History of Mathematics Reader
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§ R1. Astrology

B. L. van der Waerden, *Science Awakening II*, Springer, 1973.

Early Egyptian astronomy seems to have been of a practical origin. It was noticed that Sirius was the "herald of the flood" (8). "The flooding of the Nile over its banks is the most important event in the Egyptian agricultural year. It gives new life to the parched land. This event is heralded some weeks beforehand by a striking event in the firmament, namely the first visibility of Sirius in the morning sky." (9). Other practical advice based on stars include: "when strong Orion begins to set, then remember to plough"; and "fifty days after the solstice is the right time for men to go sailing" (12). The stars were also used to tell time at night. "In the course of the centuries these Stars of Time became Gods of Time and Destiny." (14). "'From their might derives everything that humanity encouters in the way of disasters,' says the revelation of Hermes Trismegistos." (29). "According to Hermes Trismetgistos the decans can also be called 'horoskopoi'—hour indicators. The decan that rises in the hour of the birth of a child determines the nature of the child." (32).

Babylonian astronomy, on the other hand, seems to be linked to, and largely dictated by, astrology as far back as the record goes. "The oldest cuneiform texts giving the positions of the planets in the zodiac date from the second half of the fifth century B.C. To just this period, and to Babylon too, belongs the oldest horoscope that has been preserved." (2). Of course Babylonian astronomy is much older than this, but precise knowledge of planetary positions were not important as long as astrology was impersonal, perhaps for the reasons given below. Indeed, "Old-Babylonian astrology was not interested, or at least not in the first place, in the fate of the individual. Its principal interest was the well-being of the country. Its predictions concern the weather and the harvest, drought and famine, war or peace and of course also the fate of the Kings." (48-49).

The rationale for impersonal astrology may have included the following. "Just as the great Gods Sin (the moon) and Shamash (the sun) are obviously responsible for the regular procession of months, days and years, and thus influence our entire life, so it was thought that the Goddess Ishtar [Venus] communicates important things to us by her appearances and disappearances." (57). Above we saw some examples of apparently important influences of the stars, in the spirit of which one will say things like "O Ursa major ... Put truth for me" (58), as one prayer reads. A further consideration is the plausibility of the idea of a strictly periodic universe (of course the world would be periodic if it was determined by the heavens, which are paradigmatically periodic). As Eudemos was later to relate, "If we are to believe the Pythagoreans, I shall in the future, even as everything recurs according to the Number, again tell you tales here, holding this little stick in my hand, while you will sit before me as you do now; and likewise everything else will be the same." (114). The periodicity at which the world repeats is presumably a common multiple of all planetary periods.

The rationale for individual astrology seems to have included the following. The idea that the souls of the dead rise to the heavens is an old one. Not the first example is that "the inscription for the fallen at the battle of Potidea (-431) says: 'The aether will receive their souls, as the earth receives their bodies'" (146). From here it is a rather short step to the idea that, as expressed for example "in Servius' commentary on Aeneid VI 714, the souls before birth go down through the planetary spheres, acquiring thereby from Saturn inertia, from Mars wrath, from Venus lust, from Mercury avarice, from Jupiter ambition" (144).

Another argument in support of this view is that the heavens are the paradigm of self-motion, which is not displayed by soulless objects. As Plato puts it: "the soul which has lost its wings is borne along until it gets hold of something solid, ... taking upon itself an earthly body, which seems to be selfmoving, because of the power of the soul within it" (147, Phaedrus 246b-c).

CARL SAGAN, *Cosmos*, TV show, PBS, 1980.

"Our language preserves an astrological consciousness … The word 'disaster' comes from the Greek for 'bad star'. The Italians once believed that disease was caused by the influence of the stars. It's the origin of our word 'influenza'. …

Astrology developed into a strange discipline, a mixture of careful observations, mathematics and record keeping with fuzzy thinking and pious fraud. Nevertheless, astrology survived and flourished. Why? Because it seems to lend a cosmic significance to the routine of our daily lives. It pretends to satisfy our longing to feel personally connected to the universe. Astrology suggests a dangerous fatalism. If our lives are controlled by a set of traffic signals in the sky, why try to change anything? …

Astrology can be tested by the lives of twins. There are many real cases like this: One twin is killed in childhood in, say, a riding accident, or is struck by lightning but the other lives to a prosperous old age. Suppose that happened to me. My twin and I would be born in precisely the same place and within minutes of each other. Exactly the same planets would be rising at our births. If astrology were valid how could we have such profoundly different fates? …

Also, how could it possibly work? How could the rising of Mars at the moment of my birth affect me then or now? I was born in a closed room. Light from Mars couldn't get in. The only influence of Mars which could affect me was its gravity. But the gravitational influence of the obstetrician was much larger than the gravitational influence of Mars. Mars is a lot more massive but the obstetrician was a lot closer."

ROGER BECK, *A Brief History of Ancient Astrology*, Blackwell, 2007.

"Here then are [2nd-century astrologer] Vettius Valens' sketches of the Seven [heavenly bodies]…:

'The Sun is the overseer of all; he is fiery, he is the light of the intellect and the instrument of the soul's perception. In a horoscope he indicates kingship, leadership, … the father, the master …

The Moon … indicates human life at birth, the body, the mother …, nurture … housekeeping, the queen, the mistress …

Saturn makes those born under him … solitary, … robed in black, importunate, miserable … He causes … laziness, inactivity, hindrances, long drawn out litigation, reversals, secrets, oppression, fetters, griefs, accusations, tears, loss of parents, captivity, banishment. He makes … tax collectors … He brings things to completion … He makes people single or widowed, orphaned or childless.

Jupiter indicates … abundance, salaries, large gifts, good crop yields, justice, rulership, … release from chains, freedom …

Mars indicates violence, wars, plundering, uproar, excess, adultery …; masculinity, perjury, error, negotiations on bad terms; those who work with fire or iron, artisans, masons. He makes military commanders …

Venus is desire and erotic love. She indicates the mother and the nurturer. She causes … reconciliations for good ends, marriages, refined arts and crafts, good singing voices, music, sweetness of melody, beauty of form, painting …

Mercury indicates education, letters, argumentation, logic, brotherhood, … calculations, geometry, commerce, youth, play, theft, … discoveries … He is the giver of discernment and judgment. He is in charge of brothers, younger children, …'"

AUGUSTINE, *The Confessions*, c. 400. Translated by Henry Chadwick, Oxford University Press, 1991.

"I now wished to attack and with ridicule to refute … one of those charlatans who make money out of astrology. … I therefore gave attention to those who are born twins. Most of them emerge from the womb in succession at a brief interval of time. They may contend that in the realm of nature this interval has considerable consequences. But it cannot be recorded by human observation and noted in the tables that the astrologer will inspect to give a true forecast. … Someone inspecting the identical tables ought to have been able to say that Esau and Jacob would have the same destiny. Yet things turned out differently in each case." (VII.vi.10)

§ R2. Ancient cosmology

PLATO, *Timaeus*, c. –360.

"He [the creator] spun it [the heavens] round uniformly in the same spot and within itself and made it move revolving in a circle; and all the other six motions [up, down, right, left, back, forth] He took away and fashioned it free from their aberrations." (34a) (Aristotle makes the same point in *De Caelo*, II.6, where also the possibility of acceleration and deceleration in the heavens is explicitly rejected.)

ARISTOTLE, *De Caelo (On the Heavens)*, c. –350. Translated by J. L. Stocks.

"Bodies are either simple or compounded of such; and by simple bodies I mean those which possess a principle of movement in their own nature, such as fire and earth ... For if the natural motion is upward, it will be fire or air, and if downward, water or earth. ... It follows that circular movement also must be the movement of some simple body. For the movement of composite bodies is, as we said, determined by that simple body which preponderates in the composition. These premises clearly give the conclusion that there is in nature some bodily substance other than the formations we know, prior to them all and more divine than they." (I.2)

PTOLEMY, *Almagest*, c. 150.

"It is our purpose to demonstrate for the five planets that all their apparent anomalies can be represented by uniform circular motions, since these are proper to the nature of divine beings, while disorder and non-uniformity are alien [to such beings]." (IX.2)

PTOLEMY, *Planetary Hypotheses*, c. 150. Quoted from Bernard R. Goldstein, The Arabic version of Ptolemy's Planetary Hypotheses, *Transactions of the American Philosophical Society*, 57(4), 1967..

"The distances of the ... planets may be determined without difficulty from the nesting of the spheres, where the least distance of a sphere is considered equal to the greatest distance of a sphere below it." (7) That is to say, according to the epicyclic planetary models presented in the Almagest, each planet sways back and forth between a nearest and a furthest distance from the earth. The "sphere" of each planet must be just thick enough to contain these motions. This argument assumes that "there is no space between the greatest and least distances [of adjacent spheres]," which "is most plausible, for it is not conceivable that there be in Nature a vacuum, or any meaningless and useless thing." (8)

§ R3. Babylonia

JENS HØYRUP, *Lengths, Widths, Surfaces: A Portrait of Old Babylonian Algebra and Its Kin*, Springer, 2002.

The quadratic equation example in the lecture is taken from Høyrup's translation, p. 50. The translation of these text is far from straightforward. In fact, Høyrup's translation is quite radically different from previous ones. Babylonian mathematical sources are extremely sparse in words, as is understandable since they are meticulously inscribed on small clay tablets. This extremely condensed, telegraphic style, allows for a considerable scope of interpretation. A typical phrase such as "30 a-na 7 ta-na-sima 210" (5) can be interpreted alternately in purely arithmetical ("multiply 30 by 7; the result is 210") or geometrical terms ("form a rectangle with sides 30 and 7; its area is 210"). The former option is the traditional one adopted by Neugebauer et al. The primary argument for this interpretation is the fact that the sources often add lengths and areas together, which is geometrically nonsensical. On the other hand the terms used for multiplication seem to point, linguistically speaking, to the geometrical interpretation ("to raise", "to hold", etc.), and indeed certain words for "multiplication" are only used for multiplying two lengths, never areas. Furthermore, certain words that can be read as "protrude", "break", etc., can be understood quite literally in the geometrical reading, whereas they are basically ignored in the arithmetical readings:

"We may say that the received interpretation made sense of the numbers occurring in the text. But it obliterated the distinction made in the texts which after all need not be synonymous unless the arithmetical interpretation is taken for granted; ... and it had to dismiss some phrases as irrelevant ... or to explain them by gratuitous ad-hoc hypotheses." (13)

The difference between the two readings is quite important since only according to Høyrup's reading does it follow that "The procedure is ... algorithm and proof in one. ... [It] performs all steps in such a way that their correctness is obvious." (98)

But the geometrical reading does not mean that the procedure is applicable to geometrical problems only. On the contrary, the procedure is "functionally abstract": there are examples where a segment represents a number, an area, a volume, or a commercial rate (280). Virtually all texts "use the Sumerograms *us* and *sag* unerringly for the lengths and widths of the standard representation; 'real' linear dimensions (the length of a wall, the distance bricks are to be carried, the width of a canal), in contrast, may as well be written in syllabic Akkadian ... This suggests strongly that the Old Babylonian authors were explicitly aware of the functionally abstract character of their standard representation." (280-281) "In this sense, Neugebauer was right in considering the us and sag as equivalents of the symbols of modern algebra." (10)

ELEANOR ROBSON, *Mathematics in Ancient Iraq: A Social History*, Princeton University Press, 2008.

"From about 6000 BCE, long before writing, Neolithic villagers used simple geometric counters in clay and stone to record exchange transactions, funded by agricultural surpluses. As societies and economies grew in size and complexity, ever more strain was placed on trust and memory. By the late fourth millennium the intricacies of institutional management necessitated both an increasing numerical sophistication and the invention of written signs for the commodities, agents, and actions involved in controlling them." (27) "Mesopotamian city states had implemented an extensible and powerful literate technology for the quantitative control and management of their assets and labour force. In doing so, they had created in parallel a new social class—in Uruk called the umbisag 'accountant/scribe'—who was neither economically productive nor politically powerful, but whose role was to manage the primary producers on the elite's behalf." (40)

"It was the ... state bureaucracy in which the scribes were embedded that ... drove the need for ... the sexagesimal place value system ... by imposing increasingly high calculational standards on its functionaries through the demand for complex annual balanced accounts." (83) This went hand in hand with "centrally imposed reforms of weights and measures throughout the third millennium" (84). "None of these newly invented units of measure was recorded with compound metrological numerals, but always written as numbers recorded according to the discrete notation system followed by a separate sign for the metrological unit," (76) unlike the earliest sources, where, "while there was a single word for 'ten'," "there was no single numeral but different signs for 'ten-discrete-objects', 'ten-units-of-grain', and 'ten-units-of-land'" (33).

"In the early Old Babylonian period [c. 1850 BCE], elementary scribal training underwent a revolution ... in which emphasis was more on the ability to manipulate imaginary lines and areas in almost algebraic ways than on the ability to count livestock or calculate work rates." (86) "Topics range from apparently abstract 'naive-geometrical algebra', via plane geometry, to practical pretexts for setting a problem—whether agricultural labour, land inheritance, or metrological conversions. Even the most abstract problems may be dressed up with 'practical' scenarios." (89)

This was "a style of mathematics that encapsulated the principles of ... justice" on which the society was based; "in solving abstruse puzzles about measured space, the true scribe demonstrated his or her technical capability ... for upholding justice and maintaining social and political stability on behalf of king and god" (266).

As one scribe put it: "When I go to divide a plot, I can divide it; when I go to apportion a field, I can apportion the pieces, so that when wronged men have a quarrel I soothe their hearts ... Brother will be at peace with brother." (122)

"The Sumerian word for justice was nig-si-sa, literally 'straightness, equality, squareness', Akkadian misarum 'means of making straight'. The royal regalia of justice were the measuring rod and rope ... In [this] light ... Old Babylonian mathematics, with its twin preoccupation of land and labour management on the one hand and cut-and-paste geometrical algebra on the other, becomes truly comprehensible." (123-124)

This tradition effectively came to an end with "the collapse of the Old Babylonian kingdom in c. 1600 BCE" (151), though "traces of Old Babylonian mathematical learning lingered on long after the political ideology that it supported had disappeared" (181). "Evidence suggests that mathematics ... was still a vital component of Babylonian intellectual life" (151) for a while, but ultimately a massive decline followed. "In the first half of the first millennium we find a low level of mathematical sophistication in school, consumer, and professional contexts." (212)

"Mathematics and mathematical astronomy were central components of the last flowering of cuneiform culture." (261) "From the mid-seventh century [BCE] onwards, ... compilers of eclipse records and astronomical diaries had begun to think in terms of divine quantification. ... Apparently random events of great ominous significance were observed, quantified, and recorded in the hope that numerical patterns could be detected amongst them. The ultimate aim was to understand the will of the gods, to ensure that they were propitiated and would act benignly to the king and humanity. Thus in later Babylonia mathematics became a priestly concern." (268)

These priests "comprised a tiny number of individuals from a restricted social circle, intermarrying, working closely together to train each successive generation, and highly valuing privacy and secrecy" (261-262). "Their sole aim was to uphold the belief systems and religious practices of ancient times." "In this context [they] developed increasingly mathematically sophisticated means to ensure the calendrical accuracy of their rituals." (262)

§ R4. Early numeration

GRAHAM FLEGG, *Numbers Through the Ages*, Sheridan House, 1989.

"There is evidence of tallying (carving notches on bone or wood) going back more than thirty thousand years" (37)—the oldest signs of mathematical activity in the historical record.

