge of a thing through the cause on account of which the thing is. … However, demonstration (I speak of the most perfect kind of demonstration) must depend upon those things which are ‘per se’ and proper to that which is demonstrated; indeed, those things which are accidental and in common are excluded from perfect demonstrations. But the mathematician neither considers the essence of quantity, nor treats of its affections as they flow from such essence, nor declares them by the proper causes on account of which they are in quantity, nor makes his demonstrations from proper and ‘per se’ but from common and accidental predicates. Thus mathematical doctrine is not properly science.” (Pereyra, 1562; 13)

“The geometer proves [*Elements* I.32] that the triangle has three angles equal to two right ones on account of the fact that the external angle which results from extending the side of that triangle is equal to two angles of the same triangle which are opposed to it. Who does not see that this middle is not the cause of the property which is demonstrated? … [The external angle] is related in an altogether accidental way to [the angle sum of the triangle]. Indeed, whether the side is produced and the external angle is formed or not, or rather even if we imagine that the production of the one side and the bringing about of the external angle is impossible, nonetheless that property will belong to the triangle; but, what else is the definition of an accident than what may belong or not belong to the thing without its corruption?" (Pereyra, 1562; 15)

"Piccolomini [1547] had given several arguments aimed at showing that mathematical propositions could not be proven by causal proofs. One of the arguments appealed to the authority of Proclus, who had remarked that in geometry there are propositions of equal value and dignity that can be proved from each other. [E.g., *Elements*] 1.35 and 1.36, which are the converse of each other. In Piccolomini's opinion this argues for the noncausality of such theorems. Were they causal, Piccolomini said, the same proposition would be the cause of itself. But this is a contradiction since nothing can be its own cause. The problem that Piccolomini is addressing here is the following. In mathematics one can often prove that A implies B and that B implies A—that A and B are equivalent. However, if we read 'implication' as being more than just logical consequence, and we demand the antecedent acts as the 'cause' of the conclusion then we run into trouble in trying to make sense of those theorems in which antecedents and consequences can be reversed, since in the intuitive reading of the relation of causality if A 'causes' B then B cannot 'cause' A. For, were the latter allowed, one could infer that A 'causes' A. But this goes against the intuitive reading of causality since, as Piccolomini remarks, nothing can be its own cause." (25)

ARTHUR SCHOPENHAUER, *Die Welt als Wille und Vorstellung*, 1819. Quoted from *The World as Will and Idea*, translated by R.B. Haldane & J. Kemp, 1909.

"If now with our conviction that perception is the primary source of all evidence, and that only direct or indirect connection with it is absolute truth; and further, that the shortest way to this is always the surest, as every interposition of concepts means exposure to many deceptions; if, I say, we now turn with this conviction to mathematics, as it was established as a science by Euclid, and has remained as a whole to our own day, we cannot help regarding the method it adopts, as strange and indeed perverted. We ask that every logical proof shall be traced back to an origin in perception; but mathematics, on the contrary, is at great pains deliberately to throw away the evidence of perception which is peculiar to it, and always at hand, that it may substitute for it a logical demonstration. This must seem to us like the action of a man who cuts off his legs in order to go on crutches, or like that of the prince in the *Triumph der Empfindsamkeit* who flees from the beautiful reality of nature, to delight in a stage scene that imitates it. …

Instead of … giving a thorough insight into the nature of the triangle, [Euclid] sets up certain disconnected arbitrarily chosen propositions concerning the triangle, and gives a logical ground of knowledge of them, through a laborious logical demonstration, based upon the principle of contradiction. Instead of an exhaustive knowledge of these space-relations we therefore receive merely certain results of them, imparted to us at pleasure, and in fact we are very much in the position of a man to whom the different effects of an ingenious machine are shown, but from whom its inner connection and construction are withheld. We are compelled by the principle of contradiction to admit that what Euclid demonstrates is true, but we do not comprehend why it is so. We have therefore almost the same uncomfortable feeling that we experience after a juggling trick, and, in fact, most of Euclid's demonstrations are remarkably like such feats. The truth almost always enters by the back door, for it manifests itself per accidens through some contingent circumstance. Often a reductio ad absurdum shuts all the doors one after another, until only one is left through which we are therefore compelled to enter. Often, as in the proposition of Pythagoras, lines are drawn, we don't know why, and it afterwards appears that they were traps which close unexpectedly and take prisoner the assent of the astonished learner. … The proposition of Pythagoras teaches us a qualitas occulta of the right-angled triangle. …

This specially empirical and unscientific knowledge is like that of the doctor who knows both the disease and the cure for it, but does not know the connection between them. But all this is the necessary consequence if we capriciously reject the special kind of proof and evidence of one species of knowledge, and forcibly introduce in its stead a kind which is quite foreign to its nature. However, in other respects the manner in which this has been accomplished by Euclid deserves all the praise which has been bestowed on him through so many centuries, and which has been carried so far that his method of treating mathematics has been set up as the pattern of all scientific exposition. Men tried indeed to model all the sciences after it, but later they gave up the attempt without quite knowing why. Yet in our eyes this method of Euclid in mathematics can appear only as a very brilliant piece of perversity." (Volume I, Book I, §15)

§ R13. Did Greek mathematics have a hidden method?

R13.1. Why would the Greeks have hidden their methods?

R13.2. What "historical methodology" did these mathematicians use to make inferences about Greek mathematical thought?

EVANGELISTA TORRICELLI, *Quadratura Parabolae per novam indivisibilium Geometriam pluribus modis absoluta*, 1644, translation by Andrew Leahy, *Convergence*, February 2017.

"For my part I would not dare to assert that this Geometry of Indivisibles is a thoroughly new invention. Rather, I would have believed that the old geometers used this one method in the discovery of the most difficult theorems, although they would have produced another way more acceptable in their demonstrations, either for concealing the secret of the art or lest any opportunity for contradiction be proffered to envious detractors."

RENÉ DESCARTES, quoted from *Philosophical Writings*, Modern Library, 1958, 15.

"We have sufficient evidence that the ancient geometers made use of a certain analysis which they applied in the resolution of their problems, although, as we find, they grudged to their successors knowledge of this method."

PIERRE DE FERMAT, quoted from Michael Sean Mahoney, *The Mathematical Career of Pierre de Fermat*, Princeton University Press, 1973, 119.

"Those long buried monuments of geometry in which so many great findings of the Ancients lie with the roaches and worms."

GOTTFRIED WILHELM LEIBNIZ, quoted from *New Essays on Human Understanding*, translated by P. Remnant & J. Bennett, Cambridge University Press, 1996, 489.

"The art of symbols is a marvellous aid, in that it unburdens the imagination. If we look at Diophantus's *Arithmetic* and the geometrical treatises of Apollonius and Pappus, we shall not doubt that the ancients had something of it."

JOHN WALLIS, *A treatise of algebra*, London, 1685.

"It is to me a thing unquestionable, That the Ancients had somewhat of like nature with our Algebra; from whence many of their prolix and intricate Demonstrations were derived. And I find other modern Writers of the same opinion therein. … But this their Art of Invention, they seem very studiously to have concealed: contenting themselves to demonstrate by Apagogical Demonstrations, (or reducing to Absurdity, if denied,) without shewing us the method, by which they first found out those Propositions, which they thus demonstrate by other ways.

Of which, Nunes … speaks thus: 'O how well had it been if those Authors, who have written in Mathematics, had delivered to us their Inventions, in the same way, and with the same discourse, as they were found out! And not as Aristotle says of Artificers in Mechanics, who shew us the Engines they have made, but conceal the Artifice, to make them the more admird! The method of Invention, in divers Arts, is very different from that of Tradition, wherein they are delivered. Nor are we to think, that all these Propositions in Euclid and Archimedes were in the same way found out, as they are now delivered to us.'" (3)

§ R14. Ancient cosmology

R14.1. Why did the Greeks describe planetary motions by combinations of circles?

R14.2. Were Greek astronomers "realists" (i.e., believed their planetary models corresponded to physical reality) or "instrumentalists" (i.e, considered their theories to be nothing more than recipes for calculation and prediction)?

R14.3. How has our view of the relation between heaven and earth changed over time?

PTOLEMY, *Almagest*, c. 150. Translated by G. Toomer.

"It is our purpose to demonstrate for the five planets that all their apparent anomalies can be represented by uniform circular motions, since these are proper to the nature of divine beings, while disorder and non-uniformity are alien [to such beings]." (IX.2)

ARISTOTLE, *De Caelo (On the Heavens)*, c. –350. Translated by J. L. Stocks.

"Bodies are either simple or compounded of such; and by simple bodies I mean those which possess a principle of movement in their own nature, such as fire and earth … For if the natural motion is upward, it will be fire or air, and if downward, water or earth. … It follows that circular movement also must be the movement of some simple body. For the movement of composite bodies is … determined by that simple body which preponderates in the composition. These premises clearly give the conclusion that there is in nature some bodily substance other than the formations we know, prior to them all and more divine than they." (I.2)

PTOLEMY, *Planetary Hypotheses*, c. 150. Quoted from Bernard R. Goldstein, The Arabic version of Ptolemy's Planetary Hypotheses, *Transactions of the American Philosophical Society*, 57(4), 1967.

"The distances of the … planets may be determined without difficulty from the nesting of the spheres, where the least distance of a sphere is considered equal to the greatest distance of a sphere below it." (7) That is to say, according to the epicyclic planetary models presented in the Almagest, each planet sways back and forth between a nearest and a furthest distance from the earth. The "sphere" of each planet must be just thick enough to contain these motions. This argument assumes that "there is no space between the greatest and least distances [of adjacent spheres]," which "is most plausible, for it is not conceivable that there be in Nature a vacuum, or any meaningless and useless thing. … But if there is space or emptiness between the [spheres], then it is clear that the distances cannot be smaller, at any rate, than those mentioned." (8)

§ R15. Kepler

R15.1. Did Kepler's religious and mystical beliefs help or hinder his science?

R15.2. What aspects of Kepler's worldview can be reconciled with a modern atheistic outlook?

R15.3. Was Kepler similar to Plato?

EDWARD ROSEN (ED.), *Kepler's Conversation with Galileo's Sidereal Messenger*, Johnson Reprint Corp., 1965.

The recent telescopic discovery of the moons of Jupiter lead to the conclusion that Jupiter is inhabited. Why else would it have moons? "For whose sake, the question arises, if there are no people on Jupiter to behold this wonderfully varied display with their own eyes?" (40) "We deduce with the highest degree of probability that Jupiter is inhabited." (44)

But the earth is still privileged (45-46): (1) It is in the middle (three bodies below, three above). (2) Its orbit touches the icosahedron and the dodecahedron, which is the most distinguished position. (3) It sees all the planets. On Jupiter they cannot see Mercury because it is too close to the sun. "Will anyone then deny that, to make up for the planets concealed from the Jovians but visible to us earth-dwellers, four other planets are allocated to Jupiter, to match the four inferior planets … which revolve around the sun within Jupiter's orbit. Let the Jovian creatures, therefore, have something with which to console themselves."

MAX CASPAR, *Kepler*, Dover, 1993.

"Aesthetic-artistic consideration of the universe" (382). "I consider it my duty and task … to advocate … what I … have recognized as true and whose beauty fills me with unbelievable rapture on contemplation." (298). "I may say with truth that whenever I consider in my thoughts the beautiful order, how one thing issues out of and is derived from another, then it is as though I had read a divine text, written onto the world itself … saying: Man, stretch thy reason hither, so that thou mayest comprehend these things" (152).

Mathematics a means to this end. "Kepler consciously renounced [Archimedean] rigor and wanted to take over from Archimedes only so much as 'is sufficient for the pleasure of the lovers of geometry.'" (234). "Don't sentence me completely to the treadmill of mathematical calculations and leave me time for philosophical speculations, which are my sole delight. Each one has his own particular pleasure, one the tables and nativities, I the flower of astronomy, the artistic structure of the motions." (308).

Man's cognitive abilities designed for this purpose. "The world partakes of quantity and the mind of man grasps nothing better than quantities for the recognition of which he was obviously created." (96). "Nature loves these relationships in everything that is capable of thus being related. They are also loved by the intellect of man who is an image of the Creator." (94).

The universe designed for this purpose. "The earth's axis in inclined to the ecliptic in consideration of the people distributed over the whole surface of the earth, so that the change of the heavenly phenomena should extend to all places on the earth and consequently all people have a share in it. … Sun and moon have the same apparent sizes, so that the eclipses, one of the spectacles arranged by the Creator for instructing observing creatures in the orbital relations of the sun and the moon, can occur. The earth moves around the sun to make it possible for man to get to know the world and its dimensions." (296).

Reception of the above. These ideas were quite well received e.g. in the case of the *Mysterium Cosmographicum*: "Professor Georg Limnäus in Jena … is ecstatic that at last someone had again revived the time-honoured Platonic art of philosophising. … [Tycho Brahe] takes unusual pleasure in the book: … the zeal, the fine understanding and acumen ought to be praised [even though] certain details give him pause." (69-70). It was different with the more modern physics of the *Astronomia Nova*: "Kepler ran up against rejection and lack of understanding on all sides. Maestlin, Fabricius, Longomontanus and others shook their heads." (135).

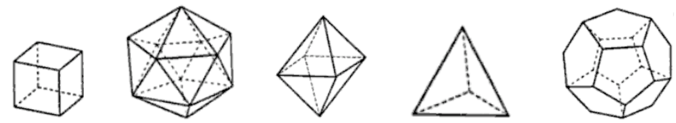
CHARLOTTE METHUEN, *Kepler's Tübingen: Stimulus to a Theological Mathematics*, Scolar Press, 1998.

The human mind is created to do mathematics. According to Melanchthon, the atomistic doctrines of creation by chance "wage war against human nature, which was clearly founded to understand divine things" (76); astronomical observations are as natural to a human being as "swimming to a fish or singing to a nightingale" (85).

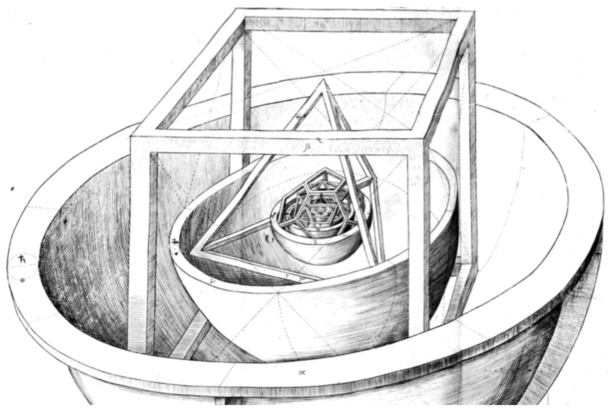
The purpose of scientific study is therefore twofold. (1) "inflaming their souls with love and enthusiasm for the truth and rousing them to understanding of the noblest things" (Melanchthon, 73). (2) Astronomers are "priests of the book of nature" (Kepler, 206n3). The existence of God follows from the universe's "beauty, order, and all things which have been founded for settled purposes" (Heerbrand, 137). "God desired that knowledge of the wonderful courses and powers should lead us towards knowledge of the divine" (Melanchthon, 76).