Tallying remained a widespread record of transactions well into modern times, as witnessed for instance in Shakespeare (48): "There shall be no money, all shall eat and drink on my score." (Henry VI) "He parted well and paid his score." (Macbeth)

"Tally sticks were admitted as legal documents even as late as the eighteenth and nineteenth centuries." (43) For instance, "Napoleon's book of civil law ... of 1804" states: "The tally sticks which match their stocks have the force of contracts between persons who are accustomed to declare in this manner the deliveries they have made or received." (44)

"The word for 'contract' in Chinese is written by means of two characters at the top, one for a notched stick and one for a knife, and another at the bottom which means 'large'. Thus a contract in Chinese is symbolised as 'a large tally stick'." (44)

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As for verbal representation of numbers, "Over most of aboriginal Australia one finds essentially only two number words, 'one' and 'two': many of the tribes are reported definitively as not counting beyond 2, and as indicating higher multiplicities by the word 'many', while others go a little further by compounding these words—for example, expressing 3 as 'two-one', 4 as 'two-two' ... A similar method of counting, the so-called 2-system, is found in New Guinea, in South America, and in South Africa." (7, quoting Seidenberg)

"At a very early stage of counting, the 'number' of something that was counted was felt to be one of its attributes." (57) Therefore, "In Gothic as well as High and Middle German the number words for 2 and 3 do appear as adjectives; they have separate masculine, feminine and neuter forms, and they are inflected like adjectives. The same is true for Latin and Greek." (58)

However, "In all known Indo-European languages, numbers beyond 4 are not treated as adjectives." (59) "It is quite possible that in the original Indo-European language 'one', 'two', 'three', and 'four' were earlier number words, which were more closely connected with the counted objects and treated like adjectives, whereas 'five', 'six', etc. were later number words, perhaps taken over from a foreign language, which were no longer considered as adjectives and hence remained unchanged in all cases." (60)

"Another argument in favour of [this interpretation] is given by the form of the number words for 8 in Latin and Greek. In Latin the word is octo, in Greek oktô; the ending -ô in Greek is a dual ending. In the Gothic and Sanskrit words for 8, ahtaú and astau, the ending -au is also a dual ending. So it seems that 8 was originally conceived as a dual, i.e. as two fours ... In many primitive languages, the number word for 8 is formed as 'twice 4'. So it is quite possible that in the mother language of the Indo-European family the independent number words originally ended at 4 and that 8 was expressed as twice 4." (60)

"It is also possible that at one time counting stopped at 8. This would occur naturally when two handsbreadths had been used up in measurement. After the doubling of 'four' to give 'eight', there would be a need for a 'new' number [or hand] before counting could continue. It is a striking fact that there is a similarity in most Indo-European languages between the word for 9 and the word for 'new'." (60)

EDWARD SAPIR, Notes on the Takelma Indians of Southwestern Oregon, *American Anthropologist*, New Series, 9(2), 1907, pp. 251-275.

Number words of "the Takelma numeral system" have these literal meanings: "Four is evidently nothing but 'two two'; … six, seven, eight, and nine are respectively equivalent to 'one finger in,' 'two fingers in,' 'three fingers in,' and 'four fingers in' …; ten is 'two hands' … twenty is quite transparently 'one person' …, i. e., 'two hands and two feet'." (266)

§ R5. Greek Antiquity: Beginnings

STRABO, *Geographica*, c. -10, quoted from *The Geography of Strabo*, translated by H.C. Hamilton & W. Falconer, London, 1903.

An exact and minute division of the country was required by the frequent confusion of boundaries occasioned at the time of the rise of the Nile, which takes away, adds, and alters the various shapes of the bounds, and obliterates other marks by which the property of one person is distinguished from that of another. It was consequently necessary to measure the land repeatedly. Hence it is said geometry originated here, as the art of keeping accounts and arithmetic originated with the Phoenicians, in consequence of their commerce. (17.1.3)

PROCLUS, *A Commentary on the First Book of Euclid's Elements*, c. 450, translated by Glenn R. Morrow, Princeton University Press, 1992.

"Geometry was first discovered by the Egyptians and originated in the remeasuring of their lands. This was necessary for them because the Nile overflows and obliterates the boundary lines between their properties. It is not surprising that the discovery of this and the other sciences had its origin in necessity, since everything in the world of generation proceeds from imperfection to perfection. Thus they would naturally pass from sense-perception to calculation and from calculation to reason. Just as among the Phoenicians the necessities of trade gave the impetus to the accurate study of number, so also among the Egyptians the invention of geometry came about from the cause mentioned." (52)

Some people mistakenly believe that greater perimeter means greater area. "Such a misconception is held by geographers who infer the size of a city by the length of its walls. And the participants in a division of land have sometimes misled their partners in the distribution by misusing the longer boundary line; having acquired a lot with a longer periphery, they later exchanged it for lands with a shorter boundary and so, while getting more than their fellow colonists, have gained a reputation for superior honesty." (318)

HERODOTUS, *The Histories*, c. -440, translation by A. D. Godley.

"This king [Sesostris] also (they said) divided the country among all the Egyptians by giving each an equal parcel of land, and made this his source of revenue, assessing the payment of a yearly tax. And any man who was robbed by the river of part of his land could come to Sesostris and declare what had happened; then the king would send men to look into it and calculate the part by which the land was diminished, so that thereafter it should pay in proportion to the tax originally imposed. From this, in my opinion, the Greeks learned the art of measuring land." (2.109)

PAPPUS, *Collection*, c. 340, quoted from Ivor Thomas (ed.), *Selections illustrating the History of Greek Mathematics*, Vol. 2, Loeb Classical Library, Harvard University Press, 1941.

"Though God has given to men … the best and most perfect understanding of wisdom and mathematics, He has allotted a partial share to some of the unreasoning creatures as well. To men, as being endowed with reason, He granted that they should do everything in the light of reason and demonstration, but to the other unreasoning creatures He gave only this gift, that each of them should, in accordance with a certain natural forethought, obtain so much as is needful for supporting life. This instinct may be observed to exist in many other species of creatures, but it is specially marked among bees. Their good order and their obedience to the queens who rule in their commonwealths are truly admirable, but much more admirable still is their emulation, their cleanliness in the gathering of honey, and the forethought and domestic care they give to its protection. Believing themselves, no doubt, to be entrusted with the task of bringing from the gods to the more cultured part of mankind a share of ambrosia in this form, they do not think it proper to pour it carelessly into earth or wood or any other unseemly and irregular material, but, collecting the fairest parts of the sweetest flowers growing on the earth, from them they prepare for the reception of the honey the vessels called honeycombs, [with cells] all equal, similar and adjacent, and hexagonal in form.

That they have contrived this in accordance with a certain geometrical forethought we may thus infer. They would necessarily think that the figures must all be adjacent one to another and have their sides common, in order that nothing else might fall into the interstices and so defile their work. Now there are only three rectilineal figures which would satisfy the condition, I mean regular figures which are equilateral and equiangular, inasmuch as irregular figures would be displeasing to the bees. [Pappus goes on to argue that only triangles, squares or hexagons fit around a point.] … The bees in their wisdom chose for their work that which has the most angles, perceiving that it would hold more honey than either of the two others.

Bees, then, know just this fact which is useful to them, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material in constructing each. But we, claiming a greater share in wisdom than the bees, will investigate a somewhat wider problem, namely that, of all equilateral and equiangular plane figures having an equal perimeter, that which has the greater number of angles is always greater, and the greatest of them all is the circle having its perimeter equal to them." (588-593)

H. D. P. Lee (ed.), *Zeno of Elea: A Text, with Translation and Notes*, Cambridge University Press, 1936.

Zeno (c. -450) most famously argued that motion is impossible. His "dichotomy" argument is perhaps most famous of all:

"An object in motion must move through a certain distance; but since every distance is infinitely divisible the moving object must first traverse half the distance through which it is moving, and then the whole distance; but before it traverses the whole of the half distance, it must traverse half of the half, and again the half of this half. If then these halves are infinite in number, because it is always possible to halve any given length, and if it is impossible to traverse an infinite number of positions in a finite time ... [then] therefore it is impossible to traverse any magnitude in a finite time." (45; Simplicius 1013.4)

There is also the "Achilles" form of the argument:

"The argument is called the Achilles because of the introduction into it of Achilles, who, the argument says, cannot possibly overtake the tortoise he is pursuing. For the overtaker must, before he overtakes the pursued, first come to the point from which the pursued started. But during the time taken by the pursuer to reach this point, the pursued always advances a certain distance; even if this distance is less than that covered by the pursuer, because the pursued is the slower of the two, yet none the less it does advance, for it is not at rest. And again during the time which the pursuer takes to clever this distance which the pursued has advanced, the pursued again covers a certain distance ... And so, during every period of time in which the pursuer is covering the distance which the pursued ... has already advanced, the pursued advances a yet further distance; for even though this distance decreases at each step, yet, since the pursued is also definitely in motion, it does advance some positive distance. And so ... we arrive at the conclusion that not only will Hector never be overcome by Achilles, but not even the tortoise." (51; Simplicius 1014.9)

This seems to be basically a literary elaboration of the dichotomy argument which adds little in terms of substance. However, unlike the dichotomy, it does not assume an absolute notion of distance, so it could be seen as an improvement on the former insofar as it strikes equally agains purely relativistic notions of motion.

A very different argument is the "arrow" argument:

"If everything is either at rest or in motion, but nothing is in motion when it occupies a space equal to itself, and what is in flight is always at any given instant occupying a space qual to itself, then the flying arrow is motionless." (53; Aristotle Physics Z 9. 239b)

The precise meaning of this argument is not very clear, but one possible interpretation is this: if time is made up of instants and if the arrow is at any given instant occupying a fixed place, then how can it move? Analogously one might argue: if a line segment is made up of points, and a point has no length, how can the line segment have a length?

Finally, there is the "stadium" argument:

"The fourth [of Zeno's arguments against motion] is the one about the two rows of equal bodies which move past each other in a stadium with equal velocities in opposite directions ... This, [Zeno] thinks, involves the conclusion that half a given time is equal to its double [i.e. the whole time]." (55; Aristotle Physics Z. 9. 239b)

For if the speed of the first row is measured relative to the fixed stadium, and the speed of the second row relative to the first, then it will appear that the second row is moving twice as fast. According to Aristotle "the fallacy lies in assuming that a body takes an equal time to pass with equal velocity a body that is in motion and a body that is at rest, an assumption which is false."

Árpád Szabó, *The Beginnings of Greek Mathematics*, Springer, 1978.

Szabo's main thesis is that "mathematics itself grew out of the more ancient subject of dialectic" (245). "Quite obviously it was dialectic which came first. The mere fact that all those terms which relate to the foundations of mathematics are of dialectical origin should have led us to this conclusion. Dialectic did not borrow any of its vocabulary from mathematics; instead, perfectly ordinary expressions from dialectic were transformed into technical mathematical terms. Hence early Greek mathematics, at least when it is viewed as an elaborately constructed system of knowledge, can properly be called a branch of dialectic." (253-254)

The terms referred to here are those for axiom, postulate, hypothesis, etc., all of which, in dialectic, essentially stood for some variant of "concessions which the participants in a discussion have agreed to make" (238). Even in the best sources the uses of such terms are not very systematic. For example, a clear distinction between definition and axiom is often lacking. On this basis "we can ... conjecture that the earliest foundations were composed only of definitions. After all, we know that the word hypotheseis (as a mathematical term) had two meanings; it could denote either a fundamental principle or just a definition. This in itself seems to indicate that fundamental principles were at one time identified with definitions. Furthermore, it is obvious that the very first hypotheseis on which the partners in a dialectical debate have to agree take the form of definitions." (255)

"We know that the term aitema [=postulate] came from dialectic where it was used to denote a 'demand' about which the second partner in a dialogue had reservations. Let us see whether there is any connection between this early meaning of the word and Euclid's postulates. At first glance, Postulates 1-3 appear to be such simple, self-evident and easily fulfilled 'demands' that one is tempted to disregard the literal meaning of their name." (276) But they involve movement (of compasses etc.), which is a problematic notion especially in the view of Zeno and the Eleatics. "If we bear this in mind, it is easy to understand why Euclid's first three postulates had to be laid down. ... They really are demands (aitemata) and not agreements (homologema); for they postulate motion, and anyone who adhered consistently to Eleatic teaching would not have been able to accept statements of this kind as a basis for further discussion." (279)

"Our text of Euclid" has a separate heading called common notions, but this was not a well-entrenched term and these principles "obviously bore the name axioma in pre-Euclidean times" (281), and "the noun axioma, when used as a dialectical term, was originally synonymous with aitema and just meant a 'demand' or 'request'" (286). Indeed, Euclid's common notions "all assert properties of a relation (equality) which must have been regarded (by the Eleatics at least) as 'self-contradictory'. It is by no means evident that two distinct things (i.e. two things which are not the same) can ever be 'equal to one another'; furthermore, it makes no sense to speak of two things unless they can be distinguished from one another in some way." (290)

The common notions are also dubious in that they "are assertions which are justified by practical experience and, in some cases, directly by sense-perception. Axiom 7, for example, states that 'things which coincide with one another are equal to one another'. It can literally be seen that plane figures which coincide are actually equal; hence this axiom is verified by sensory experience." (290) Therefore the common notions "could not have been accepted by the Eleatics, who required that all knowledge be obtained by purely intellectual means and without appealing to the senses. These principles were originally called demands (axiomata) because the other party in a dialectical debate had reservations about accepting them as a basis for further inquiry or, in other words, because their acceptance could only be demanded." (301)

"After Plato's time, however, the essentials of Eleatic dialectic were no longer very well understood; hence the ancient term axioma acquired a new meaning. Since it had always been used to refer to a group of principles which, from the viewpoint of common sense, were evidently valid, it came now to denote those statements whose truth was 'accepted as a matter of course'." (301)

The anti-empirical stance of the Eleatics also ties into Szabo's speculations about early geometry being based on visual reasoning. For example, Szabo argues that the original meaning of the word for "prove" (deiknymi) was "show" or "point out" in a visual sense (although "we also find deiknymi used with the meaning 'to make known by words' in texts as early as the Odyssey" (188)). "Mathematics was at one time a practical science whose subject matter was empirical and whose principles were established by empirical methods"; indeed "the sudden turn away from empiricism [in mature Greek mathematics] seems somewhat surprising" (216): one must "explain why a predominantly empirical mathematical tradition suddenly and for no apparent reason became anti-empirical and anti-visual" (217).

Szabo attributes this radical break to "the decisive influence of the Eleatic school of philosophy" (217). The notion of proof by contradiction provides the link between mathematics and the Eleatics. "It is apparent that indirect arguments played a very important part in Eleatic philosophy. Without them it would not have been possible to establish such central doctrines as that there is no motion, no change, no becoming, no perishing, no space and no time. Of course, these doctrines contradict the evidence of our senses and are incompatible with empiricism, nonetheless the Eleatics, bolstered by their belief that reason was the only guide to truth, accepted them." (218)

Mathematicians were drawn into this school of philosophy since "the existence of incommensurability could not be conclusively proved by practical or empirical methods. Hence a complex of problems associated with incommensurability made it necessary to adopt Eleatic techniques of proof in geometry." (316-317) Thus the earlier sense of proof, with its visual connotations, was replaced by axiomatic foundations in the sense of dialectic, and anti-empiricism came with the bargain.

§ R6. Greek Antiquity: Maturity

PROCLUS, *A Commentary on the First Book of Euclid's Elements*, c. 450, translated by Glenn R. Morrow, Princeton University Press, 1992.