JOHANNES KEPLER, *Mysterium Cosmographicum: The Secret of the Universe [1596]*, ABaris Books, 1981.

"Greetings, friendly reader. The nature of the universe, God's plan for creating it, God's source for the numbers, … the reason why there are six orbits, the spaces which fall between al the spheres … —here Pythagoras reveals all this to you by five figures." (49) Namely, there are five regular polyhedra:



And the spheres of the planets are nested in such a way that the regular polyhedra fit precisely between them:



The ordering of the polyhedra in this arrangement is justified by a myriad arguments, of which the following is a representative sample:

"[The regular polyhedra] are classified into three primaries, the cube, tetrahedron and dodecahedron, and two secondaries, the octahedron and the icosahedron. For the correctness of this distinction, note the properties of each class. … 2. Every one of the primaries has its particular type of face: the cube has the square, the pyramid the triangle, the dodecahedron the pentagon; the secondaries borrow the triangular face from the pyramid. … 6. It is characteristic of the primaries to stand upright, of the secondaries to balance on a vertex. For if you roll the latter onto their base, or stand the former on a vertex, in either case the onlooker will avert his eyes at the awkwardness of the spectacle. … Therefore, since there was an obvious distinction between the solids, nothing could be more appropriate than that our Earth, the pinnacle and pattern of the whole universe, and therefore the most important of the moving stars, should by its orbit differentiate between the two classes stated, and should be allotted the position which we have attributed to it above." (105)

What is the purpose of astronomy? "As we do not ask what hope or gain makes a little bird warble, since we know that it takes delight in singing because it is for that very singing that a bird was made, so there is no need to ask why the human mind undertakes such toil in seeking out these secrets of the heavens. … The reason why there is such a great variety of things, and treasuries so well concealed in the fabric of the heavens, is so that fresh nourishment should never be lacking for the human mind, and it … should have in this universe an inexhaustible workshop in which to busy itself." (55)

§ R16. Descartes

R16.1. What did Descartes see as the key aspects of geometrical reasoning that were to be generalised to other domains of thought?

R16.2. What is the relation between God and geometry, according to Descartes?

R16.3. What is the relation between geometry and practical construction, according to Descartes?

R16.4. Descartes is famous for introducing algebraic methods in geometry. What did he see as the purpose of this?

René Descartes, *A Discourse on the Method*, 1637, translated by Ian Maclean, Oxford University Press, 2008.

"I was most keen on mathematics, because of its certainty and the incontrovertibility of its proofs; but I did not yet see its true use. Believing as I did that its only application was to the mechanical arts, I was astonished that nothing more exalted had been built on such sure and solid foundations." (9 = AT 7)

"The long chains of reasonings, every one simple and easy, which geometers habitually employ to reach their most difficult proofs had given me cause to suppose that all those things which fall within the domain of human understanding follow on from each other in the same way, and that as long as one stops oneself taking anything to be true that is not true and sticks to the right order so as to deduce one thing from another, there can be nothing so remote that one cannot eventually reach it, nor so hidden that one cannot discover it. And I had little difficulty in determining those with which it was necessary to begin, for I already knew that I had to begin with the simplest and the easiest to understand; and considering that of all those who had up to now sought truth in the sphere of human knowledge, only mathematicians have been able to discover any proofs, that is, any certain and incontrovertible arguments, I did not doubt that I should begin as they had done." (17-18 = AT 19)

Descartes's general method:

"The first [principle of my method] was never to accept anything as true that I did not incontrovertibly know to be so; that is to say, carefully to avoid both prejudice and premature conclusions; and to include nothing in my judgements other than that which presented itself to my mind so clearly and distinctly, that I would have no occasion to doubt it." (17 = AT 18)

"The second was to divide all the difficulties under examination into as many parts as possible, and as many as was required to solve them in the best way." (17 = AT 18)

"The third was to conduct my thoughts in a given order, beginning with the simplest and most easily understood objects, and gradually ascending, as it were step by step, to the knowledge of the most complex; and positing an order even on those which do not have a natural order or precedence." (17 = AT 18)

"The last was to undertake such complete enumerations and such general surveys that I would be sure to have left nothing out." (17 = AT 19)

RENÉ DESCARTES, *Principles of Philosophy*, 1644, translated by R. P. Miller, Springer, 1982.

Descartes's philosophical method is modelled on the method of Euclid's Elements, as is clear from Descartes's preface:

"One must begin by searching for … first causes, that is, for Principles [which] must be so clear and so evident that the human mind cannot doubt of their truth when it attentively considers them … And then, one must attempt to deduce from these Principles the knowledge of the things which depend upon them, in such a way that there is nothing in the whole sequence of deductions which one makes from them which is not very manifest." (xvii-xviii)

But Descartes is not content with merely adopting the Euclidean method—he also justifies it. He does this by showing that it survives even the most critical examination possible, namely that announced in the first sentence of the text: "whoever is searching for truth must, once in his life, doubt all things" (I.1).

The Euclidean method is the only philosophical method to survive this critical abyss, by the following chain of reasoning.

First we prove our own existence. "We can indeed easily suppose that there is no God, no heaven, no material bodies; and yet even that we ourselves have no hands, or feet, in short, no body; yet we do not on that account suppose that we, who are thinking such things, are nothing: for it is contradictory for us to believe that that which thinks, at the very time when it is thinking, does not exist. And, accordingly, this knowledge, *I think, therefore I am*, is the first and most certain to be acquired by and present itself to anyone who is philosophizing in correct order." (I.7)

"The knowledge of remaining things depend on a knowledge of God," because the next things the mind feels certain of are basic mathematical facts, but it cannot trust these judgments unless it knows that its creator is not deceitful. Thus "the mind … discovers [in itself] certain common notions, and forms various proofs from these; and as long as it is concentrating on these proofs it is entirely convinced that they are true. Thus, for example, the mind has in itself the ideas of numbers and figures, and also has among its common notions, *that if equals are added to equals, the results will be equal*, and other similar ones; from which it is easily proved that the three angles of a triangle are equal to two right angles, etc." But the mind "does not yet know whether it was perhaps created of such a nature that it errs even in those things which appear most evident to it." Therefore "the mind sees that it rightly doubts such things, and cannot have any certain knowledge until it has come to know the author of its origin." (I.13)

The existence of God is established to Descartes's satisfaction by several dubious arguments, most notably the following. "Just as, for example, the mind is entirely convinced that a triangle has three angles which are equal to two right angles, because it perceives that the fact that its three angles equal two right angles is necessarily contained in the idea of a triangle; so, solely because it perceives that necessary and eternal existence is contained in the idea of a supremely perfect being, the mind must clearly conclude that a supremely perfect being exists." (I.14) And all the more since it is "very well know from [our] natural enlightenment" "that that which is more perfect is not produced by an efficient and total cause which is less perfect; and moreover that there cannot be in us the idea or image of anything, of which there does not exist somewhere (either in us or outside us), some Original, which truly contains all its perfections. And because we in no way find in ourselves those supreme perfections of which we have the idea; from that fact alone we rightly conclude that they exist, or certainly once existed, in something different from us; that is, in God." (I.18)

"It follows from this that all the things which we clearly perceive are true, and that the doubts previously listed are removed" (I.30), since "God is not the cause of errors," owing to his perfection, seeing as "the will to deceive certainly never proceeds from anything other than malice, or fear, or weakness; and, consequently, cannot occur in God." (I.29) "Thus, Mathematical truths must no longer be mistrusted by us, since they are most manifest." (I.30)

In the same way we can be sure that material objects exist, since otherwise "it would be impossible to devise any reason for not thinking Him a deceiver" (II.1). But the argument forces upon us the restriction "that the nature of body does not consist in weight, hardness, color, or other similar properties; but in extension alone" (II.4), since a body can easily be conceived to be deprived of its secondary properties (cf. also II.11), but not its extension.

Physics, therefore, must be based on a theory of extended matter and nothing else. Two key characteristics of Cartesian physics follow quite naturally from this starting point, and are indeed introduced almost immediately: relativity of space (II.13-14) and contact mechanics (II.36-52).

The first is a quite unavoidable corollary of Descartes's starting point, since his perspective does not admit the possibility of space as a concept separate from body. Thus he is compelled to argue that "the names 'place' or 'space' do not signify a thing different from the body which is said to be in the place; but only designate its size, shape and situation among other bodies" (II.13). "So when we say that a thing is in a certain place, we understand only that it is in a certain situation in relation to other things" (II.14).

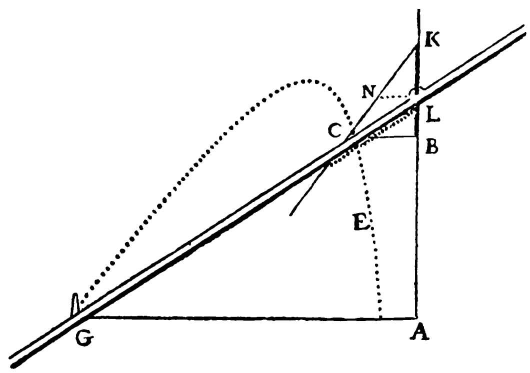
A second rather straightforward consequence of Descartes's starting point is that contact mechanics is the fundamental phenomena in terms of which all other physics must be construed. And indeed Descartes offers a detailed account of contact mechanics almost at once, in II.36-52.

René Descartes, *The Geometry*, translated by D. E. Smith & M. L. Latham, Dover, 1954.

The main theme of Descartes's geometry is the justification of algebraic methods in terms of the standards of classical, construction-based geometry.

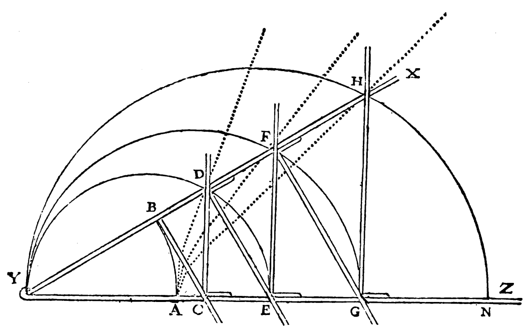
"To treat all the curves I mean to introduce here [i.e., all algebraic curves], only one additional assumption [beyond ruler and compasses] is necessary, namely, [that] two or more lines can be moved, one [by] the other, determining by their intersection other curves. This seems to me in no way more difficult [than the classical constructions]." (43)

For example, here the rigid triangle KNL is made to move vertically along the axis $ABLK$:



This in turn moves the ruler attached at L, which is also constrained by the peg fixed at G. Therefore the ruler makes a mostly rotational motion as the triangle moves upwards. The intersection C of the ruler and the extension of KN defines the traced curve, in this case a hyperbola.

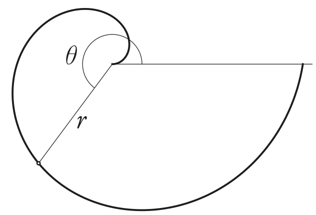
Similarly, here the ruler YZ is fixed and the ruler YX is turning counter-clockwise about its fixed point Y:



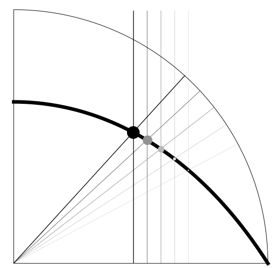
The ruler BC is attached perpendicularly to YX at B. As YX turns, BC pushes the next ruler CD rightwards (as its foot is constrained so as to move along YZ only). The point D traces the desired curve (dotted). By attaching further rulers, each of which is in turn pushed by the previous one (DE by CD, EF by DE, and so on), the mesolabe can generate curves of higher and higher order (traced by F, H, etc.).

"I could give here several other ways of tracing and conceiving a series of curved lines, each curve more complex than any preceding one, but I think the best way to group together all such curves and then classify them in order, is by recognizing the fact that all points of those curves which we may call 'geometric', that is, those which admit of precise and exact measurement, must bear a definite relation to all points of a straight line, and that this relation must be expressed by means of a single equation" (48). In other words, the legitimate curves of exact geometry are precisely those representable by algebraic equations in rectilinear coordinates.

In contrast to algebraic curves, "the spiral,



the quadratrix,



and similar curves … are not among those curves that I think should be included here, since they must be conceived of as described by two separate movements whose relation does not admit of exact determination" (44). In other words, these kinds of curves involve independent linear and circular motions whose speeds need to be coordinated, but how is this coordination supposed to be achieved?

René Descartes, *Rules for the direction of the mind*, Liberal Arts Press, 1961.

Mathematics is the model for all knowledge. "In seeking the correct path to truth we should be concerned with nothing about which we cannot have a certainty equal to that of the demonstrations of arithmetic and geometry." (II, 367) "And although I speak a good deal here of figures and numbers … nevertheless anyone who pays close attention to my meaning will easily observe that I am not thinking at all of common mathematics, but I am setting forth a certain new discipline … broad enough to bring out the truths of any subject whatsoever." (IV, 375) The power of mathematics stems from "certain basic roots of truth implanted in the human mind by nature, which we extinguish in ourselves daily by reading and hearing many varied errors" (IV, 377).

Proofs should be intuited as wholes. Deductions "may sometimes be accomplished through such a long chain of inferences that when we have arrived at the conclusions we do not easily remember the whole procedure which led us to them … Because of this, I have learned to consider each of these steps by a certain continuous process of the imagination … Thus I go from first to last so quickly that by entrusting almost no parts of the process to the memory, I seem to grasp the whole series at once." (VII, 388-389) "In this way our knowledge is made much more certain and the capacity of our minds is increased as much as possible." (XI, 408)

To achieve this end "we must make use of every assistance of the intellect, the imagination, the senses, and the memory" (XII, 411). "By the aid of each faculty … human efforts can serve to repair the deficiencies of the mind." (XII, 417). For example, "if the intellect proposes to examine something which can be related to the body it should produce in the imagination the most distinct idea of it possible; and in order to do this more readily, the object which this idea represents should be exhibited to the external senses." (XII, 417-418). "We are to do nothing from this point on without the aid of the imagination." (XIV, 444)

Algebra and analytic geometry is intended to be precisely such an aid to the intuition. For having recorded the steps of a proof in algebraic terms, "we can run through all of them in a very rapid movement of thought and grasp as many as possible at the same time" (XVI, 456). Intuiting the whole in this way is important to us "who are seeking evident and distinct knowledge of things; but not the arithmeticians, who are satisfied if the have discovered the number sought even though they have not noticed how it depends upon the given facts, although this latter is the only point in which science truly lies" (XVI, 459).

Another benefit of algebra in this regard. By algebra "we translate what we understand to be affirmed about magnitudes in general into that particular magnitude that we can most easily and distinctly picture in our imagination" (XIV, 442); specifically, "a magnitude should never be regarded in the imagination otherwise than as a line or a surface, even though it may be called a 'cube' or a 'biquadratic'" (XVI, 457).