"Proclus' commentary on book I of Euclid's Elements is almost certainly a written version of lectures which he presented to students and associates in Athens in the mid-fifth century. ... Readers of the commentary should always bear in mind that, although it is the work of Proclus, it is also a record of an educational and intellectual tradition." (ix)

Mathematics stems from the soul, not sense experience. "Should we admit that [the objects of mathematics] are derived from sense objects, either by abstraction, as is commonly said, or by collection from particulars to one common definition?" (10) No, because "The unchangeable, stable, and incontrovertible character of the proportions [of mathematics] shows that it is superior to the kinds of things that move about in matter." (3) "And how can we get the exactness of our precise and irrefutable concepts from things that are not precise? ... We must therefore posit the soul as the generatrix of mathematical forms and ideas." (11)

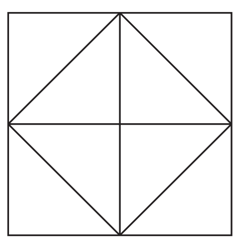
Nevertheless (somehow) mathematics has many uses. For although we just argued that it is "immaterial and theoretical; when it touches on the material world it delivers out of itself a variety of sciences—such as geodesy, mechanics, and optics—by which it benefits the life of mortals. ... and many things incredible to men it has made credible to all. Recall what Hieron of Syracuse is said to have remarked about Archimedes, who had built a three masted vessel ... When all the Syracusans together were unable to launch it and Archimedes made it possible for Hieron alone to move it down to the shore [by a system of pulleys], he exclaimed, in his amazement: 'From this day forth we must believe everything that Archimedes says.' ... Many of our predecessors have recorded such things in praise of mathematics." (50-51)

"There are nevertheless contentious persons who endeavor to detract from the worth of this science, ... declaring that the empirical sciences concerned with sense objects are more useful than the general theorems of mathematics. Mensuration, they say, is more useful than geometry, popular arithmetic than the theory of numbers, and navigation than general astronomy. For we do not become rich by knowing what wealth is but by using it, nor happy by knowing what happiness is but by living happily. Hence we shall agree, they say, that the empirical sciences, not the theories of the mathematicians, contribute most to human life and conduct. Those who are ignorant of principles but practiced in dealing with particular problems are far and away superior in meeting human needs to those who have spent their time in the schools pursuing theory alone." (22)

Reply to this objection. "We do not think it proper ... to measure its utility by looking to human needs and making necessity our chief concern. ... We must therefore posit mathematical knowledge and the vision that results from it as being worthy of choice for their own sakes, and not because they satisfy human needs. And if we must relate their usefulness to something outside them, it is to intellectual insight that they must be said to be contributory. For to that they lead the way and prepare us by purifying the eye of the soul and removing the hindrances that the senses present to our knowing the whole of things. ... Consequently instead of crying down mathematics for the reason that it contributes nothing to human needs ... we should, on the contrary, esteem it highly because it is above material needs and has its good in itself alone." (23-24)

PLATO, *Meno*, translated by W. K. C. Guthrie, Penguin, 1956.

Socrates wants to illustrate the nature of mathematical knowledge by leading an uneducated slave boy to discover how to double a given square of area four square feet. In the course of the dialog below he draws this figure in the sand, starting with one of the four small squares:



"SOCRATES: Tell me, boy, is not this our square of four feet? You understand? BOY: Yes. SOCRATES: Now we can add another equal to it like this? BOY: Yes. SOCRATES: And a third here, equal to each of the others? BOY: Yes. SOCRATES: And then we can fill in this one in the corner? BOY: Yes. SOCRATES: Then here we have four equal squares? BOY: Yes. SOCRATES: And how many times the size of the first square is the whole? BOY: Four times. SOCRATES: And we want one double the size. You remember? BOY: Yes. SOCRATES: Now does this line going from corner to corner cut each of these squares in half? BOY: Yes. SOCRATES: And these are four equal lines enclosing this area? BOY: They are. SOCRATES: Now think. How big is this area? BOY: I don't understand. SOCRATES: Here are four squares. Has not each line cut off the inner half of each of them? BOY: Yes. SOCRATES: And how many such halves are there in this figure? BOY: Four. SOCRATES: And how many in this one? BOY: Two. SOCRATES: And what is the relation of four to two? BOY: Double. SOCRATES: How big is this figure then? BOY: Eight feet. SOCRATES: On what base? BOY: This one. SOCRATES: The line which goes from corner to corner of the square of four feet? BOY: Yes. SOCRATES: The technical name for it is 'diagonal'; so if we use that name, it is your personal opinion that the square on the diagonal of the original square is double its area. BOY: That is so, Socrates. SOCRATES: What do you think, Meno? Has he answered with any opinions that were not his own? MENO: No, they were all his. SOCRATES: Yet he did not know, as we agreed a few minutes ago. MENO: True. SOCRATES: But these opinions were somewhere in him, were they not? MENO: Yes. SOCRATES: So a man who does not know has in himself true opinions on a subject without having knowledge. ... This knowledge will not come from teaching but from questioning. He will recover it for himself." (84d-85d)

PLATO, *Timaeus*, translated by Donald J. Zeyl, Hackett, 2000.

Polyhedral theory of the elements. "Let us now assign to fire, earth, water, and air the [regular polyhedra]. To earth let us give the cube, because of the four kinds of bodies earth is the most immobile and the most pliable ... And of the solid figures that are left, we shall next assign the least mobile of them to water, to fire the most mobile, and to air the one in between" (55d-56a). The dodecahedron "still remained, and this one the god used for the whole universe" (55c).

Applications of the polyhedral theory ("a moderate and sensible diversion," 59d). Water=icosahedron has 20 equilateral triangles as its sides, while fire=tetrahedron has 4 and air=octahedron 8, so "when water is broken up into parts by fire or even by air, it could happen that the parts recombine to form one corpuscle of fire and two of air" (56d), i.e., steam is two parts air and one part fire. A second example may illustrate how the relative sizes of the polyhedra matter (61a). Fire is of course the smallest, followed by air. Thus, for example, water can normally be dissolved by air (evaporation) by air octahedra slipping in between the water icosahedra. But since the fire tetrahedra are smaller they dissolve water much more efficiently. And if the water is sufficiently packed (ice) then air cannot dissolve it at all since only fire can get through the cracks.

Human anatomy is an appendix to the soul. "The entire body" was created "as its vehicle" (69c), and its properties were designed to serve the soul, e.g., "They wound the intestines round in coils to prevent the nourishment from passing through so quickly that the body would of necessity require fresh nourishment just as quickly, there by rendering it insatiable. Such gluttony would make our whole race incapable of philosophy and the arts, and incapable of heeding the most divine part within us." (73a). Even eyesight was created not for worldly purposes but primarily to give us the mind the idea of number and time:

Origins of human understanding. "Our ability to see the periods of day-and-night, of months and of years, of equinoxes and solstices, has led to the invention of number and has given us the idea of time and opened the path to inquiry into the nature of the universe. These pursuits have given us philosophy, a gift from the gods to the mortal race whose value neither has been nor ever will be surpassed. I'm quite prepared to declare this to be the supreme good our eyesight offers us." (47a-b).

Generation of animals. "[Birds] descended from ... simple-minded men, men who studied the heavenly bodies but in their naiveté believed that the most reliable proofs concerning them could be based upon visual observation. Land animals ... came from men who had no tincture of philosophy and who made no study of the heavens whatsoever ... As a consequence ... they carried their forelimbs and their heads dragging toward the ground." (91d-92a).

ARISTOTLE, *Posterior Analytics*, translated by Jonathan Barnes, Oxford University Press, 1994.

The essence Aristotle's view of the axiomatic-deductive method is summed up in the following sentence: "Demonstrative understanding ... must proceed from items which are true and primitive and immediate and more familiar than and prior to and explanatory of the conclusions." (71b)

Three notable consequences of this thesis are:

The axiomatic-deductive method is much more than mere logic. "There can be a deduction even if these conditions are not met, but there cannot be a demonstration—for it will not bring about understanding." (71b)

There is a fundamental distinction between "demonstrations which are said to demonstrate and those which lead to the impossible" (85a), i.e., proofs by contradiction, which must be seen as intrinsically inferior (87a).

Axioms stem from perception. "I call prior and more familiar in relation to us items which are nearer perception" (72a), so immediate perception must be the ultimate foundations of "demonstrative understanding." "We must get to know the primitives [i.e., axioms] by induction; for this is the way in which perception instills universals." (100b) However, "for the principles [i.e., axioms] a geometer as geometer should not supply arguments" (77b). Note the two coextensive words for "axiom"—indeed, "I call the same things principles and primitives" (72a), since immediately given truths and logical starting points of a deductive system should be the same thing.

JOHN AUBREY, *Brief Lives*, late 17th century.

"[Thomas Hobbes] was 40 years old before he looked on geometry; which happened accidentally. Being in a gentleman's library, Euclid's Elements lay open, and 'twas the 47 El. libri I [Pythagorean Theorem]. He read the proposition. By God, sayd he (he would now and then sweare an emphaticall Oath by way of emphasis), this is impossible! So he reads the Demonstration of it, which referred him back to such a Proposition; which proposition he read. That referred him back to another, which he also read. Et sic deinceps, that at last he was demonstratively convinced of that trueth. This made him in love with geometry."

REVIEL NETZ, *The Shaping of Deduction in Greek Mathematics*, Cambridge University Press, 2003.

"THEAANDTHEBTAKENTOGETHERAREEQUAL-TOTHECANDTHED This is how the Greeks would write A+B=C+D, had they written in English. And it becomes clear that only by going beyond the written form can the reader realise the structural core of the expressions. Script must be transformed into pre-written language, and then be interpreted through the natural capacity for seeing form in language. Greek mathematical formulae are post-oral, but pre-written. They no longer rely on the aural; they do not yet rely on the layout." (163)

"The lettered diagram is a distinctive mark of Greek mathematics. ... No other culture developed it independently." (58) "The overwhelming rule in Greek mathematics is that propositions are individuated by their diagrams" (38), contrary to the economy of using the same diagram for several propositions, and contrary even to plain sense, it would seem, in the use of completely functionless diagrams for number-theoretic propositions (41).

But the diagrams were schematic only, with for example conic sections being crudely represented by circular arcs (34). The diagrams were also static since "of the media available to the Greeks ... none had ease of writing and rewriting" (14). Standard media were papyri and wax tablets, and, for larger audiences, such as Aristotle's lectures, "the only practical option was wood ... painted white" (16). "None of these [ways of representing figures] is essentially different from a diagram as it appears in a book. ... The limitations of the media available suggest ... the preparation of the diagram prior to the communicative act—a consequence of the inability to erase." (16) "This, in fact, is the simple explanation for the use of perfect imperatives in the references to the setting out—'let the point A have been taken'. It reflects nothing more than the fact that, by the time one comes to discuss the diagram, it has already been drawn." (25)

"Many Greek mathematical works were originally set down within letters." (13) Which is understandable since "in every generation ... a few dozens at most of active mathematicians ... thinly spread across the eastern Mediterranean ... had to discover each other. ... Alexandria may indeed be the exception to my rule—an exception not to be overestimated, since there were never more than a handful of Alexandrian mathematicians. They formed a literary tradition, not a school." (291)

LUCIO RUSSO, *The Forgotten Revolution: How Science Was Born in 300 BC and Why it Had to Be Reborn*, Springer, 2004.

The "Scientific Revolution" took place not in the 17th century but in Hellenistic times. Its root was the marriage of philosophy and technology:

"Despite all the achievements of their culture, the Greeks of the classical age were still behind the Egyptians and Mesopotamians from the technological point of view." (28) "The technological development of all three cultures – classical Greece, Egypt and Mesopotamia – having proceeded by a gradual accumulation and transmission of empirical knowledge, it is natural that the extra millennia would give the two older civilizations a technological advantage." (29) "The Greeks who moved to the new kingdoms that arose from Alexander's conquests had to administer and control these more advanced economies and technologies with which they were not familiar; their one crucial advantage and guide consisted in the sophisticated methods of rational analysis developed by the Greek cultural tradition during the preceding centuries. It is in this situation that science is born." (29)

The practical and technological aspect of Hellenistic thought and science is often not fully appreciated because: "Lack of interest in applied science is of course documented among many classical-era Greek thinkers (who lived before the full blossoming of the scientific method) and among imperial-era Roman intellectuals (to whom the scientific method remained alien). ... If one believes in a homogeneous attitude of the 'Ancients' regarding science, one can be tempted to reconstruct it by dismissing as unrepresentative all the true scientists ... This misunderstanding is further compounded by the fact that most of what we know about Hellenistic scientists comes to us through the sieve of imperial-era writers." (198) Furthermore, modern academia is full of airheads who read only philosophers and don't understand mathematical writers (194, 202), leading them to completely miss the point of Hellenistic thought because they are too obsessed with Plato and Aristotle. This, of course, has been a prevalent disease in Europe from the Roman conquest onwards. But in Hellenistic times leading thinkers were happy to ignore Aristotle (233) and even the so-called Platonic features of Euclid's Elements are likely later insertions (320-326).

—Physical astronomy.

Archimedes is the quintessential Hellenistic scientist: a first-rate mathematician who is also deeply immersed in practical engineering. That this combination gave rise to brilliant science is proved perhaps most clearly by Archimedes's hydrostatics of paraboloids. This is an exact deductive theory that starts with natural laws and derives detailed, quantitative, highly non-trivial, empirically verifiable results (74). What more could one ask for? It is quite simply an outstanding masterpiece of science by the standards of any age. Only the mathematically illiterate could fail to grasp its immense significance – as indeed they have.

You might object that the hydrostatics of paraboloids is a very narrow topic, and try to write it off on those grounds. But you would be wrong. In the same work, "Archimedes showed that simple postulates on gravity (essentially that gravity is a spherically symmetric pull toward the center of the earth, as Aristotle thought), together with simple postulates about fluids, necessarily imply the spherical shape of the oceans (in rest conditions). ... There is no doubt that Archimedes' demonstration was also used to explain the form of the earth as a whole ... Indeed, the idea that the earth was originally fluid is reported in several sources, and in particular by Diodorus Siculus, who explicitly relates the earth's shape to gravity. ... This is a good example of how exact science can connect apparently distant subjects through logical ties: Archimedes' theorem not only cast light on the earth's geological past, but also had important astronomical and cosmological consequences[, for] Once gravity is used to explain the roundness of the earth, the next step is inevitable, namely explaining in the same way the obvious spherical shape of the sun and of the moon." (303)

Thus we are led naturally to the idea of each heavenly body having its own gravitational attraction. "Plutarch says explicitly: 'Just as the sun attracts to itself the parts of which it consists, so does the earth.'" (304) From here it is a short step to the idea that the sun has a pull not only on its own parts but also on other bodies such as the earth. Indeed the Greeks were well aware that tides can be explained this way: ancient sources "characterize without doubt the lunisolar theory" (308) of tides—that is to say, the correct explanation of tides—postulating the causal role of the sun and moon, and describing the effects in extensive and accurate detail.

Now, if every heavenly body pulls on every other, this leads to the idea of a dynamic theory of planetary motions, seeing planetary motions as composed of rectilinear inertia and gravitational pull. Indeed we read in Vitruvius: "the sun's powerful force attracts to itself the planets by means of rays projected in the shape of triangles; as if braking their forward movement or holding them back, the sun does not allow them to go forth but [forces them] to return to it" (297). Pliny likewise has it that planets are "prevented by a triangular solar ray from following a straight path" (298). The reference to triangles suggests an underlying mathematical treatment, and indeed there are further traces of this (298-302). Furthermore, "the technical tool of vector addition for displacements is present in Heron and in the pseudo-Aristotelian Mechanics, and indeed it is used in this latter work to explain how a uniform circular motion can be regarded as a continuous superposition of a displacement 'according to nature', along the tangent, with one 'contrary to nature', directed toward the center." (301-302)

Moreover, "Simplicius ... tells us that Hipparchus wrote a work on gravity titled On bodies thrust down because of gravity. This is the same terminology used several times by Plutarch" (291) in describing the question posed by "the folks who introduced the thrust toward the center" as to whether "boulders thrust through [a tunnel into] the depths of the earth, upon reaching the center, should stay still with nothing touching or supporting them; [or whether] if thrust down with impetus they should overshoot the center and turn back again and keep bobbing back and forth" (287). An indication that this was a serious mathematical question is Simplicius's statement that "Hipparchus contradicts Aristotle regarding weight, as he says that the further something is, the heavier it is." (292) "The only way to make [this statement] comprehensible is to suppose that Hipparchus meant the weight of bodies inside the earth, recognizing that it decreases as the body nears the center." (293)

Another link between celestial and everyday mechanics is this. Plutarch: "To help the moon, that it may not fall, there is its motion itself and the whizzing nature of its rotation, just as objects placed in a sling are prevented from falling by the circular motion. ... For this reason the moon does not follow its weight, which is cancelled by the counterweight of the rotation." (286) This metaphor has a suggestive consequence, for we may observe that "Anyone who has tried to spin around a weight at the end of a string has noticed that it is impossible to do this while keeping perfectly still; likewise a hammer thrower never remains immobile, but swings his own body in a small circle as he spins the hammer in a larger one." (314) Perhaps this is what Seleucus has in mind when, according to Aetius: "Seleucus the mathematician (also one of those who think the earth moves) says that the moon's revolution counteracts the whirlpool motion of the earth." (315) For perhaps "what is meant is the earth's revolution along a very small circle (that is, around the earth-moon barycenter), a wobbling very much like that of a largish object caught near the center of a whirlpool. So the moon's revolution counteracts, or is counterposed to, the earth's wobbling: as the moon describes a large orbit, the earth describes a small one, both remaining always opposed in relation to the center of the orbit, just as the hammer thrower moves in a small circle, keeping diametrically opposite the projectile in its circular trajectory." (315)

This dynamical perspective gives a further argument for heliocentrism. Seneca: "You are mistaken in thinking that any star [=planet] stops on its track or turns backward. Heavenly bodies cannot be detained or turned back; they forever move forth; as they once were sent on their way, so they continue; ... if ever [these bodies] stop, they will fall upon one another." (294) Just as a rock in a sling would fall to the ground if not held in orbit by its rotational speed, so the planetary system, if robbed of speed, would collapse into a point under mutual gravitational attraction. "Heliocentrism is able to solve the dynamical problem mentioned by Seneca: the sling argument can be applied to planetary motion exactly as to lunar motion, by making the sun, rather than the earth, be the center." (295) For on this account the planets are never actually stationary but only appear to be so when the earth is overtaking them in its orbit. This goes hand in hand with "Seneca's statement that planetary stations are just an illusion and the ship analogy [illustrating relativity of motion, which Seneca discusses]" (295).

It is well-known that Hellenistic astronomers advocated heliocentrism (80-82, 294-295, 297). Less clear is how far they worked out a complete quantitative theory of the planets using a heliocentric model. If anyone did it would most likely have been Hipparchus, and indeed there are indications that he did (285-286, 293-294). Also "several technical elements of Ptolemaic astronomy can only be explained as derivatives of an earlier heliocentric model" (317).

It is in any case known for a fact that important mathematical works from this era, and Hipparchus in particular, are completely lost almost without a trace. One clear example is this correct solution to an advanced combinatorial problem, incidentally transmitted in complete isolation: "in Plutarch's dialogues we find this remark: 'Chrysippus said that the number of intertwinings obtainable from ten simple statements is over one million. Hipparchus contradicted him, showing that affirmatively there are 103,049 intertwinings.'" (281)

—Unity of mathematics and applied science.

Philo of Byzantium says on artillery design that "in this techne many calculations are needed, and someone who makes a small departure in the individual parts causes a large error in the result." (280) At the same time, however, "everything cannot be accomplished through pure thought and the methods of mechanics, but much is found also by experiment" (111). "Thus Hellenistic scientists had already enunciated explicitly the relationship between mathematics and experiments that is usually considered typical of the Galilean method. Soon after this passage Philo gives the formula for the diameter of the opening that the spring (tension rope) goes through, and hence the diameter of the spring itself, as a function of the weight of the projectile that one wishes to throw a given distance; the diameter is proportional to the cube root of the weight, the proportionality constants being given by Philo." (111)

We also see in this example how this practical engineering problem connects directly to the most abstract mathematics of the day: "The famous problem of the doubling of the cube (extraction of cube roots) thus reveals its practical interest in the task of 'calibrating' catapults. An ingenious instrument, the mesolabe, was designed by Eratosthenes to perform the extraction. ... Eratosthenes mentions the usefulness of his instrument in designing catapults." (111)

More generally, "The modern distinction between physical and mathematical sciences was alien to Hellenistic science, which was unitary. ... Just as works on statics and optics bear a clear relation to concrete activities such as the use of balances and optical instruments ..., the exact same relation ... obtains between Euclidean geometry and drawing with ruler and compass." (189) Indeed, the other classical construction problems of geometry likewise "had some practical interest in antiquity": "the trisection of the angle ... to draw divisions corresponding to the hours in sundials" and "the quadrature of the circle ... to compute trigonometric functions, essential in topography and astronomy" (201).

"Consider the following two propositions: Construct an equilateral triangle on a given segment. With a given force move a given weight by means of gears. The first is taken from the Elements, the second from Heron's Mechanics. From the point of view of Hellenistic science these two statements (or 'problems') are strictly analogous: both are followed by an exposition of the necessary construction and then the demonstration that, based on propositions already known, the construction does satisfy the statement's conditions." (185-186) Thus for example when "Archimedes ... solved the problem of lifting a given weight with a given force" (71) this must have meant giving a concrete construction recipe, and indeed Heron describes a machine to this end in detail (99).

In Europe, "Renaissance intellectuals were not in a position to understand Hellenistic scientific theories, but, like bright children whose lively curiosity is set astir by a first visit to the library, they found in the manuscripts many captivating topics, especially those that came with illustrations. ... The most famous intellectual attracted by all these 'novelties' was Leonardo da Vinci." (335) "Leonardo's 'futuristic' technical drawings ... was not a science-fiction voyage into the future so much as a plunge into a distant past. Leonardo's drawings often show objects that could not have been built in his time because the relevant technology did not exist. This is not due to a special genius for divining the future, but to the mundane fact that behind those drawings ... there were older drawings from a time when technology was far more advanced." (336)

LUCIO RUSSO, The Definitions of Fundamental Geometric Entities Contained in Book I of Euclid's Elements, *Archive for History of Exact Science*, 52 (1998), 195–219.

Definition 4 of Euclid's Elements reads: "a straight line is [a line] which lies uniformly in respect to [all] its points." "Definitions like these are today considered useless and their inclusion in the Elements is usually seen as a serious flaw in the Elements." But "the presence of the above definitions in our manuscripts of the Elements is ... far from warranting their authenticity, in view of the scant reliability of the textual tradition." (196) Manuscript evidence suggests that "the original text of Book I of Euclid's Elements did contain some of the definitions," but that "Euclid did not hesitate in using geometrical terms he had not defined in advance" (such as "circumference")—a practice "avoided in the Imperial age" when "more definitions were again included in textbooks." (198) In the works of Archimedes and Apollonius (who "belong to the same scientific tradition" as Euclid) "there is nothing analogous to the pseudo-definitions of fundamental geometrical entities contained in the Elements. The introduction of terms implicitly defined through postulates is instead frequent." (209–210)

Heron of Alexandria (c. 50) wrote a work devoted to, as he says, "describing and sketching for you as briefly as possible ... the technical terms premised in the elements of geometry" (213). Heron's description of a straight line begins as follows: "a straight line is [a line] which, uniformly in respect to [all] its points, lies upright and stretched to the utmost towards the ends, such that, given two points, it is the shortest of the lines having them as ends." "Archimedes ... had in fact assumed that among all lines with the same ends the straight line has the minimum length. It is worth noting that Archimedes' statement was not a 'definition', but [a postulate]. In order to draw a 'definition' from Archimedes' postulate, Heron, however, could not restrict his statement to only one couple of points; he had to require that Archimedes' property should be verified uniformly in respect to all its points ... Heron's sentence is therefore completely clear." (215)

"We know that the obscure scholar who compiled the list of definitions in the form in which they now appear in Book I of the Elements was not a mathematician of any value. We have supposed that he had decided to use as definitions of elementary geometrical entities some excerpts from Heron's long illustrations. In our case he might have truncated Heron's first sentence as soon as he could get a syntactically correct sentence, even if empty of mathematical meaning." (215) The goal in doing so may have been "to get a set of short 'definitions' suitable to be learnt by heart in the schools ... If such a list was usually premised to the Elements, it could hardly avoid being eventually confused with Euclid's text." (203)

A similarly "suspicious" aspect of the definitions in the Elements is "that point and line are both defined twice (point in definitions 1 and 3 and line in definitions 2 and 6). The insertion of two independent definitions of the same term is an evident logical incongruity and it is strange that such an incongruity could have escaped Euclid." "These duplicated definitions can be easily explained as being derived from Heron, who had reported many different characterizations of point and line. If the compiler of the list of definitions afterwards included in the Elements had to decide which of Heron's sentences to keep as 'definitions' to insert in the text, it is understandable that sometimes the choice was not easy and to keep two of them could appear the best decision. Let us illustrate the situation in the particular case of the point. In order to draw from Heron's long description ... a short 'definition' of point, the most obvious device would have been the one of truncating Heron's passage, transcribing only his first proposition. The first five words of Heron's passage (a point is that which has no part) actually constitute definition 1 of the Elements. It should have been very tempting, however, to retain some of the other characterizations of point too and in particular the one of points as extremities of lines. This second characterization is also included in the Elements, as definition 3." (214)

§ R7. Conic sections

ZEUTHEN, H. G., *Die Lehre von den Kegelschnitten im Altertum*, A. F. Höst & Sohn, 1886.

In chapter 21 Zeuthen proposes the following hypothesis as to the origins of the theory of conic sections. Very little is known about the early history of conic sections, but arguably the two main facts known about it are: (i) At an early stage conics were used for the duplication of the cube. (ii) In the early period, cones were defined as line segments rotated about an axis and conic sections as the intersection of a cone with a plane perpendicular to its side.

The perpendicularity restriction in (ii) at first appears very artificial and strange. It makes no sense in terms of the natural application of conic section theory in astronomical gnomonics, nor does it make any theorems about conics easier to prove. This suggests that the study of conic sections was not originally an end in itself, but only a way of interpreting curves already necessitated elsewhere. The solution of (i) came first, and the notion of a conic section was concocted as a way of explicating the curves involved in this important construction. From this point of view the perpendicularity restriction is reasonable, since it leads to definite and clear constructions of all conic curves; greater generality is superfluous and would only muddle the matter.

NEUGEBAUER, OTTO, The Astronomical Origin of the Theory of Conic Sections, *Proceedings of the American Philosophical Society*, 92(3) (1948), 136–138.

Contrary to Zeuthen's theory, Neugebauer proposed that the theory of conic sections originated in gnomonic astronomy after all, and that in fact the perpendicularity condition in the early definition of conics is evidence not against this hypothesis but in favour of it. "The strange condition of perpendicularity of the intersecting plane always seemed to me to point to only one explanation, the theory of sundials. The generating line must be the 'gnomon' ... adjusted in such a way that it always points to the sun when it culminates. The plane onto which the shadow is cast is perpendicular to the gnomon." (136) In other words, the gnomon is pointing towards the highest (i.e., noon) position of the sun in any given day, and the plane recording its shadow is perpendicular to this gnomon. This arrangement, sure enough, produces conic sections consistent with the perpendicularity condition, for in this case the gnomon is contained within the surface of the cone defined by the circular path of the sun and the tip of the gnomon, whence the plane perpendicular to the gnomon is also perpendicular to the side of the cone, as required.

"Though I feel confident that the above explanation gives the real motivation for the early Greek theory of conic sections, I must admit that I do not know of the existence of sundials of this type." (138) It is true that Neugebauer's sundial has certain theoretically pleasing properties, as he notes with the insinuation that this ought to have been appreciated by ancient astronomers. But these pleasing properties, it seems to me, do not come from this sundial being more natural or superior but simply from removing much of the complexity of a traditional sundial by the mere stipulation that the gnomon is always pointing toward the highest point of the sun. Certainly, if this stipulation was workable in practice this sundial would have much to commend it. But it would be very cumbersome in practice to keep realigning the direction of the sundial every day so as to keep up with the sun's changing position across the seasons. If anything, the easiest way to determine the highest point of the sun is by means of a fixed sundial, so Neugebauer's sundial has a distinct note of circularity in its setup. It is a mathematician's fantasy more than a realistic proposal. Neugebauer disagrees: "The adjustment towards the culminating point is very easy to control: one must merely prevent the noon shadow from becoming visibly different from zero. Thus the whole construction is very simple in practical execution." (136) But the again how do you know when noon is? Again we are back to needing a second sundial to regulate the first.

§ R8. China

ROGER HART, *The Chinese Roots of Linear Algebra*, Johns Hopkins University Press, 2010.

The ancient Chinese had counting-board methods for solving systems of linear equations which correspond very closely to the modern determinant and Gaussian elimination methods. The earliest source for this (and in fact for Chinese mathematics altogether) is a set of bamboo strips found in a tomb dating from the year -186. The mathematics in question was probably already centuries old by then. Some of these strips are photographically reproduced on p. 29.

The strips explain how to solve two linear equations in two unknowns, for example: "In dividing coins, if each person receives 2, there is a surplus of 3; if each person receives 3, there is a shortage of 2. It is asked how many persons and coins are there?" (46)

Or: "[Fine] rice costs 3 coins for 2 dou; coarse rice costs 2 coins for 3 dou. Now given 10 dou of coarse and fine rice [combined] that is sold for 13 coins, how much coarse rice and fine rice is there?" (53)

The solutions correspond to the modern determinant method. For systems of higher order the Chinese used the Gaussian elimination method. This method is described in the first-century treatise Nine Chapters on the Mathematical Arts. A sample problem:

"[We are to ascend a mountain carrying a weight of 40 dan] given one superior horse, two common horses, and three inferior horses. ... The superior horse together with one common horse, the [group of two] common horses together with one inferior horse, and the [group of three] inferior horses together with one superior horse, are all able to ascend. Problem: How much weight do the superior horse, common horse, and inferior horse each have the strength to pull?" (165)

The solution is based on forming the "coefficient matrix" on the counting board and then bringing it into triangular form by row manipulations. One is reminded of those Mancala board games where marbles are placed in a grid of pits on a wooden board—a delightfully concrete version of a matrix, and one very well suited for these kinds of computations.

JOSEPH NEEDHAM, *Science in Traditional China*, Harvard University Press, 1981.

Chapter 5 considers some aspects of the question why "Chinese civilisation did not spontaneously develop modern natural science as Western Europe did, though China had been much more advanced in the fifteen pre-Renaissance centuries" (131).