A quip on why we should denote the answers we seek by a letter such as x. "It frequently happens that individuals are so eager to investigate problems that they apply their capricious intelligence to finding a solution before they have determined by what signs they will recognize the object of their search, if they should stumble upon it by accident; these persons are no less foolish that would be a boy, sent somewhere by his master, who was so eager to obey that he started to run without waiting for instructions, and without knowing where he was ordered to go." (XIII, 435)

JOHN AUBREY, *Brief Lives*, late 17th century.

"[Descartes] was so eminently learned that all learned men made visits to him, and many of them would desire him to show them his store of instruments. He would draw out a little drawer under his table and show them a pair of compasses with one of the legs broken; and then for his ruler, he used a sheet of paper folded double."

§ R17. Hobbes

R17.1. What did Hobbes see as the key aspects of geometrical reasoning that were to be generalised to other domains of thought?

R17.2. In what ways is Hobbes similar or dissimilar to Descartes?

JOHN AUBREY, *Brief Lives*, late 17th century.

"[Thomas Hobbes] was 40 years old before he looked on geometry; which happened accidentally. Being in a gentleman's library, Euclid's Elements lay open, and 'twas the 47 El. libri I [Pythagorean Theorem]. He read the proposition. By God, sayd he (he would now and then sweare an emphaticall Oath by way of emphasis), this is impossible! So he reads the Demonstration of it, which referred him back to such a Proposition; which proposition he read. That referred him back to another, which he also read. Et sic deinceps, that at last he was demonstratively convinced of that trueth. This made him in love with geometry."

THOMAS HOBBES, *Leviathan*, 1651. Quoted from *The English Works of Thomas Hobbes of Malmesbury*, Volume III.

"Geometry … is the only science that it hath pleased God hitherto to bestow on mankind" (23-24).

"There can be nothing so absurd, but may be found in the books of philosophers. And the reason is manifest. For there is not one of them that begins his ratiocination from the definitions, or explications of the names they are to use; which is a method that hath been used only in geometry; whose conclusions have thereby been made indisputable." (33)

"For all men by nature reason alike, and well, when they have good principles. For who is so stupid, as both to mistake in geometry, and also to persist in it, when another detects his error to him?" (35)

"By Philosophy is understood the knowledge acquired by reasoning, from the manner of the generation of any thing, to the properties: or from the properties, to some possible way of generation of the same; to the end to be able to produce, as far as matter, and human force permit, such effects, as human life requireth. So the geometrician, from the construction of figures, findeth out many properties thereof; and from the properties, new ways of their construction, by reasoning; to the end to be able to measure land, and water; and for infinite other uses." (664)

THOMAS HOBBES, *De Corpore*, 1655. Quoted from *The English Works of Thomas Hobbes of Malmesbury*, Volume I.

"Philosophy is such knowledge of effects or appearances as we acquire by true ratiocination from the knowledge we have first of their causes or generation." (3)

This definition is explicitly modelled on mathematics: "How the knowledge of any effect may be gotten from the knowledge of the generation thereof, may easily be understood by the example of a circle: for if there be set before us a plain figure, having, as near as may be, the figure of a circle, we cannot possibly perceive by sense whether it be a true circle or no ... [But if] it be known that the figure was made by the circumduction of a body whereof one end remained unmoved" then the properties of a circle become evident. (6)

Another way of putting it is that "The subject of Philosophy, or the matter it treats of, is every body of which we can conceive any generation." (10) Just as the domain of geometry is the set of constructible curves.

As the circle example shows, motion is a basic form of generation. Indeed motion is the foundation of geometry and physics alike. "First we are to observe what effect a body moved produceth, when we consider nothing in it besides its motion; ... from which kind of contemplation sprung that part of philosophy which is called geometry." (71) Next "we are to pass to the consideration of what effects one body moved worketh upon another; ... that is, when one body invades another body which is either at rest or in motion, what way, and with what swiftness, the invaded body shall move; and, again, what motion this second body will generate in a third, and so forwards. From which contemplation shall be drawn that part of philosophy which treats of motion ... [and ultimately] comprehend that part of philosophy which is called physics. (71-72)

"And, therefore, they that study natural philosophy, study in vain, except they begin at geometry; and such writers or disputers thereof, as are ignorant of geometry, do but make their readers and hearers lose their time." (73)

"After physics we must come to moral philosophy; in which we are to consider the motions of the mind, namely, appetite, aversion, love, benevolence, hope, fear, anger, emulation, envy, &c." (72) "For the causes of the motions of the mind are known ... And, therefore, ... by the synthetical method, and from the very first principles of philosophy, [one] may by proceeding in the same way [as in geometry and physics], come to the causes and necessity of constituting commonwealths, and to get the knowledge of what is natural right, and what are civil duties; and, in every kind of government, what are the rights of the commonwealth, and all other knowledge appertaining to civil philosophy." (73-74)

Carefully enunciated definitions are another prominent aspect of mathematics that is to be carried over into general philosophy. "Whatsoever the common use of words be, yet philosophers, who were to teach their knowledge to others, had always the liberty ... of taking to themselves such names as they please for the signifying of their meaning, if they would have it understood. Nor had mathematicians need to ask leave of any but themselves to name the figures they invented, parabolas, hyperboles, cissoeides, quadratrices, &c. or to call one magnitude A, another B." (16) "But definitions of things, which may be understood to have some cause, must consist of such names as express the cause or manner of their generation, as when we define a circle to be a figure made by the circumduction of a straight line in a plane, &c." (81-82)

Logical, syllogistic reasoning is another distinctive attribute of mathematics. In fact, "They that study the demonstrations of mathematicians, will sooner learn true logic, than they that spend time part in reading the rules of syllogizing which logicians have made; no otherwise than little children learn to go, not by precepts, but by exercising their feet." (54-55)

THOMAS HOBBES, *Six Lessons to the Savilian Professors of the Mathematics*, 1656. Quoted from *The English Works of Thomas Hobbes of Malmesbury*, Volume VII.

"Of arts, some are demonstrable, others indemonstrable; and demonstrable are those the construction of the subject whereof is in the power of the artist himself, who, in his demonstration, does no more but deduce the consequences of his own operation. The reason whereof is this, that the science of every subject is derived from a precognition of the causes, generation, and construction of the same; and consequently where the causes are known, there is place for demonstration, but not where the causes are to seek for. Geometry therefore is demonstrable, for the lines and figures from which we reason are drawn and described by ourselves; and civil philosophy is demonstrable, because we make the commonwealth ourselves." (183-184)

§ R18. Leibniz

R18.1. Did Leibniz agree or disagree with Descartes on the foundations of geometry?

R18.2. Did Leibniz agree or disagree with Descartes on the purpose of algebra in geometry?

R18.3. In what ways was Leibniz a traditionalist regarding the foundations of geometry? Why?

VIKTOR BLÅSJÖ, *Transcendental curves in the Leibnizian calculus*, Elsevier, 2017.

"Leibniz ... made it his mission in mathematics to do Descartes one better. While Descartes had pushed the boundaries of geometry to include all algebraic curves, Leibniz would push them further still and include also the curves that went beyond, or transcended, algebra—the transcendental curves. ... Thus

Leibniz faced … the problem of providing these curves with a Euclidean-style, construction-based foundation.

Leibniz arguably considered this *the* foundational problem of the day, and he did so with good reason. Transcendental curves and the quantities constructible with their aid were at this time being found indispensable in numerous branches of mathematics and physics, such as the brachistochrone in dynamics, the catenary in statics, the cycloidal path of the optimal pendulum clock in horology, the loxodrome in navigation, caustics in optics, arc lengths of ellipses in astronomy, and logarithms in computational mathematics. … But these new [curves] were profoundly incompatible with the norms of mathematical rigour of their day, as the very epithet 'transcendental' attests: though the literal meaning of this term, coined by Leibniz himself, is that these curves 'transcend all algebraic equations,' this meant by extension that they transcended geometry itself as far as the authoritative vision of Descartes was concerned. In this way these new transcendental curves exerted a profound strain on the foundations of the subject. Simply letting all transcendental curves through the gates of geometry *en masse* would be an unthinkable betrayal of what geometry had always stood for. Geometry was defined by its foundational stringency, minimalism and constructivism; this was the source of all its credibility. So to suddenly open the floodgates for transcendental curves would be much more than a bold extension of geometry: it would be, arguably, to stop doing geometry altogether in any meaningful sense of the term." (12–13)

Against this background we can easily understand why Leibniz was so eager to stress that: "I do not in the least pretend to the glory of being an innovator … On the contrary I normally find that the oldest and commonly received opinions are the best. And I do not think one can be accused of being an innovator when on produces only a few new truths, without overturning established opinions. For this is what geometers do and all who penetrate more deeply." (26)

But at the same time "Leibniz was certainly very impressed by the recent triumphs of analytical methods, which 'reduce everything from imagination to analysis'. Indeed he envisioned this as a model for stringent reasoning in general. … It was in these kinds of terms that Leibniz saw the greatness of his infinitesimal calculus: 'As far as the differential calculus is concerned, I admit that there is much in common between it and the things which were explored by both you [Wallis] and Fermat and others, indeed already by Archimedes himself. Yet now the matter is perhaps carried much further, so that now those things can be accomplished which in the past seemed closed even to the greatest geometers as Huygens himself recognised. The matter is almost the same in the analytical calculus applied to conical curves or higher: Who does not consider Apollonius and other ancients to have had theorems which present matters for the equations by which Descartes later preferred to designate curves. In the meantime the matter has been reduced to calculation by the method of Descartes, so that now conveniently and without trouble that can be done which formerly required much effort of contemplation and imagination. In the same way, by our differential calculus, transcendentals too, which Descartes himself excluded in the past, are subjected to analytical operations.'

Or more succinctly: 'For what I love most in this calculus is that it gives us the same advantage over the ancients in the geometry of Archimedes as Viete and Descartes gave us in the geometry of Euclid and Apollonius; and it dispenses with the efforts of the imagination.'

In sum, there can be no doubt that Leibniz attributed the utmost importance to the analytical side of mathematics. To him it was absolutely essential that whatever solution of the problem of transcendental curves one may come up with, it must in any case be accompanied by a successful analytical method comparable to that of Descartes.

However, despite this—despite analytic expressions being 'what I love most'—Leibniz would not let this displace the construction paradigm as the foundations of geometry. To him, as to Descartes, curves were properly defined and made geometrical only by construction; their analytic representations were but a welcome bonus. Thus when Leibniz needs to justify the inclusion of transcendental curve in geometry he falls back on their construction by motion. For example…: '[Certain problems] transcend all algebraic equations. Yet since these problems can nevertheless actually be proposed in geometry, nay should even be considered among the foremost ones, … it is therefore certainly necessary to receive such curves into geometry, by which alone [such problems] can be constructed. And since they can be drawn exactly by a continuous motion, as is clear for the cycloid and similar [curves], they are to be considered not mechanical but geometrical, especially since by their usefulness they leave the curves of ordinary geometry (if you except the line and the circle) far behind, and have properties of the greatest importance, which are entirely capable of geometrical demonstrations.'

[Or again:] 'Descartes, in order to maintain the universality and sufficiency of his method, found it appropriate to exclude from geometry all the problems and all the curves which could not be subjected to this method, under the pretext that these things were only mechanical. Since, however, these problems and lines can be constructed or conceived by means of certain exact motions, and have important properties, and nature often uses them, one may say that he commits the same error as one who criticises some ancients for restricting themselves to constructions for which one needs nothing but ruler and compass, as if all the rest was mechanical.'

In short, transcendental curves are ultimately justified in terms of their construction, not in terms of their analytical representations. This insistence on retaining both the analytic and construction-based paradigms leads to a fundamental conflict acknowledged, somewhat reluctantly, by Leibniz: 'And I must admit that, other things being equal, I like constructions by motion better than pointwise ones, and when the motion is of proper simplicity I consider it not as mechanical but as geometrical. The pointwise construction does indeed lend itself more conveniently to analytical calculation. But properly speaking one is not concerned about this in geometry.'

The point here is that an equation of the form $y = f(x)$ is effectively a recipe for pointwise construction: pick some point x on the axis, raise a perpendicular above it, and mark off the height $f(x)$ on this perpendicular. Though no one minds this anymore, it is still true today: the y-values of the graphs of, say, a trigonometric function are defined not in terms of a single generation of this graph but in terms of separate circle-measurements for each x-value. We may have a difficult time seeing this as a drawback today but Huygens makes a compelling case:

'One cannot say that the description of a curved line through found points is geometrical, that is to say complete, or that lines so described can serve as a geometrical construction for some problems, because for this, in my opinion, no curved lines can serve except those that can subsequently be described by some instrument, as the circle by a pair of compasses; and the conic sections, conchoids and others by the instruments invented thereto. For the lines drawn by hand from point to point can only give the sought quantity approximately and consequently not according to geometrical perfection. For what does it help to find as many points as one wishes, in case one does not find the one point that is sought?'

By extension, then, this is a case against accepting formulas such as $\cos(x)$, $\arcsin(x)$, $\log(x)$, e^x, etc., as legitimate solutions of geometrical problems. In the 18th century these kinds of expressions were increasingly seen as self-sufficient, but Leibniz's generation would accept nothing of the sort, since doing so would mean giving up the construction-based paradigm and with it all the accumulated credibility of classical geometry."(14–17)

"Constructions are important because they reduce curves to motion, i.e., to a primitive, intuitively given notion. 'Those real definitions are most perfect which resolve the thing into simple primitive notions understood in themselves', and motion is ultimately the core primitive notion of geometry. Thus, for example, Leibniz speaks of 'pure mathematics, that is, mathematics which contains only numbers, figures, and motions'." "Leibniz's focus on constructions as the foundations of mathematics allows him a way out of [an] Aristotelian conundrum [namely that of §R12]: '[Geometry] does demonstrate from causes. For it demonstrates figures from motion; from the motion of a point a line arises, from the motion of a line a surface, from the motion of a surface a body. … Thus the constructions of figures are motions, and the properties of figures, being demonstrated from their constructions, therefore come from motion, and hence, *a priori*, from a cause.' Thus, basing geometry on constructions imposes a natural order—a causal hierarchy, as it were—on its theorems whence Aristotle's ideal of demonstrative understanding can be maintained." (45)

§ R19. Rationalism versus empiricism

R19.1. What aspects of classical geometry would you highlight if you wanted to justify the rationalistic interpretation? The empiricist one?

R19.2. Newton and Leibniz disagreed on what it means to treat gravity scientifically. In what way does their disagreement parallel their views on the nature of geometry?

VIKTOR BLÅSJÖ, *Transcendental curves in the Leibnizian calculus*, Elsevier, 2017.

"For two millennia the method embodied in Euclid's *Elements* was the gold standard of exact reasoning. By the time of the Renaissance and the scientific revolution it had also passed the test of time with flying colours: while it seemed that all other teachings invariably crumbled in the face of expanding knowledge and experience, the Euclidean edifice not only stood without a scratch but also proved an indispensable foundation for the most exciting new advances in the understanding of the world. The obvious message was not lost on reflective minds: If you seek certain and eternal truth then you better do whatever it was that Euclid did.

But what was it about the Euclidean method that made is so uniquely successful, and how could it be generalised beyond its traditional scope? Today the phrase 'axiomatic-deductive method' is often used to try to capture its essence, and indeed it is based on a small set of axioms, and indeed it proceeds meticulously through short, stringently verified deductive steps. But 17th century eyes saw something more in Euclid, something to which subsequent generations have grown increasingly blind. To them the ideal of the Euclidean method represented not a specialised, formal way of studying geometry, but a model of reasoning in general and our only reliable window toward an understanding of the nature of knowledge.

There were in fact two competing interpretations of the Euclidean method in the 17th century. They are summarised and contrasted in [Table 1]. As we see, these interpretations generalise the Euclidean method not only to an expanded view of geometry but also to physics and even philosophy in general. Descartes's famous phrase *cogito ergo sum* ('I think therefore I am') encapsulates his view: one starts in complete ignorance and nothingness and can only build up one's knowledge from the most immediate and undeniable principles. Newton's view, by contrast, is summed up in his statement: 'As in mathematics, so in natural philosophy, the investigation of difficult things by the method of analysis ought ever to precede the method of composition.' That is to say, instead of the Cartesian method of 'composing' all knowledge from intuitive starting principles, Newton advocates its opposite: analysis, i.e., starting with all the things one wants to understand and then trying to reduce them to simple principles. Euclid's *Elements* and Newton's *Principia* both start with a few simple axioms and deduce increasingly more complex results from them, but this, according to Newton, is not to be seen as mirroring the process of acquiring knowledge. This 'method of composition,' or synthesis, is but a mode of presentation adopted after the fact, for the sake of consolidating and clarifying logically the insights gained through analysis." (205–206)

"The defining characteristic of Cartesian physics is its insistence on explaining everything in terms of contact mechanics. Leibniz agreed completely: 'A body is never moved naturally except by another body that touches and pushes it . . . Any other kind of operation on bodies in either miraculous or imaginary.' Whence his famous conflict with Newton on the nature of gravity. Leibniz condemns very fiercely the notion of gravity as a primitive cause: 'I maintain that the attraction of bodies, properly called, is a miraculous thing, since it cannot be explained by the nature of bodies.' ... [Newton on the other hand was] largely content to simply stipulate gravity. 'With the cause of gravity [I] meddle not', says Newton, since 'I have so little fancy to things of this nature'." (58)

Another way of putting it is this. It would be very convenient to postulate:

- In classical geometry, solutions of classical construction problems.
- In early modern geometry, formulas as definitions of curves.
- In physics (as shown by Newton), action at a distance and absolute space.

But, from an operationalist point of view, it would be epistemologically irresponsible to do so. Instead one must speak only of what is:

- Generated by very simple processes.

  Because then the intuitive warrant of starting principles propagate through the entire theory.

  Geometrical constructions by ruler, compass, and other basic tools have this property. As does the mechanistic program of reducing physics to the collisions of bodies.

  Newtonian action at a distance, by contrast, violates this principle.

- Reducible to concrete operations grounded in physical experience.

  Because operational statements are meaningful, knowable, testable; non-operational statements often not. Experience is consistent; speculative thought often not.

  The operational reading of geometrical theorems has exactly these properties, as we have seen. In physics, the same logic leads to the insistence that spatial relations are always relative because that is the only empirical knowledge we can have of them: by measurements we can determine whether a body is moving with respect to another, but not whether it is "really" moving in some absolute sense. Newton, on the contrary, stipulated an absolute space as a universal frame of reference for his physics, so that the velocity and position of a body have a fixed meaning in and of itself, not only in relation to other bodies, which violates the principle and is not an operationally meaningful notion.

ISAAC NEWTON, The Author's Preface to *Philosophiae Naturalis Principia Mathematica* (1687); trans. Andrew Motte (1729).

"The description of right lines and circles, upon which Geometry is founded, belongs to Mechanics. Geometry does not teach us to draw these lines, but requires them to be drawn. For it requires that the learner should first be taught to describe these accurately, before he enters upon Geometry; then it shews how by these operations problems may be solved. To describe right lines and circles are problems, but not geometrical problems. The solution of these problems is required from Mechanics; and by geometry the use of them, when so solved, is shewn. And it is the glory of Geometry that from those few principles, fetched from without, it is able to produce so many things. Therefore Geometry is founded in mechanical practice."

§ R20. Analytical mathematics

R20.1. How has the meaning of "analysis" changed over time? Identify at least three stages (Greek, 17th century, 18th century) and explain the relations between them.

R20.2. How do 18th -century views on the role of analytic-symbolic reasoning in mathematics compare to those of Descartes (§R16) and Leibniz (§R18)?

PAPPUS, *Collection*, c. 340. Translation quoted from Heath, *A History of Greek Mathematics*, Volume 2, 400.

"Analysis, then, takes that which is sought as if it were admitted and passes from it through its successive consequences to something which is admitted as the result of synthesis: for in analysis we admit that which is sought as if it were already done and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until by so retracing our steps we come upon something already known or belonging to the class of first principles, and such a method we call analysis as being solution backwards.

But in synthesis, reversing the process, we take as already done that which was last arrived at in the analysis and, by arranging in their natural order as consequences what before were antecedents, and successively connecting them one with another, we arrive finally at the construction of what was sought; and this we call synthesis."

G. W. LEIBNIZ, *La vraie méthode*, 1677. Translation quoted from Wiener, *Leibniz selections*, 15.

"Whence it is manifest that if we could find characters or signs appropriate for expressing all our thoughts as definitely and as exactly as arithmetic expresses numbers or geometric analysis

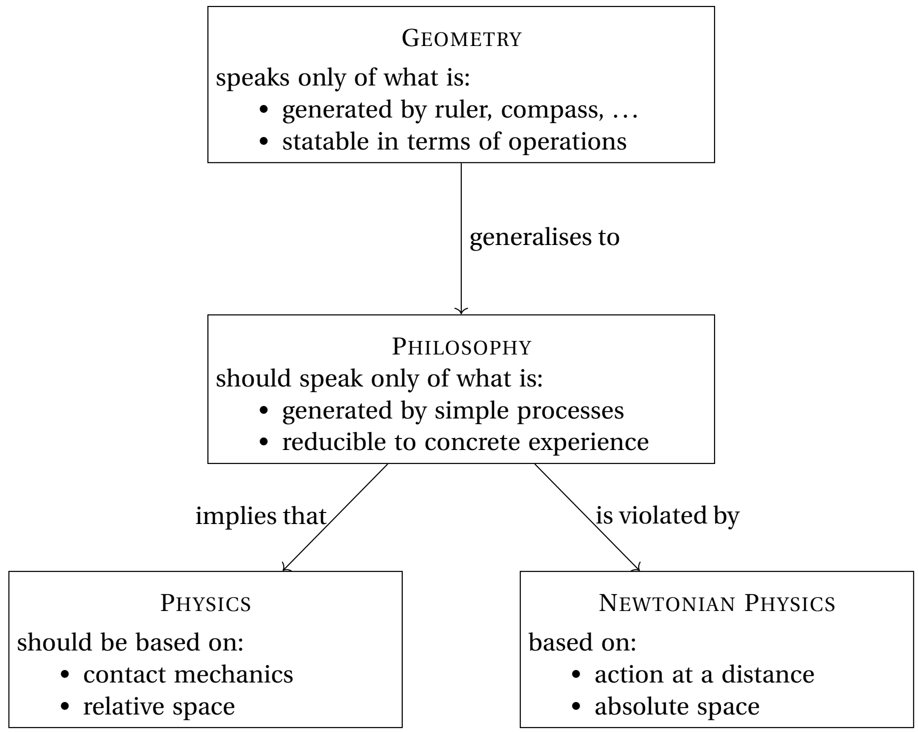


Figure 1: 17th-century operationalist view of scientific method.

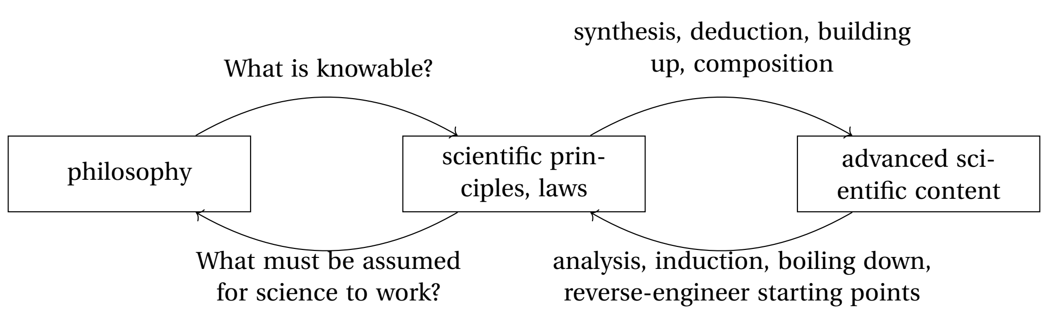


Figure 2: Opposite views on the relation between science and philosophy. Top: The view of Descartes, Leibniz, et al., according to which science must flow from philosophically justified starting principles. Bottom: The view of Newton, according to which philosophical and scientific principles are subordinated to science and retrofitted to agree with science after the fact.

	Descartes, Leibniz Continental rationalism	Newton British empiricism
The search for knowledge starts with …	intuitively clear primitive notions	the rich diversity of phenomena
… and consists in …	deducing the diversity of phenomena from them.	reducing them to a few simple principles.
Intrinsic justification of the axiomatic principles is …	immediate by their intuitive nature	external to the matter at hand
… and is therefore …	the crucial epistemological cornerstone of the entire enterprise.	of secondary importance at best.
In the case of physics, the axiomatic principles are …	the laws of contact mechanics	Newton's three force laws and the law of gravity
… which are established by means of …	their intuitively immediate nature.	induction from the phenomena.
In the case of geometry, the study of curves starts with …	the primitive intuition of local motion	the diversity of curves conceived in any exact manner whatever
… and consists in …	constructively building up a theory of all knowable curves on this basis.	investigating their properties in a systematic fashion.
Geometrical axioms are thus …	the intuitively immediate principles that define and generate the entire subject.	the outcomes of the reductive study of curves, which it was found convenient and illuminating to take as assumptions when the time came to write a systematic account.
The certainty of geometrical reasoning …	stems directly from the axioms' intuitive warrant and the constructive manner in which the rest is built up from them.	stems not from the axioms as such, but from the general method and exactitude of geometrical reasoning.

Table 1: Overview of the two competing interpretations of the Euclidean method in the 17$^{\text{th}}$ century. (From Blåsjö, 206.)

expresses lines, we could in all subjects in so far as they are amenable to reasoning accomplish what is done in Arithmetic and Geometry. For all inquiries which depend on reasoning would be performed by the transposition of characters and by a kind of calculus, which would immediately facilitate the discovery of beautiful results. For we should not have to break our heads as much as is necessary today, and yet we should be sure of accomplishing everything the given facts allow. Moreover, we should be able to convince the world what we should have found or concluded, since it would be easy to verify the calculation either by doing it over or by trying tests similar to that of casting out nines in arithmetic. And if someone would doubt my results, I should say to him: 'Let us calculate, Sir' and thus by taking to pen and ink, we should soon settle the question."

LEONHARD EULER, *Mechanica sive motus scientia analytice exposita*, 1736. Translation based on I. Bruce and M. Mahoney.

"What distracts the reader the most [in previous works on mechanics], is the fact that everything is carried out synthetically, with the demonstrations presented in the manner of the old geometry, and the analysis hidden … I always have the same trouble, when I might chance to glance through Newton's Principia … Whenever the solutions of problems seem to be sufficiently well understood by me, yet by making only a small change, I might not be able to solve the new problem using this method. Thus I have endeavoured for a long time now, to get at the analysis behind those synthetic method in order to draw out the same propositions." (Preface)

PIERRE-SIMON LAPLACE, *Exposition du système du monde*, 6th ed., Paris, 1835 = *OEuvres*, 6 (Paris, 1884). Translation quoted from Hawkins, *Emergence of the Theory of Lie Groups*, 2000, 108.

"By abandoning oneself to the operations of Analysis … one is led, by the generality of this method and by the inestimable advantage of transforming the reasoning into mechanical procedures, to results often inaccessible to synthesis. Such is the fruitfulness of Analysis that it suffices to translate particular truths into this language in order to see emerge from their very expression a multitude of new and unexpected truths." (465)

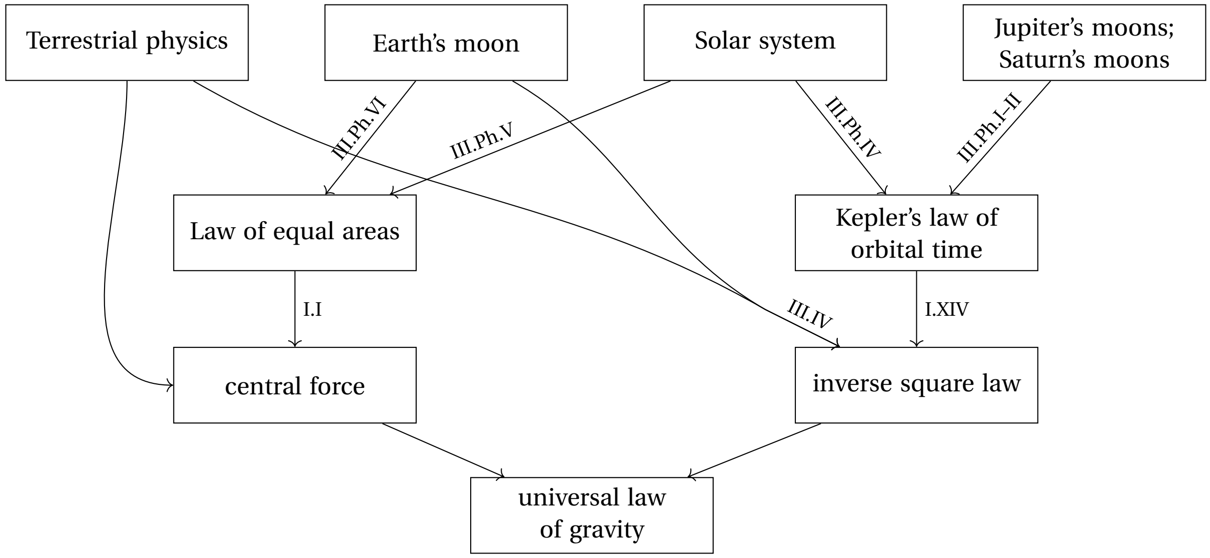


Figure 3: Flowchart illustrating Newton's method of reducing "phenomena" to universal scientific principles in the *Principia*.