A possible hypothesis as to what restrained the Chinese is that they lacked a zeal for progress owing to their "veritable deification" (109) of men of the past and "liturgical veneration" (111) for the past. However, this hypothesis is soon turned on its head, "for you can find textual evidence in every period showing that in spite of their veneration for the sages, Chinese scholars and scientific men believed that there had been progress beyond the knowledge of their distant ancestors" (116). In fact, much of the veneration was directed precisely at "ordinary men and women who conferred benefits upon posterity" (111). The Chinese also had a clear "conception of the three major technological stages of man's culture, the ages of stone bronze, and iron" (112). Furthermore, "each new emperor wanted to have a new [astronomical calendar], necessarily better and more accurate than any of those that had gone before. No mathematician or astronomer in any Chinese century would have dreamed of denying the continual progress and improvement in the sciences they professed. ... The same may also be said of the pharmaceutical naturalists, whose descriptions of the kingdoms of Nature grew and grew." (116).

A related hypothesis "set up by many philosophers and writers" (122) is that Christianity was particularly conducive to modern science owing to its linear, realist conception of time which is said to be essential for science not only in suggesting that progress is possible but also as a conceptual foundation for aspects of scientific theories themselves, such as the notion of causality. This hypothesis, however, is also readily refuted, as the Chinese conception of time was always predominantly linear (128; argued from circumstantial evidence), proving that whatever the difference between Europe and China may have been "it had nothing to do with China's attitude towards time" (131).

Another possible explanation is that capitalism is conducive to science "since the social situation in the era of the rise of capitalism greatly favoured ... the higher artisanate, where cooperation sprang quite naturally from working conditions" (117). "'Thus science', says Zilsel, '... came to be regarded as the product of a cooperation for nonpersonal ends, a cooperation in which all scientists of the past, the present, and the future have a part.' Today, he went on, this idea or ideal seems almost self-evident, yet no Brahamic, Buddhist, Muslim, or Latin scholastic, no Confucian scholar or Renaissance humanist, no philosopher or rhetor of classical antiquity ever achieved it. Zilsel would have done much better not to include the reference to the Confucian scholars until Europe knew a little more about them, for in fact it would seem that the idea of cumulative, disinterested, cooperative enterprise in amassing scientific information was much more customary in mediaeval China than anywhere in the pre-Renaissance West." (118).

Lastly, one must consider the hypothesis that the Chinese lacked Baconian empiricism, this so-called cornerstone of the so-called scientific method. Again, not so. One finds a number of such sayings as "those who can manage the dykes and the rivers ... did not learn their business from Yu the Great, they learned it from the waters" (119). We also have the following remarkable and unequivocal tribute to empiricism. "Liu Cho appealed to the throne in +604 for the authorisation of new research on solar shadow measurements, proposing a geodetic survey of a meridian arc. What he said was: 'Thus, the heavens and the earth will not be able to conceal their form, and the celestial bodies will be obliged to yield up to us their measurements. We shall excel the glorious sages of old, and resolve our remaining doubts about the universe. ...'" (120). This ambitious and costly undertaking was indeed carried out (though not until the next century). But it did not spark a scientific revolution, proving that there is much more to science than Baconian empiricism.

§ R9. Middle Ages

OTTO GEORG VON SIMSON, *The Gothic Cathedral*, Princeton University Press, 1988.

"At least one literary document survives that explains the use of geometry in Gothic architecture: the minutes of the architectural conferences held during 1391 and the following years in Milan. … The question debated at Milan is not whether the cathedral is to be built according to a geometrical formula, but merely whether the figure to be used is to be the square … or the equilateral triangle. … The minutes of one particularly stormy session relate an angry dispute between the French expert, Jean Mignot, and the Italians. Overruled by them on a technical issue, Mignot remarks bitterly that his opponents have set aside the rules of geometry by alleging science to be one thing and art another. Art, however, he concludes, is nothing without science, *ars sine scientia nihil est.* … This argument was considered unassailable even by Mignot's opponents. They hasten to affirm that they are in complete agreement as regards this theoretical point and have nothing but contempt for an architect who presumes to ignore the dictates of geometry."

TOBIAS DANTZIG, *Number: The Language of Science*, Masterpiece Science edition, Pi Press, 2005.

"There is a story of a German merchant of the fifteenth century, which I have not succeeded in authenticating, but it is so characteristic of the situation then existing that I cannot resist the temptation of telling it. It appears that the merchant had a son whom he desired to give an advanced commercial education. He appealed to a prominent professor of a university for advice as to where he should send his son. The reply was that if the mathematical curriculum of the young man was to be confined to adding and subtracting, he perhaps could obtain the instruction in a German university; but the art of multiplying and dividing, he continued, had been greatly developed in Italy, which in his opinion was the only country where such advanced instruction could be obtained." (26)

§ R10. Kepler

JOHANNES KEPLER, *Mysterium Cosmographicum: The Secret of the Universe [1596]*, ABaris Books, 1981.

"Greetings, friendly reader. The nature of the universe, God's plan for creating it, God's source for the numbers, ... the reason why there are six orbits, the spaces which fall between al the spheres...—here Pythagoras reveals all this to you by five figures." (49)

Although this theory itself is very famous, the details of Kepler's reasoning are not, so I would like to convey its flavour by means of a few examples. Consider first the issue of how to justify the ordering of the polyhedra that matches the planetary distances. This arrangement is justified by a myriad dubious arguments, of which the following is a representative sample:

"[The regular polyhedra] are classified into three primaries, the cube, tetrahedron and dodecahedron, and two secondaries, the octahedron and the icosahedron. For the correctness of this distinction, note the properties of each class. ... 2. Every one of the primaries has its particular type of face: the cube has the square, the pyramid the triangle, the dodecahedron the pentagon; the secondaries borrow the triangular face from the pyramid. ... 6. It is characteristic of the primaries to stand upright, of the secondaries to balance on a vertex. For if you roll the latter onto their base, or stand the former on a vertex, in either case the onlooker will avert his eyes at the awkwardness of the spectacle. ... Therefore, since there was an obvious distinction between the solids, nothing could be more appropriate than that our Earth, the pinnacle and pattern of the whole universe, and therefore the most important of the moving stars, should by its orbit differentiate between the two classes stated, and should be allotted the position which we have attributed to it above." (105)

Once this role of the earth as a divider is recognised, the five remaining planets become associated with one polyhedron each. This leads to "an astrological game" (119) with such conclusions as: "Woman is always fickle and capricious; and the shape of Venus [i.e. the icosahedron] is the most capricious and variable of all [i.e. has the most faces]." (117)

Lest anyone should look disparagingly on such "games," Kepler provides a beautiful statement of the purpose of astronomy:

"As we do not ask what hope or gain makes a little bird warble, since we know that it takes delight in singing because it is for that very singing that a bird was made, so there is no need to ask why the human mind undertakes such toil in seeking out these secrets of the heavens. ... The reason why there is such a great variety of things, and treasuries so well concealed in the fabric of the heavens, is so that fresh nourishment should never be lacking for the human mind, and it ... should have in this universe an inexhaustible workshop in which to busy itself." (55)

EDWARD ROSEN (ED.), *Kepler's Conversation with Galileo's Sidereal Messenger*, Johnson Reprint Corp., 1965.

The recent telescopic discovery of the moons of Jupiter lead to the conclusion that Jupiter is inhabited. Why else would it have moons? "For whose sake, the question arises, if there are no people on Jupiter to behold this wonderfully varied display with their own eyes?" (40) "We deduce with the highest degree of probability that Jupiter is inhabited." (44)

But the earth is still privileged (45-46): (1) It is in the middle (three bodies below, three above). (2) Its orbit touches the icosahedron and the dodecahedron, which is the most distinguished position. (3) It sees all the planets. On Jupiter they cannot see Mercury because it is too close to the sun. "Will anyone then deny that, to make up for the planets concealed from the Jovians but visible to us earth-dwellers, four other planets are allocated to Jupiter, to match the four inferior planets ... which revolve around the sun within Jupiter's orbit. Let the Jovian creatures, therefore, have something with which to console themselves."

MAX CASPAR, *Kepler*, Dover, 1993.

"Aesthetic-artistic consideration of the universe" (382). "I consider it my duty and task ... to advocate ... what I ... have recognized as true and whose beauty fills me with unbelievable rapture on contemplation." (Kepler, p. 298). "I may say with truth that whenever I consider in my thoughts the beautiful order, how one thing issues out of and is derived from another, then it is as though I had read a divine text, written onto the world itself ... saying: Man, stretch thy reason hither, so that thou mayest comprehend these things" (152).

Mathematics a means to this end. "Kepler consciously renounced [Archimedean] rigor and wanted to take over from Archimedes only so much as 'is sufficient for the pleasure of the lovers of geometry.'" (234). "Don't sentence me completely to the treadmill of mathematical calculations and leave me time for philosophical speculations, which are my sole delight. Each one has his own particular pleasure, one the tables and nativities, I the flower of astronomy, the artistic structure of the motions." (Kepler, p. 308).

Man's cognitive abilities designed for this purpose. "[T]he world partakes of quantity and the mind of man grasps nothing better than quantities for the recognition of which he was obviously created." (96). "Nature loves these relationships in everything that is capable of thus being related. They are also loved by the intellect of man who is an image of the Creator." (94). Cf. also p. 93 and above.

The universe designed for this purpose. "The earth's axis in inclined to the ecliptic in consideration of the people distributed over the whole surface of the earth, so that the change of the heavenly phenomena should extend to all places on the earth and consequently all people have a share in it. ... Sun and moon have the same apparent sizes, so that the eclipses, one of the spectacles arranged by the Creator for instructing observing creatures in the orbital relations of the sun and the moon, can occur. The earth moves around the sun to make it possible for man to get to know the world and its dimensions." (296).

Reception of the above. These ideas were quite well received e.g. in the case of the Mysterium Cosmographicum: "Professor Georg Limnäus in Jena ... is ecstatic that at last someone had again revived the time-honoured Platonic art of philosophising. ... [Tycho Brahe] takes unusual pleasure in the book: ... the zeal, the fine understanding and acumen ought to be praised [even though] certain details give him pause." (69-70). It was different with the more modern physics of the Astronomia Nova: "Kepler ran up against rejection and lack of understanding on all sides. Maestlin, Fabricius, Longomontanus and others shook their heads." (135).

CHARLOTTE METHUEN, *Kepler's Tübingen: Stimulus to a Theological Mathematics*, Scolar Press, 1998.

The human mind is created to do mathematics. According to Melanchthon, the atomistic doctrines of creation by chance "wage war against human nature, which was clearly founded to understand divine things" (76); astronomical observations are as natural to a human being as "swimming to a fish or singing to a nightingale" (85).

The purpose of scientific study is therefore twofold. (1) "inflaming their souls with love and enthusiasm for the truth and rousing them to understanding of the noblest things" (Melanchthon, p. 73). (2) Astronomers are "priests of the book of nature" (Kepler, p. 206n3). The existence of God follows from the universe's "beauty, order, and all things which have been founded for settled purposes" (Heerbrand, p. 137). "God desired that knowledge of the wonderful courses and powers should lead us towards knowledge of the divine" (Melanchthon, p. 76).

§ R11. Galilean mechanics

HERBERT BUTTERFIELD, *Origins of Modern Science*, Macmillan, 1957.

"Now, if we are seeking to understand [the] birth of modern science we must not imagine that everything is explained by the resort to an experimental mode of procedure, or even that experiments were any great novelty. It was commonly argued, even by the enemies of the Aristotelian system, that the system itself could never have been founded except on the footing of observations and experiments ... In one of the dialogues of Galileo, it is Simplicius, the spokesman of the Aristotelians—the butt of the whole piece—who defends the experimental method of Aristotle against what is described as the mathematical method of Galileo." (80)

Nowadays, "we learn how [Galileo] this martyr of science climbed the leaning tower of Pisa with a one-hundred-pound cannon ball under one arm and a one-pound ball under the other. ... None of the vast crowd who are supposed to have observed the experiment gave any evidence on its behalf ... and the writings of Galileo give no confirmation of the story. On the contrary, the writings of Galileo showed that he had tried the experiment several times in his youth with the opposite result" (81) "To crown the comedy, it was an Aristotelian, Coresio, who in 1612 claimed that previous experiments had been carried out from too low an altitude. In a work published in that year he described how he had improved all previous attempts—he had not merely dropped the bodies from a high window, he had gone to the very top of the tower of Pisa. The larger body had fallen more quickly than the smaller one ..., and the experiment, he claimed, had proved Aristotle to have been right all the time." (82)

Thus Galileo's law of falling bodies was not based on experiment but was in fact directly counter-experimental. Likewise, the second main law of terrestrial mechanics, the law of inertia, "was hardly a thing which the human mind would ever reach by an experiment" (84) And in the second great science of the day, astronomy, not a single experiment was performed, but it somehow made drastic progress anyway, and not infrequently by flatly disregarding empirical evidence. Meanwhile, "the science in which experiment reigned supreme [i.e. alchemy/chemistry] was remarkably slow, if not the slowest of all, in reaching its modern form" (81)

§ R12. Galileo versus the Church

, *The Bible*, King James version.

"Then spake Joshua to the Lord in the day when the Lord delivered up the Amorites before the children of Israel, and he said in the sight of Israel, Sun, stand thou still upon Gibeon; and thou, Moon, in the valley of Ajalon. And the sun stood still, and the moon stayed, until the people had avenged themselves upon their enemies. … So the sun stood still in the midst of heaven, and hasted not to go down about a whole day." (Joshua, X.12-13)

RICHARD J. BLACKWELL, *Galileo, Bellarmine, and the Bible*, University of Notre Dame Press, 1992.

The counter-reformation unwittingly provided the weapons with which Galileo would be attacked. Luther challenged church authority and emphasised reliance on and personal understanding of the Bible (sola Scriptura). The Council of Trent was formed to answer this threat. The Council decreed that: God's message was conveyed both through the Bible and unwritten traditions; interpreting the Bible is a matter for the appropriate authorities; "in matters of faith and morals" no one shall dare to interpret the Bible contrary to church tradition and "the unanimous agreement of the Fathers."

Bellarmine, the most important authority in Galileo's time, interpreted these decrees in the most unfortunate way possible for Galileo. Firstly, he took the relation of Bible and tradition to be complementary rather than concurring. Thus the Fathers' approval of geocentrism is independent rather than redundant evidence. Secondly, he took "unanimous agreement of the Fathers" to be achieved when one Father spoke on a matter while the others remained silent. Thus strengthening this independent evidence. Lastly, he interpreted "matters of faith" in such a way that everything in the Bible (geocentrism, the fact that Tobias had a dog, etc.) is a matter of faith. This is the crucial point in Bellarmine's letter to Foscarini. Foscarini and Galileo had a reasonable argument that Copernicanism was not in conflict with the Fathers or the Counsil, but these authoritative interpretations by Bellarmine effectively made any further debate impossible ("checkmate," as Blackwell says).

But Bellarmine and the church establishment had no interest in bringing about Inquisition proceedings. However, Galileo's Aristotelian enemies (with whom he had debated on floating bodies and mechanics) saw an opportunity. By persistently and prominently accusing Galileo of arguing contrary to scripture they forced him into a dilemma: either let the argument stand unopposed (thus blocking his theories from being accepted and pursued) or get involved with the dangerous matter of scriptural interpretation. Galileo chose the latter option; now all the Aristotelians had to do was to sit back and watch the vultures mince this easy pray. Caccini, a petty priest and "a thoroughly nasty person," took the bait and unwittingly executed the Aristotelians' plan. He preached against Galileo with such fervour that his superior had to send Galileo a formal apology (69). And with considerable effort and calculated deceit he managed to force the matter onto the initially reluctant Inquisition (112).