LEONHARD EULER, *Foundations of Differential Calculus*, 1755, translated by J. D. Blanton, Springer, 2000.

"103. … The general infinite series that originates from the fraction

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$$

seems to labor under most serious difficulties. If for x we successively substitute the numbers 1, 2, 3, 4, …, we obtain the following series with their sums:

A. $1+1+1+1+1+\cdots = \frac{1}{1-1} = \infty$

B. $1+2+4+8+16+\cdots = \frac{1}{1-2} = -1$

C. $1+3+9+27+81+\cdots = \frac{1}{1-3} = -\frac{1}{2}$

D. $1+4+16+64+256+\cdots = \frac{1}{1-4} = -\frac{1}{3}$

and so forth. Since each term of series B, except for the first, is greater than the corresponding term of series A, the sum of series B must be much more than the sum of series A. Nevertheless, this calculation shows that series A has an infinite sum, while series B has a negative sum, which is less than zero, and this is beyond comprehension. Even less can we reconcile with ordinary ideas the results of this and the following series C, D, and so forth, which have negative sums while all of the terms are positive."

"109. From this we conclude that series of this kind, which are called divergent, have no fixed sums, since the partial sums do not approach any limit that would be the sum for the infinite series. This is certainly a true conclusion, since we have shown the error in neglecting the final remainder. However, it is possible, with considerable justice, to object that these sums, even though they seem not to be true, never lead to error. Indeed, if we allow them, then we can discover many excellent results that we would not have if we rejected them out of hand. Furthermore, if these sums were really false, they would not consistently lead to true results; rather, since they differ from the true sum not just by a small difference, but by infinity, they should mislead us by an infinite amount. Since this does not happen, we are left with a most difficult knot to unravel."

"111. These inconveniences and apparent contradictions can be avoided if we give the word *sum* a meaning different from the usual. Let us say that the sum of any infinite series is a finite expression from which the series can be derived. In this sense, the true sum of the infinite series $1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$ is $\frac{1}{1-x}$, since this series is derived from the fraction, no matter what value is substituted for x. With this understanding, if the series is convergent, the new definition of sum agrees with the usual definition. Since divergent series do not have a sum, properly speaking, there is no real difficulty which arises from this new meaning. Finally, with the aid of this definition we can keep the usefulness of divergent series and preserve their reputations."

MATTHEW WICKMAN, *Literature After Euclid: The Geometric Imagination in the Long Scottish Enlightenment*, University of Pennsylvania Press, 2016.

"The mathematical process in the symbolical [or algebraic] method is like running a railroad through a tunnelled mountain; that in the ostensive [or geometric] like crossing the mountain on foot. The former carries us, by a short and easy transit, to our destined point, but in miasma, darkness and torpidity; whereas the latter allows us to reach it only after time and trouble, but feasting us at each turn with glances of the earth and the heavens while we inhale the pleasant breeze, and gather new strength at every effort we put forth." (Willian Hamilton, 149)

§ R21. Foundations of the calculus

R21.1. How do the views of Leibniz and Euler differ on infinitesimals? What are strengths and weaknesses of each view?

R21.2. Can the views of Leibniz and Euler on infinitesimals be related to their overall mathematical style? (Cf. §§R18, R20.)

R21.3. What does the conflict between Berkeley and Maclaurin show about the relation between mathematics and religion in the 18$^{\text{th}}$ century?

DOUGLAS M. JESSEPH, Leibniz on the Foundations of the Calculus, *Perspectives on Science*, 6 (1998), 6-40.

"I have assumed in the demonstrations incomparably small quantities … If someone does not want to employ infinitely small quantities, he can take them to be as small as he judges sufficient to be incomparable, so that they produce an error of no importance and even smaller than any given [error]." (20; Leibniz, *Tentamen de motuum coelestium causis*)

"In the end, I do not dispute whether these inassignable quantities are true or fictive; it suffices that they serve for the abbreviation of thought, and they always bring with them a demonstration in a different style; and so I observed that if someone substitutes the incomparably small or that which is sufficiently small for the infinitely small, I would not oppose it." (28; Leibniz to Wallis, 30 March 1699)

"There is no need to take the infinite here rigorously, but only as when we say in optics that the rays of the sun come from a point infinitely distant, and thus are regarded as parallel. And when there are more degrees of infinity, or infinitely small, it is as the sphere of the earth is regarded as a point in respect to the distance of the sphere of the fixed stars, and a ball which we hold in the hand is also a point in comparison with the semidiameter of the sphere of the earth. And then the distance to the fixed stars is infinitely infinite or an infinity of infinities in relation to the diameter of the ball. For in place of the infinite or the infinitely small we can take quantities as great or as small as is necessary in order that the error will be less than any given error. In this way we only differ from the style of Archimedes in the expressions, which are more direct in our method and better adapted to the art of discovery." (30; Leibniz, 1701)

"Philosophically speaking, I no more admit magnitudes infinitely small than infinitely great. … I take both for mental fictions, as more convenient ways of speaking, and adapted to calculation, just like imaginary roots are in algebra. I once demonstrated that these expressions have a great use both in abbreviating thought and aiding discovery, and that they cannot lead to error, since in place of the infinitely small one may substitute [a quantity] as small as one wishes, and since any error will always be less than this, it follows that no error can be given." (34; Leibniz to Des Bosses, 11 March 1706)

GEORGE BERKELEY, *The Analyst*, London, 1734.

"A Discourse Addressed to an Infidel Mathematician." "I am not, Sir, a stranger to the reputation you have acquired in that branch of learning which hath been your peculiar study [i.e., mathematics]; nor to the authority that you therefore assume in things foreign to your profession, nor to the abuse that you, and too many more of the like character, are known to make of such undue authority, to the misleading of unwary persons in matters of the highest concernment, and whereof your mathematical knowledge can by no means qualify you to be a competent judge. Equity indeed and good sense would incline one to disregard the judgment of men, in points which they have not considered or examined. But several who make the loudest claim to those qualities do nevertheless the very thing they would seem to despise, clothing themselves in the livery of other men's opinions, and putting on a general deference for the judgment of you, Gentlemen, who are presumed to be of all men the greatest masters of reason, to be most conversant about distinct ideas, and never to take things upon trust, but always clearly to see your way, as men whose constant employment is the deducing truth by the justest inference from the most evident principles. With this bias on their minds, they submit to your decisions where you have no right to decide. And that this is one short way of making Infidels, I am credibly informed." (§1)

"The Method of Fluxions [i.e., the calculus] is the general key by help whereof the modern mathematicians unlock the secrets of Geometry, and consequently of Nature. And, as it is that which hath enabled them so remarkably to outgo the ancients in discovering theorems and solving problems, the exercise and application thereof is become the main if not sole employment of all those who in this age pass for profound geometers. But whether this method be clear or obscure, consistent or repugnant, demonstrative or precarious, as I shall inquire with the utmost impartiality." (§3)

"As our sense is strained and puzzled with the perception of objects extremely minute, even so the imagination, which faculty derives from sense, is very much strained and puzzled to frame clear ideas of the least particles of time, or the least increments generated therein: and much more so to comprehend the moments, or those increments of the flowing quantities in statu nascenti, in their very first origin or beginning to exist, before they become finite particles. And it seems still more difficult to conceive the abstracted velocities of such nascent imperfect entities. But the velocities of the velocities, the second, third, fourth, and fifth velocities, &c., exceed, if I mistake not, all human understanding." (§4)

"All these points, I say, are supposed and believed by certain rigorous exactors of evidence in religion, men who pretend to believe no further than they can see." (§7) "It must indeed be acknowledged the modern mathematicians do not consider these points as mysteries, but as clearly conceived and mastered by their comprehensive minds. … But if we remove the veil and look underneath … we shall discover much emptiness, darkness, and confusion.” (§8)

“If a man, by methods not geometrical or demonstrative, shall have satisfied himself of the usefulness of certain rules; which he afterwards shall propose to his disciples for undoubted truths; which he undertakes to demonstrate in a subtile manner, and by the help of nice and intricate notions; it is not hard to conceive that such his disciples may, to save themselves the trouble of thinking, be inclined to confound the usefulness of a rule with the certainty of a truth, and accept the one for the other; especially if they are men accustomed rather to compute than to think.” (§10)

Critique of how derivatives are computed. The derivative of $y = x^2$ is traditionally found as follows. Let x increase by dx. Then y increases by $dy = (x + dx)^2 - x^2 = 2x\,dx + dx^2$. Therefore $dy/dx = 2x + dx$. But dx is infinitely small, so it can be discarded. Thus the final result is that the derivative is $2x$. Before the final step, “I have supposed that … [dx] is something. And I have proceeded all along on that supposition, without which I should not have been able to have made so much as one single step.” But in the final step “I now beg leave to make a new supposition contrary to the first, i.e. I will suppose that there is no increment of x, or that [dx] is nothing; which second supposition destroys my first, and is inconsistent with it, and therefore with every thing that supposeth it. I do nevertheless beg leave to retain [the expression for dy], which is an expression obtained in virtue of my first supposition, which necessarily presupposeth such supposition, and which could not be obtained without it: All which seems a most inconsistent way of arguing, and such as would not be allowed of in Divinity.” (§14)

“And what are these fluxions? The velocities of evanescent increments? And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?” (§35)

“It is with the method of fluxions as with all other methods, which presuppose their respective principles and are grounded thereon; although the rules may be practised by men who neither attend to, nor perhaps know the principles. In like manner, therefore, … as any ordinary man may solve divers numerical questions, by the vulgar rules and operations of arithmetic, which he performs and applies without knowing the reasons of them: Even so it cannot be denied that you may apply the rules of the fluxionary method: … You may operate and compute and solve problems thereby, not only without an actual attention to, or an actual knowledge of, the grounds of that method, and the principles whereon it depends, and whence it is deduced, but even without having ever considered or comprehended them.” (§32) “But then it must be remembered that in such case although you may pass for an artist, computist, or analyst, yet you may not be justly esteemed a man of science and demonstration. Nor should any man, in virtue of being conversant in such obscure analytics, imagine his rational faculties to be more improved than those of other men which have been exercised in a different manner and on different subjects; much less erect himself into a judge and an oracle concerning matters that have no sort of connexion with or dependence on those species, symbols or signs, in the management whereof he is so conversant and expert.” (§33)

COLIN MACLAURIN, Reaction to Berkeley, quoted from *Collected Letters of Colin Maclaurin*, Birkhäuser, 1982.

Berkeley’s critique is “groundless” (427) and the alleged flaws that he “pretends to discover” (427) are all due to him having “not understood” (427) the mathematics in question. “[Newton’s] notion of fluxions has nothing obscure, mysterious, unintelligible or absurd in it.” (428) “What this writer [Berkeley] advances against the foundations of the methods of Fluxions serves only to shew that he has not considered or understood what its great Author [Newton] said in their defence when he first published them; for if he had, he would have found the most material of his objections prevented & answered there.” (425)

Berkeley’s critique was religiously motivated but: “I am satisfied that the interests of true Science and true Religion are united, & that they do real prejudice to Mankind who endeavour to represent them as opposite in any measure.” (427) “I believe it will be easily granted by all who are acquainted with the History of Learning that there is no other order or Class of Learned Men that has produced fewer writers on the side of Infidelity, or fewer adversarys to natural or revealed Religion than that of the Mathematicians. The greatest Men among them have distinguished themselves as firm in the belief, and ornaments to the practice of Christianity, and particularly these men who invented or promoted the parts which this Author has so warmly attack’d.” (426)

LEONHARD EULER, *Foundations of Differential Calculus*, 1755, translated by J. D. Blanton, Springer, 2000.

“83. This theory of the infinite will be further illustrated if we discuss that which mathematicians call the infinitely small. There is no doubt that any quantity can be diminished until it all but vanishes and then goes to nothing. But an infinitely small quantity is nothing but a vanishing quantity, and so it is really equal to 0. There is also a definition of the infinitely small quantity as that which is less than any assignable quantity. If a quantity is so small that it is less than any assignable quantity, then it cannot not be 0, since unless it is equal to 0 a quantity can be assigned equal to it, and this contradicts our hypothesis. To anyone who asks what an infinitely small quantity in mathematics is, we can respond that it really is equal to 0. There is really not such a great mystery lurking in this idea as some commonly think and thus have rendered the calculus of the infinitely small suspect to so many.”

“85. These things are very clear, even in ordinary arithmetic. Everyone knows that when zero is multiplied by any number, the product is zero and that $n \cdot 0 = 0$, so that $n : 1 = 0 : 0$. Hence, it is clear that any two zeros can be in a geometric ratio, although from the perspective of arithmetic, the ratio is always of equals. Since between zeros any ratio is possible, in order to indicate this diversity we use different notations on purpose, especially when a geometric ratio between two zeros is being investigated. In the calculus of the infinitely small, we deal precisely with geometric ratios of infinitely small quantities."

"86. If we accept the notation used in the analysis of the infinite, then dx indicates a quantity that is infinitely small, so that both $dx = 0$ and $a\,dx = 0$, where a is any finite quantity. Despite this, the geometric ratio $a\,dx : dx$ is finite, namely $a : 1$. For this reason these two infinitely small quantities dx and $a\,dx$, both being equal to 0, cannot be confused when we consider their ratio. In a similar way, we will deal with infinitely small quantities dx and dy. Although these are both equal to 0, still their ratio is not that of equals. Indeed, the whole force of differential calculus is concerned with the investigation of the ratios of any two infinitely small quantities of this kind."

"87. … From this we obtain the well-known rule that the infinitely small vanishes in comparison with the finite and hence can be neglected. For this reason the objection brought up against the analysis of the infinite, that it lacks geometric rigor, falls to the ground under its own weight, since nothing is neglected except that which is actually nothing. Hence with perfect justice we can affirm that in this sublime science we keep the same perfect geometric rigor that is found in the books of the ancients."

§ R22. Absolute versus relative space

R22.1. What led Descartes and Leibniz to insist on a relativistic notion of space? And Newton on an absolutist one?

R22.2. What are the strengths and weaknesses of each view?

R22.3. How does this relate to the rationalist/empiricist divide of §R19?

RENÉ DESCARTES, *Principia philosophiae*, 1644. Quoted from *Principles of Philosophy*, Synthese Historical Library, Reidel, 1982.

"The names 'place' or 'space' do not signify a thing different from the body which is said to be in the place; but only designate its size, shape and situation among other bodies." (II.13) "So when we say that a thing is in a certain place, we understand only that it is in a certain situation in relation to other things." (II.14)

GOTTFRIED WILHELM LEIBNIZ, 1689, quoted from R. Ariew and D. Garber (eds.), *Philosophical Essays*, Hackett, 1989.

"Motion … is nothing but a change in the positions of bodies with respect to one another, and so, motion is not something absolute, but consists in a relation." (91)

ISAAC NEWTON, *Philosophiae Naturalis Principia Mathematica*, 1687, Book 1, Scholium to Definitions; trans. Andrew Motte..