MAURICE A. FINOCCHIARO (ED.), *The Galileo Affair: A Documentary History*, University of California Press, 1989.

The dispute seems to have been sparked not so much by heliocentrism as such but rather by Galileo's forays into scriptural interpretation. Galileo claims that in such matters "one must begin not with the authority of scriptural passages but with sensory experience and necessary demonstrations" (93). This because "Scripture appear to be full not only of contradictions but also of serious heresies and blasphemies; for one would have to attribute to God feet, hands, eyes, and bodily sensations, as well as human feelings like anger contrition, and hatred, and such conditions as the forgetfulness of things past and the ignorance of future ones" (an argument which, by the way, we hear him repeat three times; pp. 50, 85, 92). The clash with the interpretations of the church fathers Galileo explains by the fact that heliocentrism was not an issue at that time (108) and that such matters were considered unimportant. He quotes St. Augustine as saying that "God did not want to teach men these things which are of no use to salvation," and ask how, then, "one can now say that to hold this rather than that proposition on this topic is so important that one is a principle of faith and one is erroneous?" (95). As for actual biblical interpretation, Galileo's most prominent example is that of Joshua stopping the sun to lengthen the day. Galileo criticises the geocentric interpretation by distinguishing the "Prime Mobile" daily motion of the heavens and the annual motion of the sun along the zodiac: stopping the latter would not lengthen the day but rather shorten it. He offers instead a Copernican interpretation which is based on the assumption that the sun's rotation causes all motion, so that stopping it would stop the entire solar system. (53-54.)

It seems that it was primarily this provocation that brought the matter to the Inquisition's attention (134-135, 138). Once provoked, the Inquisition also moved to condemn holding heliocentrism as physical truth. Perhaps they did so only because of the theory's proponents' explicit polemic with the church. After all, Copernicus' book had long been permitted, and Galileo's own Letters on Sunspots of 1613 had been censored only where it referred to scripture, not where it asserted heliocentrism.

The outcomes of the first Inquisition proceedings (1615-1616) were: a condemnation of heliocentrism as "formally heretical" (146); a special injunction that Galileo must not "hold, teach or defend it in any way whatever" (147); mild censoring of Copernicus' book (viz., removal of a passage concerning the conflict with the Bible and a handful expressions which insinuated the physical truth of the theory; pp. 149, 200-202). Thus Galileo was not actually convicted, and to protect himself from slander he requested a certificate of this fact from Cardinal Bellarmine (153).

Galileo did indeed keep quiet for a number of years, but he was lured out of silence, it seems, by a false sense of security stemming from his good relations with the new Pope, Urban VIII (cf. p. 155), in light of which he "artfully and cunningly extorted" (in the words of the Inquisition, p. 290) a permission to publish the Dialogue on the Two Wold Systems in 1632.

A special commission appointed by the Pope found many inappropriate things in the Dialogue, but this was not a major issue, they noted, for such things "could be emended if the book were judged to have some utility which would warrant such a favor" (222). The problem was instead that Galileo "may have overstepped his instructions" not to treat heliocentricism (219). This is an internal document so presumably it is sincere. The same report also points out that Galileo had placed the Pope's favourite argument (that the omnipotent God could have created any universe, including a heliocentric one), which he had been asked to include, "in the mouth of a fool" (221).

This forced the second Inquisition proceedings in 1633. Galileo's defence was quite pathetic and transparently dishonest. He claimed that: in the Dialogue "I show the contrary of Copernicus's opinion, and that Copernicus's reasons are invalid and inconclusive" (262); in light of the accusations, "it dawned on me to reread my printed Dialogue," and to his surprise "I found it almost a new book by another author" (277-278); he did not recall the injunction's phrases "to teach" or "any way whatever" since these did not appear in Bellarmine's certificate, "which I relied upon and kept as a reminder" (260). Of course he was forced to abjure. The Dialogue was prohibited, but not for its contents but rather, in the words of the Inquisition's sentence, "so that this serious and pernicious error and transgression of yours does not remain completely unpunished" and as "an example for others to abstain from similar crimes" (291).

AMIR ALEXANDER, *Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World*, Scientific American / Farrar, Straus and Giroux, 2014.

The Jesuits were the intellectual leaders of the Catholic world in the 17th century. They ran hundreds of colleges across Europe, notable as much for their "sheer educational quality" (46) as for their doctrinal role "in the fight to defeat Protestantism" (41).

The Jesuit colleges placed great emphasis on Euclidean mathematics. "It was a logical sequence of studies, but for [the Jesuits] it also represented a deeper ideological commitment. Geometry, being rigorous and hierarchical, was, to the Jesuit, the ideal science. The mathematical sciences that followed—astronomy, geography, perspective, music—were all derived from the truths of geometry ... Consequently, Clavius's mathematical curriculum ... demonstrated how absolute eternal truths shape the world and govern it" (72), thereby serving as a model for their religious doctrine and worldview. "Euclidean geometry thus came to be associated with a particular form of social and political organization, which ... the Jesuits strived for: rigid, unchanging, hierarchical, and encompassing all aspects of life." (218)

For this reason, "the Jesuits reacted with ... fury to the rise of infinitesimal methods. For the mathematics of the infinitely small was everything that Euclidean geometry was not. Where geometry began with clear universal principles, the new methods began with a vague and unreliable intuition that objects were made of a multitude of minuscule parts. Most devastatingly, whereas the truths of geometry were incontestable, the results of the method of indivisibles were anything but" (175), thereby undermining "the Jesuit quest for a single, authortized, and universally accepted truth" (176).

Thus infinitesimal mathematics was dangerous to the Jesuits not for intrinsic mathematical reasons but because it was associated with diversity of thought unchecked by authority. "'Unless mind are contained within certain limits', warned Father Leone Santi, prefect of studies at the Collegio Romano ..., 'their excursions into exotic and new doctrines will be infinite', leading to 'great confusion and perturbation to the Church'." (122)

Consequently, "In a fierce decades-long campaign, the Jesuits worked relentlessly to discredit the doctrine of the infinitely small and deprive its adherents of standing and voice in the mathematical community. Their efforts were not in vain: as 1647 was drawing to a close, the brilliant tradition of Italian mathematics was coming to an end as well." (117)

According to Alexander the Jesuit campaign against infinitesimals was extremely successful and influential. As he puts it, "champions of the infinitely small (Galileo, Cavalieri, and Torricelli) pioneered new techniques that would transform the very foundations of mathematical inquiry and practice. But when the Jesuits triumphed over the advocates of the infinitely small, this brilliant tradition died a quick death," (178), leaving "no one left in Italy to carry the torch of the infinitely small" (165). "In Italy, the stage was set for centuries of backwardness and stagnation." (180) "No city or prince wished to risk the wrath of the Jesuits, and as a result, no university chairs or positions of honor at princely courts were in the offing for supporters of the infinitely small." (165)

It should be noted, however, that Cavalieri's generation did in fact have a number of direct followers in prominent university positions. Cavalieri's student "Pietro Mengoli (1626-84) succeeded Cavalieri to the mathematics chair at Bologna," (164) a position he held for the remaining 39 years of his life. Angeli was another student of Cavalieri's, and a prima facie counterexample to Alexander's thesis. "In 1662, he was appointed to the chair of mathematics at the University of Padua, a position once held by Galileo. The Jesuits, so powerful elsewhere in Italy, could only fume as the upstart ... was raised to one of the most prestigious mathematical posts in all Europe." (170-171) "Angeli ... took on the Jesuits like no one had dared since the days of Galileo himself. He called them names [and] ridiculed [them]," (170) and "published no fewer than nine books promoting and using the method of indivisibles" (174), and held his chair for the rest of his life.

§ R13. Mechanical Philosophy

ROBERT BOYLE, *A Free Enquiry into the Vulgarly Received Notion of Nature [1686]*, Cambridge University Press, 1996.

Allegorical statement of the critical method: "For I am wont to judge of opinions as of coins: I consider much less, in any one that I am to receive, whose inscription it bears, than what metal it is made of. It is indifferent enough to me whether it was stamped many years or ages since, or came but yesterday from the mint. Nor do I regard through how many, or how few, hands it has passed for current, provided I know by the touchstone or any sure trial purposely made, whether or no it be genuine, and does or does not deserve to have been current. For if upon due proof it appears to be good, its having been long and by many received for such will not tempt me to refuse it. But if I find it counterfeit, neither the prince's image or inscription, nor its date (how ancient soever), nor the multitude of hands through which it has passed unsuspected will engage me to receive it. And one disfavouring trial, well made, will much more discredit it with me than all those specious things I have named can recommend it." (5)

The fundamental conflict between the two views of nature is brought out by another allegory: "They seem to imagine the world to be after the nature of a puppet, whose contrivance indeed may be very artificial, but yet is such that almost every particular motion the artificer is fain (by drawing sometimes one wire or string, sometimes another) to guide, and oftentimes overrule, the actions of the engine; whereas, according to us, it is like a rare clock ... where all things are so skilfully contrived that the engine being once set a-moving, all things proceed according to the artificer's first design, and the motions of the little statues that at such hours performs these or those things do not require (like those of puppets) the peculiar interposing of the artificer or any intelligent agent employed by him, but perform their functions upon particular occasions by virtue of the general and primitive contrivance of the whole engine." (12-13)

As for the actual arguments against the teleological view of nature, they consist most importantly in "diverse phenomena which do not agree with the notion or representation of nature that I question" (63), such as:

Nature abhors vacuum only inconsistently. "When a glass bubble is blown very thin at the flame of a lamp and hermetically sealed while it is very hot, the cause that is rendered why it is apt to break when it grows cold, is that the inward air (which was before rarefied by the heat), coming to be condensed by the cold, lest the space deserted by the air that thus contracts itself should be left void, nature with violence breaks the glass in pieces. But, by these learned men's favour, if the glass be blown but a little stronger than ordinary, though at the flame of a lamp, the bubble (as I have often tried) will continue unbroken, in spite of nature's pretended abhorrency of a vacuum, which needs not at all to be recurred to in the case." (65) "And why does she furiously break in pieces a thin sealed bubble, such as I come from speaking of, to hinder a vacuum? If in case she did not break it, no vacuum would ensue. And on the other side, if we admit her endeavours to hinder a vacuum not to have been superfluous, and consequently foolish, we must confess that where these endeavours succeed not, there is really produced such a vacuum as she is said to abhor. So that, as I was saying, either she must be very indiscreet to trouble herself and to transgress her own ordinary laws to prevent a danger she need not fear, or her strength must be very small—that is, not able to ... break a tender glass bubble, which perhaps a pound weight on it would ... crush in pieces." (66-67)

Bouncing ball wasteful. "For if (for example) you let fall a ball upon the ground, it will rebound to a good height, proportionable to that from whence you let it fall, or perhaps will make several lesser rebounds before it come to rest. If it be now asked, why the ball, being let out of your hand, does not fall on this or that side, or move upwards, but falls directly towards the centre of the earth by that shortest line ... which is the diameter of the earth prolonged to the centre of gravity of the ball? It will be readily answered that this proceeds from the ball's gravity, i.e. an innate appetite whereby it tends to the centre of the earth the nearest way. But then I demand, whence comes this rebound, i.e. this motion upwards? For it is plain, it is the genuine consequence of the motion downwards, and therefore is increased according as that motion in the ball was increased, by falling from a greater height. So that it seems that nature does in such cases play a very odd game, since she forces a ball, against the laws of heavy bodies, to ascend divers times upwards, upon the account of that very gravity whose office it is to carry it downwards the directest way. And at least she seems, in spite of the wisdom ascribed to her, to take her measures very ill, in making the ball move downwards with so much violence, as makes it divers times fly back from the place she intended it should go to. As if a ball which a child can play with and direct as he pleases were so unwieldy a thing that nature cannot manage it, without letting it be hurried on with far greater violence than her design requires." (67-68)

Bubbles in water inconsistent with natural place theory. "For if a bubble happens to arise from the bottom of a vessel to the upper part of it, we are told that the haste wherewith the air moves through water proceeds from the appetite it has to quit that preternatural place and rejoin the element, or great mass of air detained at the very surface of the water by a very thin skin of that liquor, together with which it constitutes a bubble. Now I demand how it comes to pass, that this appetite of the air—which, when it was at the bottom of the water, and also in its passage upwards, is supposed to have enabled it to ascend with so much eagerness and force as to make its way through all the incumbent water (which possibly was very deep)—should not be able, when the air is arrived at the very top of the water, to break through so thin a membrane of water as usually serves to make a bubble, and which suffices to keep it from the beloved conjunction with the great mass of the external air, especially since they tell us that natural motion grows more quick, the nearer it comes to the end or place of rest, the appetites of bodies increasing with their approaches to the good they aspire to, upon which account falling bodies, as stones, etc., are said (though falsely) to increase their swiftness the nearer they come to the earth. But if, setting aside the imaginary appetite of the air, we attribute the ascension of bubbles to the gravity and pressure upwards of the water, it is easy hydrostatically to explicate why bubbles often move slower when they come near the surface of the water, and why they are detained there; which last phenomenon proceeds from this: that the pressure of the water being there inconsiderable, it is not able to make the air quite surmount the resistance made by the tenacity of the superficial part of the water. And therefore in good spirit of wine, whose tenacity and glutinousness is far less than that of water, bubbles rarely continue upon the surface of the liquor, but are presently broken and vanish. ... I shall add that I have often observed that water, in that state which is usually called its natural state, is wont to have store of aerial particles mingled with it ... as may appear by putting a glass full of water into the receiver of the new pneumatical engine. For the pressure of the external air being by the pump taken off, there will from time to time disclose themselves in the water a multitude of bubbles, made by the aerial particles that lay concealed in that liquor. ... so little appetite has air in general to flee all association with water and make its escape out of that liquor, though when sensible portions of it happen to be underwater, the great inequality in gravity between those two fluids makes the water press up the air." (82-83)

Nature changeable and forgetful in caring to restore springy bodies. "If, for example, you take a somewhat long and narrow plate of silver that has not been hammered or compressed ... you may bend it which way you will, and it will constantly retain the last curve figure that you gave it. But if, having again straightened this plate, you give it some smart strokes of a hammer, it will by that merely mechanical change become a springy body: so that if with your hand you force it a little from its rectitude, as soon as you remove your hand it will endeavour to regain its former straightness. ... Now upon these phenomena I demand why, if nature be so careful to restore bodies to their former state, she does not restore the silver blade or plate to its rectitude when it is bent this way or that way before it be hammered? And why a few strokes of a hammer (which, acting violently, seems likely to have put the metal into a preternatural state) should entitle the blade to nature's peculiar care, and make her solicitous to restore it to its rectitude when it is forced from it? And why, if the springy plate be again ignited and refrigerated of itself, nature abandons her former care of it, and suffers it quietly to continue in what crooked posture one pleases to put it into? ... I shall add to what I was just now saying, that even in sword blades it has been often observed that though if, soon after they are bent, the force that bent them be withdrawn, they will nimbly return to their former straightness. Yet if they ... be kept too long bent, they will lose the power of recovering their former straightness and continue in that crooked posture, though the force that put them into it cease to act. So that it seems nature easily forgets the care she was presumed to take of it at first." (86-87)

§ R14. Natural science

WILLIAM SHAKESPEARE, *As You Like It*, 1603, Act II, Scene 1.