I do not define time, space, place and motion, as being well known to all. Only I must observe, that the vulgar conceive those quantities under no other notions but from the relation they bear to sensible objects. And thence arise certain prejudices, for the removing of which, it will be convenient to distinguish them into absolute and relative, true and apparent, mathematical and common.

I. Absolute, true, and mathematical time, of itself, and from its own nature flows equably without regard to anything external. … Relative, apparent, and common time, is some sensible and external measure of duration. …

II. Absolute space, in its own nature, without regard to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position [relative] to bodies; and which is vulgarly taken for immovable space. … Because the parts of space cannot be seen, or distinguished from one another by our senses, therefore in their stead we use sensible measures of them. For from the positions and distances of things from any body considered as immovable, we define all places; and then with respect to such places, we estimate all motions. … And so, instead of absolute places and motions, we use relative ones; and that without any inconvenience in common affairs; but in philosophical disquisitions, we ought to abstract from our senses, and consider things themselves, distinct from what are only sensible measures of them. For it may be that there is no body really at rest, to which the places and motions of others may be referred.

IV. Absolute motion is the translation of a body from one absolute place into another; and relative motion, the translation from one relative place into another. Thus in a ship under sail, the relative place of a body is that part of the ship which the body possesses; or that part of its cavity which the body fills, and which therefore moves together with the ship: and relative rest is the continuance of the body in the same part of the ship, or of its cavity. But real, absolute rest, is the continuance of the body in the same part of that immovable space, in which the ship itself, its cavity, and all that it contains, is moved. Wherefore, if the earth is really at rest, the body, which relatively rests in the ship, will really and absolutely move with the same velocity which the ship has on the earth. But if the earth also moves, the true and absolute motion of the body will arise, partly from the true motion of the earth, in immovable space; partly from the relative motion of the ship on the earth; and if the body moves also relatively in the ship; its true motion will arise, partly from the true motion of the earth, in immovable space, and partly from the relative motions as well of the ship on the earth, as of the body in the ship; and from these relative motions will arise the relative motion of the body on the earth.

ERNST MACH, *The Science of Mechanics: A Critical and Historical Account of its Development*, Open Court Publishing, 1919, first edition 1883.

Absolute space and absolute motion … are pure things of thought, pure mental constructs, that cannot be produced in experience. … [They are] therefore neither a practical nor a scientific value; and no one is justified in saying that he knows aught about it. It is an idle metaphysical conception. …

All our principles of mechanics are … experimental knowledge concerning the relative positions and motions of bodies. … No one is warranted in extending these principles beyond the boundaries of experience. In fact, such an extension is meaningless, as no one possesses the requisite knowledge to make use of it. (II.VI)

MATTHIAS SCHEMMEL, *Historical Epistemology of Space: from Primate Cognition to Spacetime Physics*, Springer, 2016.

"Inertia … served Newton as a proof of the independent existence of space. In the following reception of the concept of absolute space, it was exactly this feature which led to its broad acceptance. It allowed the concept of absolute space to serve as a foundation for a theory of mechanics, which was highly successful in integrating the growing body of knowledge on terrestrial and celestial motions. This emphasis on the mechanical argument is strikingly evident in Leonhard Euler's *Reflections on Space and Time* [1748]. Euler makes no attempt to provide a metaphysical argument for the reality of space. Rather, he takes the empirical success of Newtonian mechanics as a proof that reality must pertain to space and time, since they serve as a basis for the concepts of absolute rest and uniform motion in a straight line, which are needed for the foundation of that science. He concludes that any metaphysical derivation that denies this reality can thereby be inferred to be flawed, without further analysis[:] 'I do not want to enter the discussion of the objections that are made against the reality of space and place; since having demonstrated that this reality can no longer be drawn into doubt, it follows necessarily that all these objections must be poorly founded; even if we were not in a position to respond to them.'" (76–77)

ROBERT DISALLE, *Understanding Space-Time: The Philosophical Development of Physics from Newton to Einstein*, Cambridge University Press, 2006.

Empirical knowledge of spatial relations is always relative: by measurements we can determine whether a body is moving with respect to another, but not whether it is "really" moving in some absolute sense. Descartes and Leibniz insisted on this. Newton, on the contrary, stipulated an absolute space as a universal frame of reference for his physics, so that the velocity and position of a body have a fixed meaning in and of itself, not only in relation to other bodies.

"What right did Newton have to explain the observable relative motions by an appeal to these unobservable entities? What role can such metaphysical hypotheses play in empirical science? … [In fact,] Newton's theory of space and time was never a mere metaphysical hypothesis. Instead, it was his attempt to define the concepts presupposed by the new mechanical science." (13) "[Newton] parted from his contemporaries … in his belief that these novel explications [of space and time] came from science itself—that their authority rested, not on their conformity to epistemological and metaphysical principles …, but on the role that they play in science. What must be the nature of space and time, in order for the world to be as it appears to be, and to follow the natural laws that it appears to follow? This is Newton's question." (42)

"Newton's theory was forced into confrontation with the most prominent general philosophical accounts of space and time, namely those of Descartes and Leibniz. But its rejoinder to them was only that those philosophical views could not be reconciled with their own views of physics." (55) Indeed, there is an "apparent contradiction within Leibniz's picture: … How can the idea of force as a genuine metaphysical quantity be reconciled with the relativity of motion? Leibniz himself occasionally juxtaposed these notions, apparently unaware of the conflict that others saw quite clearly. … For example, he wrote, '… each [body in a group of interacting bodies] does truly have a certain degree of motion, or, if you wish, of force, in spite of the equivalence of these hypotheses about their motion'. … As Euler put it, 'Therefore I am least afraid of those philosophers who reduce everything to relations, since they themselves attribute so much to motion that they regard moving force as something substantial'." (61)

Similarly, the law of inertia—that a body under no external influence continues to move in the same direction with the same speed—seems almost impossible to formulate coherently in a relativistic framework. As Euler put it: "For if space and place were nothing but the relation among co-existing bodies, what would be the same direction? … Identity of direction, which is an essential circumstance in the general principles of motion, is … not to be explicated by the relation … of co-existing bodies." (37)

"For the [relativists], then, there were only three legitimate ways to resist Newton's … argument. One would be to acknowledge it, but to insist on its limitations: … [it falls short of proving that absolute space is unavoidable, leaving the possibility open that] a weaker structure than absolute space [might suffice]. … A second way is to argue that, since the laws of motion do presuppose something like the Newtonian conception of space, time, and motion, they ought to be replaced by laws that don't presuppose any such thing. … The third way is … to maintain that no matter what spatio-temporal concepts might be required by physics, the authority to pronounce upon the true nature of space, time, and motion ultimately belongs to metaphysics." (52)

Options 1 and 2 were far from viable in the 17th century, though eventually, in the late 19th century, they proved the way forward. "But we can point to a lasting philosophical accomplishment of Newton's work. He showed that the philosophical understanding of space and time has to start, not from general philosophical principles, but from a critical analysis of what we presuppose in our observation and reasoning about the physics of motion. The eventual overthrow of Newton's theory was made possible by the further pursuit, in a different theoretical and empirical context, of the same kind of analysis." (53)

§ R23. Kant

R23.1. What is the relation between geometry and empirical data regarding spatial relations, according to Kant?

R23.2. What does Kant's view accomplish?

R23.3. How does Kant's view relate to the issues of rationalism versus empiricism (§R19) and absolute versus relative space (§R22)?

R23.4. Is geometrical knowledge in some sense specifically human or tied to human nature? Compare Kant with Plato, Kepler, Descartes in this regard.

IMMANUEL KANT, *Inaugural Dissertation*, 1770.

"Space is not something objective and real …; instead, it is subjective and ideal, and originates from the mind's nature … as a scheme, as it were, for coordinating everything sensed externally." (Ak 2: 403)

IMMANUEL KANT, *The Critique of Pure Reason*, second edition, 1787, translated by J. M. D. Meiklejohn.

"A question which requires close investigation [is:] whether there exists a knowledge altogether independent of experience, and even of all sensuous impressions? Knowledge of this kind is called *a priori*, in contradistinction to empirical knowledge, which has its sources *a posteriori*, that is, in experience." (Introduction, §1)

"Certain of our cognitions rise completely above the sphere of all possible experience, and by means of conceptions, to which there exists in the whole extent of experience no corresponding object, seem to extend the range of our judgements beyond its bounds. … In this transcendental or supersensible sphere … experience affords us neither instruction nor guidance." (Introduction, §3)

"Mathematical science affords us a brilliant example, how far, independently of all experience, we may carry our *a priori* knowledge. … [It] thus leads us to form flattering expectations with regard to others, though these may be of quite a different nature. (Introduction, §3)

"Deceived by such a proof of the power of reason, we can perceive no limits to the extension of our knowledge. The light dove cleaving in free flight the thin air, whose resistance it feels, might imagine that her movements would be far more free and rapid in airless space. Just in the same way did Plato, abandoning the world of sense because of the narrow limits it sets to the understanding, venture upon the wings of ideas beyond it, into the void space of pure intellect. He did not reflect that he made no real progress by all his efforts; for he met with no resistance which might serve him for a support, as it were, whereon to rest, and on which he might apply his powers, in order to let the intellect acquire momentum for its progress. It is, indeed, the common fate of human reason in speculation, to finish the imposing edifice of thought as rapidly as possible, and then for the first time to begin to examine whether the foundation is a solid one or no. Arrived at this point, all sorts of excuses are sought after, in order to console us for its want of stability, or rather, indeed, to enable us to dispense altogether with so late and dangerous an investigation. But what frees us during the process of building from all apprehension or suspicion, and flatters us into the belief of its solidity, is this. A great part, perhaps the greatest part, of the business of our reason consists in the analysation of the conceptions which we already possess of objects. By this means we gain a multitude of cognitions, which although really nothing more than elucidations or explanations of that which (though in a confused manner) was already thought in our conceptions, are, at least in respect of their form, prized as new introspections; whilst, so far as regards their matter or content, we have really made no addition to our conceptions, but only disinvolved them. But as this process does furnish a real priori knowledge, which has a sure progress and useful results, reason, deceived by this, slips in, without being itself aware of it, assertions of a quite different kind; in which, to given conceptions it adds others, *a priori* indeed, but entirely foreign to them, without our knowing how it arrives at these, and, indeed, without such a question ever suggesting itself. I shall therefore at once proceed to examine the difference between these two modes of knowledge. (Introduction, §3)

"Of the difference between analytical and synthetical judgments. … Analytical judgments … add in the predicate nothing to the conception of the subject, but only analyse it into its constituent conceptions, which were thought already in the subject, although in a confused manner; [synthetical judgments] add to our conceptions of the subject a predicate which was not contained in it, and which no analysis could ever have discovered therein. For example, when I say, 'all bodies are extended', this is an analytical judgment. For I need not go beyond the conception of body in order to find extension connected with it, but merely analyse the conception, that is, become conscious of the manifold properties which I think in that conception, in order to discover this predicate in it: it is therefore an analytical judgment. On the other hand, when I say, 'all bodies are heavy', the predicate is something totally different from that which I think in the mere conception of a body. By the addition of such a predicate, therefore, it becomes a synthetical judgment." (Introduction, §4)

"Mathematical judgments are always synthetical. … 'A straight line between two points is the shortest', is a synthetical proposition. For my conception of straight, contains no notion of quantity, but is merely qualitative. The conception of the shortest is therefore wholly an addition, and by no analysis can it be extracted from our conception of a straight line." (Introduction, §5)

"Geometry is a science which determines the properties of space synthetically, and yet *a priori*." (Transcendental Aesthetic, §3)

"It has hitherto been assumed that our cognition must conform to the objects; but all attempts to ascertain anything about these objects *a priori*, by means of conceptions, and thus to extend the range of our knowledge, have been rendered abortive by this assumption. Let us then make the experiment whether we may not be more successful in metaphysics, if we assume that the objects must conform to our cognition. This appears, at all events, to accord better with the possibility of our gaining the end we have in view, that is to say, of arriving at the cognition of objects *a priori*, of determining something with respect to these objects, before they are given to us. … If the intuition must conform to the nature of the objects, I do not see how we can know anything of them *a priori*. If, on the other hand, the object conforms to the nature of our faculty of intuition, I can then easily conceive the possibility of such an *a priori* knowledge." (preface to 2nd ed.)

"It is therefore from the human point of view only that we can speak of space, extended objects, etc. If we depart from the subjective condition, under which alone we can obtain external intuition, or, in other words, by means of which we are affected by objects, the representation of space has no meaning whatsoever. This predicate is only applicable to things in so far as they appear to us, that is, are objects of sensibility." (Transcendental Aesthetic, §4)

ROBERT DISALLE, *Understanding Space-Time: The Philosophical Development of Physics from Newton to Einstein*, Cambridge University Press, 2006.

"[Newton's approach] cannot be viewed as a complete philosophical account of space and time, … because it treats space and time solely from the perspective of classical mechanics—that is as concepts implicitly presupposed by the classical mechanical understanding of causality and force. A philosophically thorough treatment of the problem would embrace, not only the implicit metaphysics of physics, but the general epistemological problem of space and time and the ways in which physics, and human knowledge generally, have some access to them." (55) In Kant's own words, his goal was to "provide, not engineers, as Euler had in mind, but geometers themselves with a convincing ground … for claiming the actuality of their absolute space." (62)

"[Newton had showed that] the concept of absolute motion … was implicitly assumed in … physics, and was in a sense a 'condition of possibility' of … reasoning about the motions of the solar system and their physical causes. … [But this is a limited form of argument since it shows only that] the concept is necessary, relative to a certain well-established practice of scientific reasoning about a certain kind of phenomenon. Kant, instead, argues that the Newtonian concept, and the laws of motion, [are necessary in much more general sense:] They are the *only* basis on which the concept of causality can be applied to the universe at large. They are the only basis, indeed, on which the phenomena of the heavens can be grasped as something more than mere appearances—as the appearance of genuine physical objects that stand in objective geometrical and causal relations." (71)

§ R24. Space and perception

R24.1. What is the relation between geometry and sensory perception?

R24.2. How, and under what conditions, can a mind without geometrical preconceptions be led to impose a geometrical structure on sensory perceptions?

R24.3. Does the experience of blind people becoming sighted support this account?

R24.4. What is the relation between scientific developments (Newton, Einstein) and the philosophy of space (Kant, Poincaré)?

R24.5. What is Helmholtz's critique of Kant? How might a Kantian reply to it?

HENRI POINCARÉ, *La Valeur de la Science*, 1905, quoted from *The Foundations of Science*, translated by George Bruce Halsted, The Science Press, 1913.

"The movements of our body [plays] the preponderant role … in the genesis of the notion of space. For a being completely immovable there would be neither space nor geometry; in vain would exterior objects be displaced about him, the variations which these displacements would make in his impressions would not be attributed by this being to changes of position, but to simple changes of state; this being would have no means of distinguishing these two sorts of changes, and this distinction, fundamental for us, would have no meaning for him.

… We are led to distinguish the changes produced by our own motions and we easily discriminate them for two reasons: because they are voluntary; because they are accompanied by muscular sensations. So we naturally divide the changes that our impressions may undergo into two categories ...: the internal changes, which are voluntary and accompanied by muscular sensations; the external changes, having the opposite characteristics.

We then observe that among the external changes are some which can be corrected, thanks to an internal change which brings everything back to the primitive state; others can not be corrected in this way (it is thus that, when an exterior object is displaced, we may then by changing our own position replace ourselves as regards this object in the same relative position as before, so as to reestablish the original aggregate of impressions; if this object was not displaced, but changed its state, that is impossible). Thence comes a new distinction among external changes: those which may be so corrected we call changes of position; and the others, changes of state.

Think, for example, of a sphere with one hemisphere blue and the other red; it first presents to us the blue hemisphere, then it so revolves as to present the red hemisphere. Now think of a spherical vase containing a blue liquid which becomes red in consequence of a chemical reaction. In both cases the sensation of red has replaced that of blue; our senses have experienced the same impressions which have succeeded each other in the same order, and yet these two changes are regarded by us as very different; the first is a displacement, the second a change of state. Why? Because in the first case it is sufficient for me to go around the sphere to place myself opposite the blue hemisphere and reestablish the original blue sensation. ...

Another example: An object is displaced before my eye; its image was first formed at the center of the retina; then it is formed at the border. ... Why ... am I led to decide that these two sensations, qualitatively different, represent the same image, which has been displaced? It is because I can follow the object with the eye and by a displacement of the eye, voluntary and accompanied by muscular sensations, bring back the image to the center of the retina and reestablish the primitive sensation.

I suppose that the image of a red object has gone from the center to the border of the retina, then that the image of a blue object goes in its turn from the center to the border of the retina; I shall decide that these two objects have undergone the same displacement. Why? Because in both cases I shall have been able to reestablish the primitive sensation, and that to do it I shall have had to execute the same movement of the eye, and I shall know that my eye has executed the same movement because I shall have felt the same muscular sensations.

If I could not move my eye, should I have any reason to suppose that the sensation of red at the center of the retina is to the sensation of red at the border of the retina as that of blue at the center is to that of blue at the border? I should only have four sensations qualitatively different, and if I were asked if they are connected by the proportion I have just stated, the question would seem to me ridiculous, just as if I were asked if there is an analogous proportion between an auditory sensation, a tactile sensation and an olfactory sensation." (I.III.5)

RICHARD FEYNMAN, *The Feynman Lectures on Physics*, Addison-Wesley, 1963.

"When we look at an object, there is an obvious thing we might call the 'apparent width', and another we might call the 'depth'. But the two ideas, width and depth, are not fundamental properties of the object, because if we step aside and look at the same thing from a different angle, we get a different width and a different depth, and we may develop some formulas for computing the new ones from the old ones and the angles involved. ... If it were impossible ever to move, and we always saw a given object from the same position, then this whole business would be irrelevant—[width and depth] would appear to have quite different qualities, because one appears as a subtended optical angle and the other involves some focusing of the eyes ...; they would seem to be very different things and would never get mixed up. It is because we can walk around that we realize that depth and width are, somehow or other, just two different aspects of the same thing.

[In Einstein's theory of special relativity] also we have a mixture—of positions and the time. ... In the space measurements of one man there is mixed in a little bit of the time, as seen by the other. Our analogy permits us to generate this idea: The 'reality' of an object that we are looking at is somehow greater (speaking crudely and intuitively) than its 'width' and its 'depth' because they depend upon how we look at it; when we move to a new position, our brain immediately recalculates the width and the depth. But our brain does not immediately recalculate coordinates and time when we move at high speed, because we have had no effective experience of going nearly as fast as light to appreciate the fact that time and space are also of the same nature. It is as though we were always stuck in the position of having to look at just the width of something, not being able to move our heads appreciably one way or the other." (I.17-1)

WILL. CHESSELDEN, An account of some observations made by a young gentleman, who was born blind, or lost his sight so early, that he had no remembrance of ever having seen, and was couch'd between 13 and 14 Years of age, *Philosophical Transactions*, 35(402), 1728, 447–450.

"When he first saw, he was so far from making any Judgment about Distances, that he thought all Objects whatever touch'd his Eyes, (as he express'd it) as what he felt, did his Skin. ... He knew not the Shape of any Thing, nor any one Thing from another, however different In Shape, or Magnitude; but upon being told what Things were, whose Form he before knew from feeling, he would carefully observe, that he might know them again; but having too many Objects to learn at once, he forgot many of them. ... One Particular only (tho' it may appear trifling) I will relate; Having often forgot which was the Cat, and which the Dog, he was asham'd to ask; but catching the Cat (which he knew by feeling) he was observ'd to look at her stedfastly, and then setting her down, said, So Puss! I shall know you another Time. He was very much surpriz'd, that those Things which he had lik'd best, did not appear most agreeable to his Eyes, expecting those Persons would appear most beautiful that he lov'd most, and such Things to be most agreeable to his Sight that were so to his Taste. We thought he soon knew what Pictures represented, which were shew'd to him, but we found afterwards we were mistaken; for about two Months after he was couch'd, he discovered at once, they represented solid Bodies; when to that Time he consider'd them only as Party-colour'd Planes, or Surfaces diversified with Variety of Paint; but even then he was no less surpriz'd, expecting the Pictures would feel like the Things they represented, and was amaz'd when he found those Parts, which by their Light and Shadow appear'd now round and uneven, felt only flat like the rest; and ask'd which was the lying Sense, Feeling, or Seeing? … Being shewn his Father's Picture in a Locket at his Mother's Watch, and told what it was, he acknowledged a Likeness, but was vastly surpriz'd; asking, how it could be, that a large Face could be express'd in so little Room, saying, It should have seem'd as impossible to him, as to put a Bushel of any thing into a Pint."

JEAN JACQUES ROUSSEAU, *Emile, or On Education*, 1762. Quoted from the 2013 Dover Publications edition, p. 36.

"It is only by our own movements that we gain the idea of space. The child has not this idea, so he stretches out his hand to seize the object within his reach or that which is a hundred paces from him. You take this as a sign of tyranny, an attempt to bid the thing draw near, or to bid you bring it. Nothing of the kind, it is merely that the object first seen in his brain, then before his eyes, now seems close to his arms, and he has no idea of space beyond his reach."

HERMANN VON HELMHOLTZ, The Facts in Perception, 1878, quoted from *Epistemological Writings*, Reidel, 1977.

"Kant … developed the doctrine of forms of intuiting and thinking given prior to all experience … into which forms any content we may represent must necessarily be absorbed. … [Similarly,] Locke had already established established a claim for the share which our corporeal and mental makeup has in the manner in which things appear to us. In this direction, investigations into the physiology of the senses … have now brought the fullest confirmation." (118–119)

"Sensations belonging to different senses … exclude any transition from one to the other, any relationship of greater or lesser similarity. One cannot at all ask whether e.g. sweet is more similar to blue or to red. On the other hand, … between different sensations of the same sense … transition and comparison are possible. We can … e.g. declare yellow to be more similar to orange than to blue. What physiological investigations now show is that [this] deeply incisive difference does not depend, in any manner whatsoever, upon the kind of external impression whereby the sensation is excited, but is determined alone and exclusively by the the sensory nerve upon which the impression impinges." (119) "The same aether vibrations as are felt by the eye as light, are felt by the skin as heat. The same air vibrations as are felt by the skin as a quivering motion, are felt by the ear as a note." (120)

"So physiology too acknowledges the qualities of sensation to be a mere form of intuition. But Kant went further. … Kant considers spatial specifications too as belonging as little to the world of the actual—or to 'the thing in itself'—as the colours which we see [as] attributes of bodies in themselves, but [which are actually] introduced by our eye into them." (123) "Kant assumed not only that the general form of spatial intuition is transcendentally given, but that it also contains in advance, and prior to any experience, certain narrower specifications as expressed in the axioms of geometry. … [Such as:] Between two points only *one* shortest line is possible. … Through any point only one line parallel to a given straight line is possible. … Kant used the alleged fact that these propositions appeared to us as *necessarily* correct, and that we could never at all even represent to ourselves a deviating behaviour of space, directly as a proof that they had to be given prior to all experience." (128)

"When Kant asserted that spatial relationships contradicting the axioms of Euclid could never in any way be represented, he was influenced by the contemporary states of the development of mathematics and the physiology of the senses, just as he was thus influenced in his whole conception of intuition in general as a simple psychic process, incapable of further resolution." (129)

"The concepts of [non-Euclidean] spatial structures … can readily be developed only by means of calculative analytic geometry. … With this the intuitability of such spaces, in the sense [of a coherent mathematical system], has been shown. … This is however in disagreement with the older concept of intuition, which only acknowledges something to be given through intuition if its representation enters consciousness at once with the sense impression, and without deliberation and effort. Our attempts to represent [non-Euclidean] spaces indeed do not have the ease, rapidity and striking self-evidence with which we for example perceive the form of a room which we enter for the first time, together with the arrangement and forms of the objects contained in it, the materials of which these consist, and much else besides. Thus if this kind of self-evidence were an originally given an necessary peculiarity of all intuition, we could not up to now assert the intuitability of such spaces." (130–131)

"Yet … assurance and rapidity for the occurrence of specific representations with specific impressions can also be acquired—even when no such connexion is given by nature. One of the most striking examples of this kind is our understanding of our mother tongue. Its words are arbitrarily or accidentally chosen signs—every different language has different ones. Understanding of it is not inherited, since for a German child who was brought up amongst Frenchmen and has never heard German spoken, German is a foreign language. The child becomes acquainted with the meaning of the words and sentences only through examples of their use. In this process one cannot even make understandable to the child—until it understands the language—that the sounds it hears are supposed to be signs having a sense. Lastly, on growing up it understands these words and sentences without deliberation and effort, without knowing when, where, and through what examples it learnt them, and it grasps the finest variations of their sense—often ones where attempts at logical definition only limp clumsily behind." (131)

"Thus as concerns … the issue of the origin of the axioms of geometry: the fact that the representation of [non-Euclidean] spatial relationships is not easy when experience is lacking, cannot be claimed as a ground against their intuitability. … Kant's proof for the transcendental nature of the axioms of geometry is thus inadequate. On the other hand, investigation of the empirical facts shows that the axioms of geometry, taken in the only sense in which one is allowed to apply them to the actual world, can be empirically tested and demonstrated, or even—if the case should arise—refuted." (132)

"The assumption that there is an acquaintance with the axioms which comes from transcendental intuition is: (1) an unproved hypothesis; (2) an unnecessary hypothesis, since it does not pretend to explain anything in what in fact is our world of representations which could not also be explained without its help; (3) a wholly unusable hypothesis for explaining our acquaintance with the actual world, since the propositions laid down by it should only ever be applied to the relationships in the actual world after their objective validity has been experimentally tested and ascertained." (162)

§ R25. Non-Euclidean geometry

R25.1. What is the modern view of the relation between geometry and empirical data? What are its strengths and weaknesses compared to older conceptions?

R25.2. What elements of older views such as that of Kant can be salvaged in light of non-Euclidean geometry?

R25.3. Has the history of the epistemology of geometry been one of constant retreats? That is, has the perceived status and claim to truth of geometrical knowledge been shrinking with each century?

CARL FRIEDRICH GAUSS, Quoted from Jeremy Gray, Gauss and Non-Euclidean Geometry, in András Prékopa & Emil Molnár (eds.), *Non-Euclidean Geometries*, Springer, 2006, 63.

"The assumption that the angle sum is less than 180° leads to a geometry quite different from Euclid's, logically coherent, and one that I am entirely satisfied with. … The theorems are paradoxical but not self-contradictory or illogical." (1824) "The necessity of our [Euclidean] geometry cannot be proved. … Geometry must stand, not with arithmetic which is pure *a priori*, but with mechanics." (1817)

NIKOLAI LOBACHEVSKY, *Pangeometry*, 1855, European Mathematical Society, 2010.

"Pangeometry [i.e., non-Euclidean geometry] … proves that the assumption that the value of the sum of the three angles of any right rectilinear triangle is constant, an assumption which is explicitly or implicitly adopted in ordinary geometry, is not a consequence of our notions of space. Only experience can confirm the truth of this assumption, for instance, by effectively measuring the sum of three angles of a rectilinear triangle. … One must give preference to triangles whose edges are very large, since according to Pangeometry, the difference between two right angles and the three angles of a rectilinear triangle increases as the edges increase." (75) "The distances between the celestial bodies provide us with a means for observing the angles of triangles whose edges are very large." (76)

ALBERT EINSTEIN, Geometry and Experience, 1921. *Collected Papers*, Volume 7.

"How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality? Is human reason, then, without experience, merely by taking thought, able to fathom the properties of real things? In my opinion the answer to this question is, briefly, this: as far as the propositions of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality. It seems to me that complete clarity as to this state of things became common property only through that trend in mathematics, which is known by the name of 'axiomatics'. …

Let us for a moment consider from this point of view any axiom of geometry, for instance, the following: through two points in space there always passes one and only one straight line. How is this axiom to be interpreted in the older sense and in the more modern sense?

The older interpretation: everyone knows what a straight line is, and what a point is. Whether this knowledge springs from an ability of the human mind or from experience, from some cooperation of the two or from some other source, is not for the mathematician to decide. He leaves the question to the philosopher. Being based upon this knowledge, which precedes all mathematics, the axiom stated above is, like all other axioms, self-evident, that is, it is the expression of a part of this *a priori* knowledge.

The more modern interpretation: geometry treats of objects, which are denoted by the words straight line, point, etc. No knowledge or intuition of these objects is assumed but only the validity of the axioms, such as the one stated above, which are to be taken in a purely formal sense, i.e., as void of all content of intuition or experience. These axioms are free creations of the human mind. All other propositions of geometry are logical inferences from the axioms (which are to be taken in the nominalistic sense only). … In axiomatic geometry the words 'point', 'straight line', etc., stand only for empty conceptual schemata. That which gives them content is not relevant to mathematics. …

'Practical geometry' [arises if we] add the proposition: solid bodies are related, with respect to their possible dispositions, as are bodies in Euclidean geometry of three dimensions. Then the propositions of Euclid contain affirmations as to the behavior of practically-rigid bodies. … All length-measurements in physics constitute practical geometry in this sense, so, too, do geodetic and astronomical length measurements, if one utilizes the empirical law that light is propagated in a straight line, and indeed in a straight line in the sense of practical geometry. … I attach special importance to the view of geometry, which I have just set forth, because without it I should have been unable to formulate the theory of relativity. … From the latest results of the theory of relativity it is probable that our three-dimensional space is … approximately spherical, that is, that the laws of disposition of rigid bodies in it are not given by Euclidean geometry, but approximately by spherical geometry."