The natural scientist "Finds tongues in trees, books in the running brooks, sermons in stones, and good in everything."

JOHN KEATS, *Lamia*, part II, 1884.

"Philosophy will clip an angel's wings,
Conquer all mysteries by rule and line,
Empty the haunted air, and gnomed mine—
Unweave a rainbow..."

MARJORIE HOPE NICOLSON, *Pepys' Diary and the new science*, University Press of Virginia, 1965.

Judging by Pepys' Diary, a day in the good life in the 1660s goes something like this.

"Up by 4 o'clock in the morning, and read Cicero's Second Oration against Catiline, which pleased me exceedingly; and more I discern therein than ever I thought was to be found in him." (6)

After this we set out to "walk as far as the Temple, doing some business in my way at my bookseller's and elsewhere" (37). One does not return without "having first discoursed with Mr. Hooke a little, whom we met in the streete, about the nature of sounds, and he did make me understand the nature of musicall sounds made by strings, mighty prettily; and told me that having come to a certain number of vibrations proper to making any tone, he is able to tell how many strokes a fly makes with her wings (those flies that hum in their flying) by the note that it answers to in musique" (37).

In the afternoon, a visit to the Royal Society for some experiments. Perhaps we try to kill some animals with the vacuum pump—as the poet put it: "Out of the glasse the Ayre being screwed, Pusse dyed and ne're so much as mewed" (117)—except that "we could not quite kill her, with such a way" for "the air being in upon her revives her immediately" (65). Or perhaps we try "that notable experiment of opening a dog, and laying bare his lungs, and blowing into him with bellows, keeping him thus alive as long as we pleased" (66). Or—if we don't want to overthink it—we could just give a dog some opium: "the effects whereof became manifest ... for he immediately began to nod and reel as he walk'd; whereupon ... I order'd him to be kept awake by whipping" (69). All in the name of science, of course.

"In the evening comes Mr. Reeves, with a twelve-foote glasse" to observe the moon and the planets "till one in the morning", after which it is off "to bed mighty sleepy, but with much pleasure" (23).

A bad day, on the other hand, goes like this: "Much against my will staid out the whole church ... but I did entertain myself with my perspective glass up and down the church, by which I had the great pleasure of seeing and gazing at a great many very fine women; and with that, and sleeping, I passed away the time till sermon was done." (22-23)

J. L. HEILBRON, *Elements of Early Modern Physics*, University of California Press, 1982.

In 1703, when Newton became its president, the Royal Society "wished to revive the healthy old custom, then disused for a generation, of showing experiments at the weekly meetings. The custom had been neglected not from disinterest or poverty but because it demanded a continual inventiveness that ultimately emptied even the ablest. To arouse the interest and stimulate the thinking of the heterogeneous fellowship required demonstration luciferous in philosophy, useful in art, ingenious in contrivance, and surprising and amusing in execution." (168). Similarly, electricity was taken up by "playful German professors" (179) who liked to "kill flies with sparks from their fingers" (177).

Soon the discovery of the Leyden jar enabled scientists to administer much more powerful shocks, which they immediately tried on themselves. Musschenbroek wrote to his friends at the Paris Academy explaining "how they too could blast themselves with electricity." "'I thought I was done for,'" he wrote, "adding precise directions for realizing the 'terrible experiment' and advice not to try it." The academicians could of course not resist "blasting themselves" right away and "reported bleedings, temporary paralysis, concussions, convolutions, and dizziness," with one person even "warning that his wife was unable to walk after he used her to shorten a Layden jar." (184).

"Science is a social enterprise. Let a gentleman hold the jar and a lady the PC; both feel the shock when they touch. How many others can be inserted in the train? Academician L. G. Le Monnier tried 140 courtiers, before the king; Nollet shocked 180 gendarmes in the same presence, and over 200 Cistercians in their monastery in Paris. 'It is singular to see the multitude of different gestures, and to hear the instantaneous exclamations of those surprised by the shock.' Only persons in the train felt the commotion; those in side chains branching from the main line felt nothing. Thus electricians discovered that the discharge—to use the word they introduced for the climax of the Leyden experiment—goes preferentially along the best conducting circuit ... In one demonstration only those at the extremes of the chain felt the shock, which appeared to avoid one of the company suspected 'of not possessing everything that constitutes the distinctive character of a man.' Some wits deduced that eunuchs cannot be electrified [but] three of the king's musicians, 'whose state was not equivocal,' held hands, and jumped as other men. The shy shock was found to occur only when the train stood on moist ground; apparently the discharge went through the arms and legs of the extreme members and completed its course in the soil." (186).

MARIE BOAS HALL, *Promoting Experimental Learning: Experiment and the Royal Society*, Cambridge University Press, 2002.

The Royal Society experimented with blood transfusions in the 1660s. At first "dogs were used; the donor was bled until it died, whereupon the recipient was sewn up and when set free showed itself very lively." A flurry of inter-species blood transfusions followed, with enough success that a human subject was found who agreed to have his blood replaced with that of a sheep. "The subject, one Arthur Coga, an indigent Oxford graduate, not only survived being given the blood of a sheep, but two months later he read a paper to the Society describing the effects which he had experienced." The same experiment was tried with less success by Denis in France. "One of Denis's subjects did die after a second transfusion, whether, as seems likely, as a direct result of the experiment, or, as Denis claimed, from poison administered by his wife. In any case human blood transfusion was then legally banned in Paris and the Royal Society wisely decided to discontinue the practice."

WALTER GRATZER (ED.), *Bedside Nature 1869-1953: Genius and Eccentricity in Science*, W.H. Freeman, 1999.

"The Influence of a Tuning-Fork on the Garden Spider." "Last autumn, while watching some spiders spinning their beautiful geometrical webs, it occurred to me to try what effect a tuning-fork would have upon them. On sounding an A fork and lightly touching with it any leaf or other support of the web or any portion of the web itself I found that the spider, if at the centre of the web, rapidly slews round so as to face the direction of the fork, feeling with its fore feet along which radial thread the vibration travels. Having become satisfied on this point, it next darts along that thread till it reaches either the fork itself or a junction of two or more threads the right one of which it instantly determines as before. If the fork is not removed when the spider has arrived it seems to have the same charm as any fly for the spider seizes it, embraces it, and runs about on the legs of the fork as often as it is made to sound never seeming to learn by experience that other things may buzz besides its natural food.

If the spider is not at the centre of the web at the time that the fork is applied, it cannot tell which way to go until it has been to the centre to ascertain which radial thread is vibrating ... The spider never leaves the centre of the web without a thread along which to travel back. If after enticing a spider out we cut this thread with a pair of scissors the spider seems to be unable to get back without doing considerable damage to the web generally, gumming together the sticky parallel threads in groups of three and four.

By means of a tuning-fork a spider may be made to eat what it would otherwise avoid. I took a fly that had been drowned in paraffin and put it into a spider's web and then attracted the spider by touching the fly with a fork. When the spider had come to the conclusion that it was not suitable food and was leaving it, I touched the fly again. This had the same effect as before, and as often as the spider began to leave the fly I again touched it and by this means compelled the spider to eat a large portion of the fly.

The supposed fondness of spiders for music must surely have some connection with these observations." (63)

"Suicide of a Scorpion." "One morning a servant brought to me a very large specimen of this scorpion ['the common Black Scorpion of Southern India'], which, having stayed out too long in its nocturnal rambles, had apparently got bewildered at daybreak, and been unable to find its way home. To keep it safe, the creature was at once put into a glazed entomological case. Having few leisure moments in the course of forenoon, I thought I would see how prisoner was getting on, and to have a better view of it the case was placed in a window, in the rays of a hot sun. The light and heat seemed to irritate it very much, and this recalled to my mind a story I had read somewhere, that a scorpion, on being surrounded with fire, had committed suicide. I hesitated about subjecting my pet to such terrible ordeal, but taking a common botanical lens, I focused the rays of the sun its back. The moment this was done it began to run hurriedly about the case, hissing and spitting in a very fierce way. This experiment was repeated some four or five times with like results, but on trying it once again, the scorpion turned up its tail and plunged the sting, quick as lightning, into its own back. The infliction of the wound was followed by a sudden escape of fluid, and a friend standing by me called out, 'See ,it has stung itself; it is dead;' and sure enough in less than half a minute life was quite extinct. I have written this brief notice show (1) That animals may commit suicide; (2) That the poison of certain animals may be destructive to themselves." (41)

§ R15. Early modern attitudes towards mathematics

ROGER ASCHAM, *The Schoolmaster*, 1570.

Some wits, moderate enough by nature, be many times marred by over much study and use of some sciences, namely, music, arithmetic, and geometry. These sciences, as they sharpen men's wits over much, so they charge men's manners over sore, if they be not moderately mingled, and wisely applied to some good use of life. Mark all mathematical heads, which be wholly and only bent to those sciences, how solitary they be themselves, how unapt to serve in the world.

WILLIAM KEMPE, , translator's dedication in Ramus, *The Art of Arithmeticke*, 1592.

Take away arithmetic, ye take away the merchant's eye, whereby he seeth his direction in buying and selling; ye take the goldsmith's discretion, whereby he mixeth his metals in due quantities; ye take away the captain's dexteritie, whereby he embattaileth his army in convenient order; finally ye take from all sorts of men, the faculty of executing their functions aright. Arithmetic then teacheth unto us matters in divinity, judgeth civil causes uprightly, cureth diseases, searcheth out the nature of things created, singeth sweetly, buyeth, selleth, maketh accompts, weigheth metals and worketh them, skirmisheth with the enemy, goeth on warfare, and setteth her hand almost to every good work, so profitable is she to mankind.

LESLEY MURDIN, *Under Newton's Shadow*, CRC Press, 1985.

"Wallis in 1632 found no-one to teach him mathematics [at Oxford] ... 'For Mathematics (at that time with us) were scarce looked upon as Academical studies but rather Mechanical: as the business of Traders, Merchants, Seamen, Carpenters, Surveyors of land or the like.'" (38). In other words, the currently popular reasons to study mathematics were at the time compelling reasons not to study it at all.

AMIR ALEXANDER, *Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World*, Scientific American / Farrar, Straus and Giroux, 2014.

Wallis's work on infinite series was based on daring, unrigorous extrapolations and generalisations, which he considered "'a very good Method of Investigation ... which doth very often lead us to the early discovery of a General Rule'. Most important, 'it need not ... any further Demonstration'." (270)

Hobbes, by contrast, appealed to the authority and rigour of Euclidean geometry as a model for reasoning as well as political organisation.

"Wallis and Hobbes both believed that mathematical order was the foundation of the social and political order, but beyond this common assumption, they could agree on practically nothing else. Hobbes advocated a strict and rigorous deductive mathematical method, which was his model for an absolutist, rigid, and hierarchical state. Wallis advocated a modest, tolerant, and consensus-driven mathematics, which was designed to encourage the same qualities in the body politic as a whole." (256)

Wallis's vision of mathematics was very agreeable to the experimental scientists of the Royal Society. "Experimentalism is a humbling pursuit, very different from the brilliance and dash of systematic philosophers such as Descartes and Hobbes. It is, wrote Sprat, 'a laborious philosophy ... that teaches men humility and acquaints them with their own errors'. And that is precisely what the founders of the Royal Society liked about it. Experimentalism, as Sprat noted, 'removes all haughtiness of mind and swelling imaginations', teaching men to work hard, to acknowledge their own failures, and to recognize the contributions of others." (253)

"Mathematics, [the Royal Society founders] believed, was the ally and the tool of the dogmatic philosopher. It was the model for the elaborate systems of the rationalists, and the pride of the mathematicians was the foundation of the pride of Descartes and Hobbes. And just as the dogmatism of those rationalists would lead to intolerance, confrontation, and even civil war, so it was with mathematics. Mathematical results, after all, left no room for competing opinions, discussions, or compromise of the kind cherished by the Royal Society. Mathematical results were produced in private, not in a public demonstration, by a tiny priesthood of professionals who spoke their own language; used their own methods, and accepted no input from laymen. Once introduced, mathematical results imposed themselves with tyrannical, power, demanding perfect assent and no opposition. This, of course, was precisely what Hobbes so admired about mathematics, but it was also what Boyle and his fellows feared: mathematics, by its very nature, they believed, leads to claims of absolute truth, dogmatism, threats of tyranny." (256)

JED Z. BUCHWALD & I. BERNARD COHEN (EDS.), *Isaac Newton's Natural Philosophy*, MIT Press, 2004.

Feingold's chapter discusses a fundamental clash between mathematicians and naturalists at the Royal Society. One reads with envy about the gravity attached to matters of scientific research policy at the time: "there has been much canvassing and intrigue made use of, as if the fate of the Kingdome depended on it" (77). "On the eve of Newton's election as president, matters had deteriorated to such an extent that various fellows could be restrained only with difficulty from a public exchange of blows (or, in one case, the drawing of swords)" (93).

So what was this conflict on which "the fate of the Kingdome" depended? The "philomats" identifying with Newton attacked the naturalists thus: "That Great Man [Newton] was sensible, that something more than knowing the Name, the Shape and obvious Qualities of an Insect, a Pebble, a Plant, or a Shell, was requisite to form a Philosopher, even of the lowest rank, much more to qualifie one to sit at the Head of so great and learned a Body." (77)

The naturalists, for their part, identified with Bacon, who had complained about "the daintiness and pride of mathematicians, who will needs have this science almost domineer over Physic. For it has come to pass, I know not how, that Mathematics and Logic, which ought to be but the handmaids of Physic, nevertheless presume on the strength of the certainty which they possess to exercise dominion over it." (80)

Similar points were raised many times, as here in 1700 by a minor figure: "Mathematical Arguments, of which the World is become most immoderately fond, looking upon every thing as trivial, that bears no relation to the Compasse, and establishing the most distant parts of Humane Knowledge; all Speculations, whether Physical, Logical, Ethical, Political, or any other upon the particular results of number and Magnitude. ... In any other commonwealth but that of Learning such attempts towards an absolute monarchy would quickly meet with opposition. It may be a kind of treason, perhaps, to intimate thus much; but who can any longer forbear, when he sees the most noble, and most usefull portions of Philosophy lie fallow and deserted for opportunities of learning how to prove the Whole bigger than the Part, etc." (90)

HELENA PYCIOR, *Symbols, Impossible Numbers, and Geometric Entanglements*, Cambridge University Press, 1997.

In the 17th century, some took "the symbolical style [of algebra] as a model for terse, scientific expression." Indeed, the Royal Society "exacted from all their members a close, naked, natural way of speaking ... bringing all things as near the Mathematicall plainness as they can." (Sprat, p. 46) Others disagreed. "Symbols are poor unhandsome, though necessary, scaffolds of demonstration; and ought no more to appear in public, than the most deformed necessary business which you do in your chambers." (Hobbes, p. 145) Pages of algebra often look "as if a hen had been scraping there." (Hobbes, p. 147)

Negative numbers. "That which most perplexes narrow minds in this way of thinking, is, that in common life, most quantities lose their names when they cease to be affirmative, and acquire new ones so soon as they begin to be negative: thus we call negative goods, debts; negative gain, loss; negative heat, cold; negative descent, ascent, &c.: and in this sense indeed, it may not be so easy to conceive, how a quantity can be less than nothing, that is, how a quantity under any particular denomination, can be said to be less than nothing, so long as it retains that denomination." (Saunderson, p. 287)

Technology in teaching. "That the true way of Art is not by Instruments, but by demonstration: and that it is a preposterous course of vulgar Teachers, to beginne with Instruments, and not with the Sciences, and so in stead of Artists, to make their Schollers onely doers of tricks, and as it were jugglers." (Oughtred, p. 68)

§ R16. Mathematics as model knowledge

VIKTOR BLÅSJÖ, *Transcendental curves in the Leibnizian calculus*, Ph.D. dissertation, Utrecht University, 2016.