HENRI POINCARÉ, *La Science et l'Hypothèse*, 1902, quoted from *The Foundations of Science*, translated by George Bruce Halsted, The Science Press, 1913.

"The axioms of geometry … are neither synthetic *a priori* judgments nor experimental facts. They are conventions; our choice among all possible conventions is guided by experimental facts; but it remains free and is limited only by the necessity of avoiding all contradiction. Thus it is that the postulates can remain rigorously true even though the experimental laws which have determined their adoption are only approximative."

"In other words, the axioms of geometry … are merely disguised definitions. Then what are we to think of that question: Is the Euclidean geometry true? It has no meaning. As well ask whether the metric system is true and the old measures false; whether Cartesian coordinates are true and polar coordinates false. One geometry can not be more true than another; it can only be more convenient."

"The question has also been put in another way. If [hyperbolic] geometry is true, the parallax of a very distant star will be finite [which is equivalent to an angle sum of less than 180° for an astronomical triangle]; if [spherical geometry] is true, it will be negative. These are results which seem within the reach of experiment, and there have been hopes that astronomical observations might enable us to decide between the three geometries. But in astronomy 'straight line' means simply 'path of a ray of light'. If therefore negative parallaxes were found, … two courses would be open to us; we might either renounce Euclidean geometry, or else modify the laws of optics and suppose that light does not travel rigorously in a straight line. It is needless to add that all the world would regard the latter solution as the more advantageous. The Euclidean geometry has, therefore, nothing to fear from fresh experiments." (81)

More generally, "Is the position tenable, that certain phenomena, possible in Euclidean space, would be impossible in non-Euclidean space, so that experience, in establishing these phenomena, would directly contradict the non-Euclidean hypothesis? For my part I think no such question can be put. To my mind it is precisely equivalent to the following, whose absurdity is patent to all eyes: are there lengths expressible in meters and centimeters, but which can not be measured in fathoms, feet and inches, so that experience, in ascertaining the existence of these lengths, would directly contradict the hypothesis that there are fathoms divided into six feet?" (81–82)

ROBERT DISALLE, *Understanding Space-Time: The Philosophical Development of Physics from Newton to Einstein*, Cambridge University Press, 2006.

"Kant [was] right in a certain limited sense; the kind of a-priori principle that goes to constitute a spatio-temporal framework cannot be the same as an empirical principle, and cannot be justified by the usual sort of empirical or inductive argument—since empirical arguments … must take such principles for granted." (156) "In the history of modern physics, space and time have after all played something like the role attributed to them by Kant. Not as forms of intuition: this was only incidentally the case, in a context where the geometry of space and the intuitive means of knowing about space seemed inseparable from one another. … But they have played the quasi-Kantian role of a framework that enables physics to constructively define its fundamental concepts." (153)

§ R26. Constructivist foundations

R26.1. Why did constructivist philosophies of the foundations of mathematics gain popularity in the 20th century?

R26.2. In what ways can 20th-century constructivist philosophy be considered similar to what Euclid was doing?

L. E. J. BROUWER, *Over de Grondslagen der Wiskunde*, Ph.D. thesis, University of Amsterdam, 1907. Quoted from the translation in L. E. J. Brouwer, *Collected Works*, I, North-Holland, 1975.

Wheresoever in logic the word *all* or *every* is used, this word, in order to make sense, tacitly involves the restriction: *insofar as belonging to a mathematical structure which is supposed to be constructed beforehand.* (76)

By a propositional function … the logicians … mean a statement about x …; they reckon that by that statement a class is

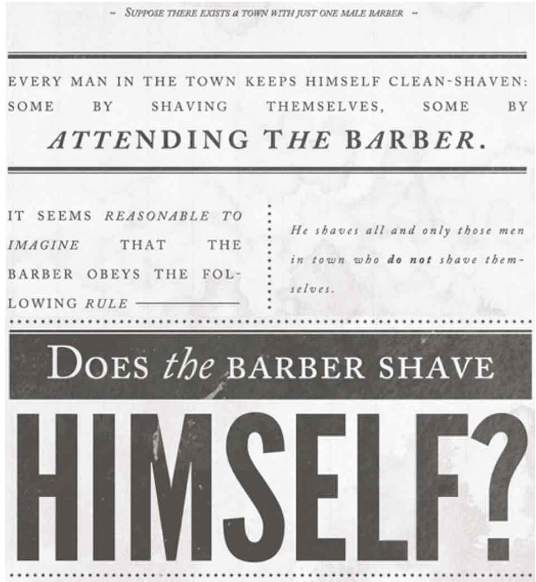


Figure 4: Popularised version of Russell's Paradox.

defined, consisting of all things … which by substitution make the statement true. [Thus they speak of] *all things for which the statement φx is true*. … As the fundamental domain of operations, within which the relations meant by the words or symbols must exist, they choose not some mathematical system, but the chimerical 'everything'. … [They thus] give a linguistic system of statements and propositional functions priority over mathematics. (88) They postulate that these sentences define classes and that it is allowed to reason about these classes according to the laws of classical logic. (89)

It is not surprising that they … came up against contradictions. … Russell discusses most amply the following contradiction: 'I consider the class of all classes which do not possess the property … to occur amongst their own elements. Does this class possess the property in question? If it does, then it occurs amongst its own elements, so it is one of the classes which do not possess the property, so it does not have the property. And vice-versa: If it does not, then it is on equal terms with its own elements, so it does have the property.' [Cf. Figure 4]

Russell suggests various methods to escape from the contradiction. … '[One] suggestion', he says, 'would be to demur to the notion of *all* objects, but in any case the notion of *every object* must be retained, for there are truths, viz. the logical principles, which hold for every object.'

But this is mistaken: the logical principles hold exclusively for words with a mathematical content. And exactly because Russell's logic is no more than a linguistic system, deprived of a presupposed mathematical system to which it would be related, there is no reason why no contradictions would appear. (89)

The language of Euclidean geometry … is reliable only because the mathematical systems and relations, which are symbolized by the words of that language as conventional signs, have been constructed beforehand independently of that language. (98)

P. W. Bridgman, *The Logic of Modern Physics*, New York: Macmillan, 1927. Paperback reprint 1960.

"We mean by any concept nothing more than a set of operations; the concept is synonymous with the corresponding set of operations." (5) "The proper definition of a concept is not in terms of its properties but in terms of actual operations." (6)

Motivation for this: Relativity theory and quantum physics show that "when experiment is pushed into new domains, we must be prepared for new facts, of an entirely different character from those of our former experience" (2). Therefore "the physicist … recognizes no *a priori* principles which determine or limit the possibilities of new experience" (3); rather we "must use in describing and correlating nature concepts of such a character that our present experience does not exact hostages of the future" (4). "For if experience is always described in terms of experience, … we need never be embarrassed" (6–7) to find that we were talking about imagined concepts with no connection to reality.

Newton's concept of absolute time violates these principles. (4-7)

"[A] consequence of the operational character of our concepts … is that it is quite possible, even disquietingly easy, to invent expressions or to ask questions that are meaningless. It constitutes a great advance in our critical attitude toward nature to realize that a great many of the questions that we uncritically ask are without meaning. If a specific question has meaning, it must be possible to find operations by which an answer may be given to it." (28)

P. W. BRIDGMAN, *The nature of some of our physical concepts*, New York: Philosophical Library, 1952.

"We do not know the meaning of a concept unless we can specify the operations which were used by us or our neighbour in applying the concept in any concrete situation" (7)

"Physicists do profitably employ concepts the meaning of which is not to be found in the instrumental operations of the laboratory, and which cannot be reduced to such operations without residue. Nearly all the concepts of theoretical and mathematical physics are of this character. … In fact, there is hardly any physical concept which does not enter to a certain extent into some theoretical edifice and which does not therefore possess to a certain degree a non-instrumental component." (8)

"I think, however, that physicists are agreed in imposing one restriction on the freedom of such operations, namely that such operations much be capable of eventually, although perhaps indirectly, making connection with instrumental reality. Only in this way can the physicist keep his feet on the ground or achieve a satisfactory degree of precision. … Politics, philosophy and religion are full of … purely verbal concepts; it is merely that such concepts are outside the field of the physicist." (10)

P. W. BRIDGMAN, *Reflections of a Physicist*, second ed., New York: Philosophical Library, 1955.

"In mathematics much the same situation arises that arises in physics; in foundation studies, in which one wants to secure the maximum awareness of what one is doing and the maximum security that he is not involving himself in contradiction, one would do well to use only concepts whose meaning is found in … unambiguously performable operations. For in this way, … the description of a situation in mathematics reduces to the description of an actual experience, … and actual experience in not self-contradictory." (17)

"One of the chief advantages of making definitions operational is that it increases precision. I find that I myself use terms without either precisely formulating the conditions which demand their use. … When my neighbour uses the term, I observe only too frequently that I cannot deduce from his use of it the occurrence of certain antecedent or subsequent occurrences which I believe are implied in his use of it. In such cases reduction of definitions to operational terms increases precision, so that operational definitions are called for when increased precision is called for." (30)

Operationalism implies questioning the Platonic philosophy of mathematics, e.g. that numbers "exist." (100)

"The very nature of meaning itself makes it impossible to get away from the human reference point" (112)

"If I am trying to describe a situation, the criterion that I have successfully communicated it, is that my fellow be able to *reconstruct* the situation. … The meaningfulness of the communication is therefore to be found in the actions which accompany it." (121)

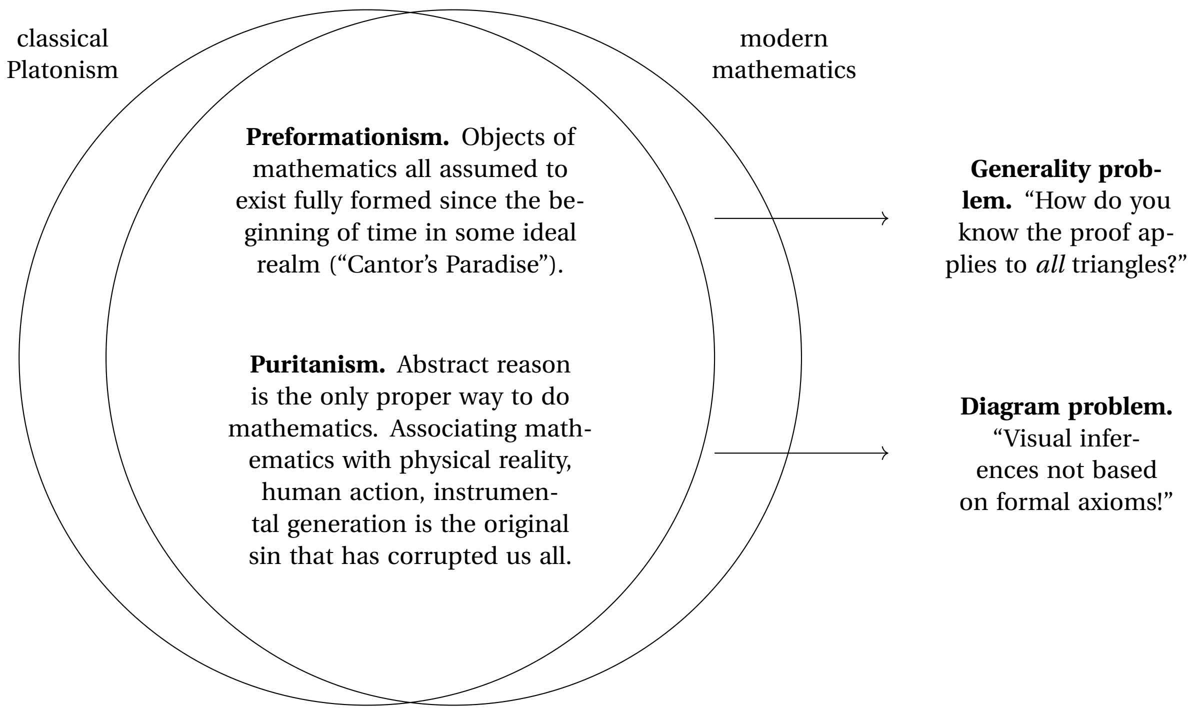


Figure 5: Shared commitments of ancient and modern Platonism.

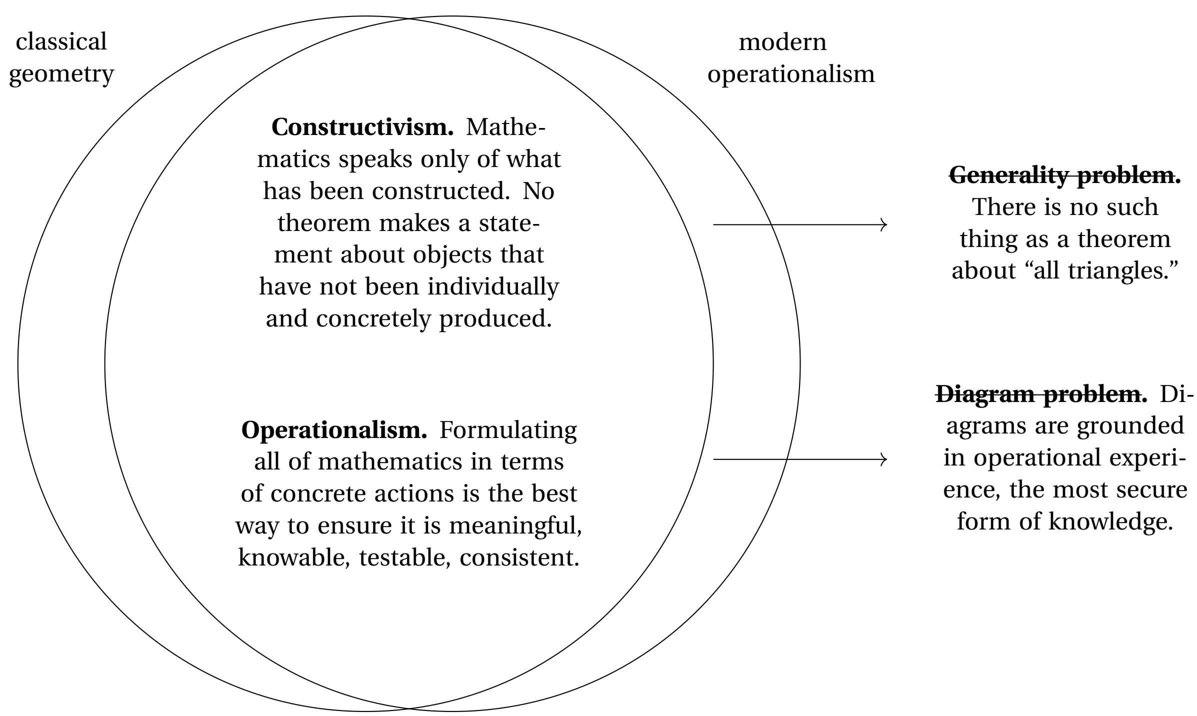


Figure 6: Shared commitments of ancient and modern operationalism.