"For two millennia the method embodied in Euclid's *Elements* was the gold standard of exact reasoning. By the time of the Renaissance and the scientific revolution it had also passed the test of time with flying colours: while it seemed that all other teachings invariably crumbled in the face of expanding knowledge and experience, the Euclidean edifice not only stood without a scratch but also proved an indispensable foundation for the most exciting new advances in the understanding of the world. The obvious message was not lost on reflective minds: If you seek certain and eternal truth then you better do whatever it was that Euclid did.

But what was it about the Euclidean method that made is so uniquely successful, and how could it be generalised beyond its traditional scope? Today the phrase 'axiomatic-deductive method' is often used to try to capture its essence, and indeed it is based on a small set of axioms, and indeed it proceeds meticulously through short, stringently verified deductive steps. But 17th century eyes saw something more in Euclid, something to which subsequent generations have grown increasingly blind. To them the ideal of the Euclidean method represented not a specialised, formal way of studying geometry, but a model of reasoning in general and our only reliable window toward an understanding of the nature of knowledge.

There were in fact two competing interpretations of the Euclidean method in the 17th century. They are summarised and contrasted in table 1. As we see, these interpretations generalise the Euclidean method not only to an expanded view of geometry but also to physics and even philosophy in general. Descartes's famous phrase *cogito ergo sum* ('I think therefore I am') encapsulates his view: one starts in complete ignorance and nothingness and can only build up one's knowledge from the most immediate and undeniable principles. Newton's view, by contrast, is summed up in his statement: 'As in mathematics, so in natural philosophy, the investigation of difficult things by the method of analysis ought ever to precede the method of composition.' That is to say, instead of the Cartesian method of 'composing' all knowledge from intuitive starting principles, Newton advocates its opposite: analysis, i.e., starting with all the things one wants to understand and then trying to reduce them to simple principles. Euclid's *Elements* and Newton's *Principia* both start with a few simple axioms and deduce increasingly more complex results from them, but this, according to Newton, is not to be seen as mirroring the process of acquiring knowledge. This 'method of composition,' or synthesis, is but a mode of presentation adopted after the fact, for the sake of consolidating and clarifying logically the insights gained through analysis." (189–190)

CHRISTOPHER CLAVIUS, *In disciplinas mathematicas prolegomena*, 1574. Translation quoted from J. M. Lattis, *Between Copernicus and Galileo*, U Chicago Press, 1994, pp. 35–36.

"The theorems of Euclid and the rest of the mathematicians, still today many years past, retain … their true purity, their real certitude, and their strong and firm demonstrations. … And thus so much do the mathematical disciplines desire, esteem, and foster truth that they reject not only whatever is false, but even anything merely probable … So there can be no doubt but that the first place among the other sciences should be conceded to mathematics."

Mathematics is "not only useful, but in fact necessary" in many fields, but "of all these benefits [of mathematical studies], perhaps the greatest is the entertainment and pleasure that fills the soul as a result of the cultivation and exercise of those arts."

THOMAS HOBBES, *De Cive*, 1642. Quoted from *The English Works of Thomas Hobbes of Malmesbury*, Volume II.

"And truly the geometricians have very admirably performed their part. For whatsoever assistance doth accrue to the life of man, whether from the observation of the heavens, or from the description of the earth, from the notation of times, or from the remotest experiments of navigation; finally, whatsoever things they are in which this present age doth differ from the rude simpleness of antiquity, we must acknowledge to be a debt which we owe merely to geometry. If the moral philosophers had as happily discharged their duty, I know not what could have been added by humane Industry to the completion of that happiness, which is consistent with humane life." (iv)

THOMAS HOBBES, *Leviathan*, 1651. Quoted from *The English Works of Thomas Hobbes of Malmesbury*, Volume III.

"Geometry ... is the only science that it hath pleased God hitherto to bestow on mankind" (23-24).

"There can be nothing so absurd, but may be found in the books of philosophers. And the reason is manifest. For there is not one of them that begins his ratiocination from the definitions, or explications of the names they are to use; which is a method that hath been used only in geometry; whose conclusions have thereby been made indisputable." (33)

"For all men by nature reason alike, and well, when they have good principles. For who is so stupid, as both to mistake in geometry, and also to persist in it, when another detects his error to him?" (35)

"By Philosophy is understood the knowledge acquired by reasoning, from the manner of the generation of any thing, to the properties: or from the properties, to some possible way of generation of the same; to the end to be able to produce, as far as matter, and human force permit, such effects, as human life requireth. So the geometrician, from the construction of figures, findeth out many properties thereof; and from the properties, new ways of their construction, by reasoning; to the end to be able to measure land, and water; and for infinite other uses." (664)

THOMAS HOBBES, *De Corpore*, 1655. Quoted from *The English Works of Thomas Hobbes of Malmesbury*, Volume I.

"Philosophy is such knowledge of effects or appearances as we acquire by true ratiocination from the knowledge we have first of their causes or generation." (3)

This definition is explicitly modelled on mathematics: "How the knowledge of any effect may be gotten from the knowledge of the generation thereof, may easily be understood by the example of a circle: for if there be set before us a plain figure, having, as near as may be, the figure of a circle, we cannot possibly perceive by sense whether it be a true circle or no ... [But if] it be known that the figure was made by the circumduction of a body whereof one end remained unmoved" then the properties of a circle become evident. (6)

Another way of putting it is that "The subject of Philosophy, or the matter it treats of, is every body of which we can conceive any generation." (10) Just as the domain of geometry is the set of constructible curves.

As the circle example shows, motion is a basic form of generation. Indeed motion is the foundation of geometry and physics alike. "First we are to observe what effect a body moved produceth, when we consider nothing in it besides its motion; ... from which kind of contemplation sprung that part of philosophy which is called geometry." (71) Next "we are to pass to the consideration of what effects one body moved worketh upon another; ... that is, when one body invades another body which is either at rest or in motion, what way, and with what swiftness, the invaded body shall move; and, again, what motion this second body will generate in a third, and so forwards. From which contemplation shall be drawn that part of philosophy which treats of motion ... [and ultimately] comprehend that part of philosophy which is called physics. (71-72)

"And, therefore, they that study natural philosophy, study in vain, except they begin at geometry; and such writers or disputers thereof, as are ignorant of geometry, do but make their readers and hearers lose their time." (73)

"After physics we must come to moral philosophy; in which we are to consider the motions of the mind, namely, appetite, aversion, love, benevolence, hope, fear, anger, emulation, envy, &c." (72) "For the causes of the motions of the mind are known ... And, therefore, ... by the synthetical method, and from the very first principles of philosophy, [one] may by proceeding in the same way [as in geometry and physics], come to the causes and necessity of constituting commonwealths, and to get the knowledge of what is natural right, and what are civil duties; and, in every kind of government, what are the rights of the commonwealth, and all other knowledge appertaining to civil philosophy." (73-74)

Carefully enunciated definitions are another prominent aspect of mathematics that is to be carried over into general philosophy. "Whatsoever the common use of words be, yet philosophers, who were to teach their knowledge to others, had always the liberty ... of taking to themselves such names as they please for the signifying of their meaning, if they would have it understood. Nor had mathematicians need to ask leave of any but themselves to name the figures they invented, parabolas, hyperboles, cissoeides, quadratrices, &c. or to call one magnitude A,

	Descartes, Leibniz Continental rationalism	Newton British empiricism
The search for knowledge starts with …	intuitively clear primitive notions	the rich diversity of phenomena
… and consists in …	deducing the diversity of phenomena from them.	reducing them to a few simple principles.
The justification of the axiomatic principles is …	immediate by their intuitive nature	external to the matter at hand
… and is therefore …	the crucial epistemological cornerstone of the entire enterprise.	of secondary importance at best.
In the case of physics, the axiomatic principles are …	the laws of contact mechanics	Newton's three force laws and the law of gravity
… which are established by means of …	their intuitively immediate nature.	induction from the phenomena.
In the case of geometry, the study of curves starts with …	the primitive intuition of local motion	the diversity of curves conceived in any exact manner whatever
… and consists in …	constructively building up a theory of all knowable curves on this basis.	investigating their properties in a systematic fashion.
Geometrical axioms are thus …	the intuitively immediate principles that define and generate the entire subject.	the outcomes of the reductive study of curves, which it was found convenient and illuminating to take as assumptions when the time came to write a systematic account.
The certainty of geometrical reasoning …	stems directly from the axioms' intuitive warrant and the constructive manner in which the rest is built up from them.	stems not from the axioms as such, but from the general method and exactitude of geometrical reasoning.

Table 1: Overview of the two competing interpretations of the Euclidean method in the 17$^{\text{th}}$ century. (From Blåsjö, 198.)

another B." (16) "But definitions of things, which may be understood to have some cause, must consist of such names as express the cause or manner of their generation, as when we define a circle to be a figure made by the circumduction of a straight line in a plane, &c." (81-82)

Logical, syllogistic reasoning is another distinctive attribute of mathematics. In fact, "They that study the demonstrations of mathematicians, will sooner learn true logic, than they that spend time part in reading the rules of syllogizing which logicians have made; no otherwise than little children learn to go, not by precepts, but by exercising their feet." (54-55)

THOMAS HOBBES, *Six Lessons to the Savilian Professors of the Mathematics*, 1656. Quoted from *The English Works of Thomas Hobbes of Malmesbury*, Volume VII.

"Of arts, some are demonstrable, others indemonstrable; and demonstrable are those the construction of the subject whereof is in the power of the artist himself, who, in his demonstration, does no more but deduce the consequences of his own operation. The reason whereof is this, that the science of every subject is derived from a precognition of the causes, generation, and construction of the same; and consequently where the causes are known, there is place for demonstration, but not where the causes are to seek for. Geometry therefore is demonstrable, for the lines and figures from which we reason are drawn and described by ourselves; and civil philosophy is demonstrable, because we make the commonwealth ourselves." (183-184)

§ R17. Descartes

RENÉ DESCARTES, *A Discourse on the Method*, translated by Ian Maclean, Oxford University Press, 2008.

Descartes makes it quite clear that his intention is to widen the scope of the mathematical method to philosophy in general: "I was most keen on mathematics, because of its certainty and the incontrovertibility of its proofs; but I did not yet see its true use. Believing as I did that its only application was to the mechanical arts, I was astonished that nothing more exalted had been built on such sure and solid foundations." (9 = AT 7)

Indeed, Descartes's definitive statement of his method is such an apt description of the Elements that it could easily have been written by Euclid himself as a preface to this work. Here I quote it in its entirety and point out the obvious parallels with Euclid.

"The first [principle of my method] was never to accept anything as true that I did not incontrovertibly know to be so; that is to say, carefully to avoid both prejudice and premature conclusions; and to include nothing in my judgements other than that which presented itself to my mind so clearly and distinctly, that I would have no occasion to doubt it." (17 = AT 18) This is of course a perfect description of the way Euclid bases his entire work on a few evident postulates and common notions.

"The second was to divide all the difficulties under examination into as many parts as possible, and as many as was required to solve them in the best way." (17 = AT 18) Just as, e.g., Euclid's proof of the Pythagorean theorem relies on some 28 previous propositions, and so on for all other theorems.

"The third was to conduct my thoughts in a given order, beginning with the simplest and most easily understood objects, and gradually ascending, as it were step by step, to the knowledge of the most complex; and positing an order even on those which do not have a natural order or precedence." (17 = AT 18) Again it is hard to imagine how any work could fit this description more perfectly than Euclid's Elements. The last point in particular is something of a peculiarity of mathematics. In mathematics, when faced with two equivalent statements, one picks arbitrarily which to prove first and which to derive as a corollary, and this has nothing to do with any kind of causal hierarchy between them.

"The last was to undertake such complete enumerations and such general surveys that I would be sure to have left nothing out." (17 = AT 19) Cf., for example, Euclid's exhaustive and systematic treatments of irrational magnitudes in Book X, and regular polyhedra in Book XIII.

Descartes immediately goes on the emphasise again that his method is modelled on mathematics:

"The long chains of reasonings, every one simple and easy, which geometers habitually employ to reach their most difficult proofs had given me cause to suppose that all those things which fall within the domain of human understanding follow on from each other in the same way, and that as long as one stops oneself taking anything to be true that is not true and sticks to the right order so as to deduce one thing from another, there can be nothing so remote that one cannot eventually reach it, nor so hidden that one cannot discover it. And I had little difficulty in determining those with which it was necessary to begin, for I already knew that I had to begin with the simplest and the easiest to understand; and considering that of all those who had up to now sought truth in the sphere of human knowledge, only mathematicians have been able to discover any proofs, that is, any certain and incontrovertible arguments, I did not doubt that I should begin as they had done." (17-18 = AT 19; cf. 16-19 generally)

RENÉ DESCARTES, *Principles of Philosophy*, translated by R. P. Miller, Springer, 1982.

Descartes's philosophical method is modelled on the method of Euclid's Elements, as is clear from Descartes's preface:

"One must begin by searching for ... first causes, that is, for Principles [which] must be so clear and so evident that the human mind cannot doubt of their truth when it attentively considers them ... And then, one must attempt to deduce from these Principles the knowledge of the things which depend upon them, in such a way that there is nothing in the whole sequence of deductions which one makes from them which is not very manifest." (xvii-xviii)

But Descartes is not content with merely adopting the Euclidean method—he also justifies it. He does this by showing that it survives even the most critical examination possible, namely that announced in the first sentence of the text: "whoever is searching for truth must, once in his life, doubt all things" (I.1).

The Euclidean method is the only philosophical method to survive this critical abyss, by the following chain of reasoning.

First we prove our own existence. "We can indeed easily suppose that there is no God, no heaven, no material bodies; and yet even that we ourselves have no hands, or feet, in short, no body; yet we do not on that account suppose that we, who are thinking such things, are nothing: for it is contradictory for us to believe that that which thinks, at the very time when it is thinking, does not exist. And, accordingly, this knowledge, *I think, therefore I am*, is the first and most certain to be acquired by and present itself to anyone who is philosophizing in correct order." (I.7)

"The knowledge of remaining things depend on a knowledge of God," because the next things the mind feels certain of are basic mathematical facts, but it cannot trust these judgments unless it knows that its creator is not deceitful. Thus "the mind ... discovers [in itself] certain common notions, and forms various proofs from these; and as long as it is concentrating on these proofs it is entirely convinced that they are true. Thus, for example, the mind has in itself the ideas of numbers and figures, and also has among its common notions, *that if equals are added to equals, the results will be equal*, and other similar ones; from which it is easily proved that the three angles of a triangle are equal to two right angles, etc." But the mind "does not yet know whether it was perhaps created of such a nature that it errs even in those things which appear most evident to it." Therefore "the mind sees that it rightly doubts such things, and cannot have any certain knowledge until it has come to know the author of its origin." (I.13)

The existence of God is established to Descartes's satisfaction by several dubious arguments, most notably the following. "Just as, for example, the mind is entirely convinced that a triangle has three angles which are equal to two right angles, because it perceives that the fact that its three angles equal two right angles is necessarily contained in the idea of a triangle; so, solely because it perceives that necessary and eternal existence is contained in the idea of a supremely perfect being, the mind must clearly conclude that a supremely perfect being exists." (I.14) And all the more since it is "very well know from [our] natural enlightenment" "that that which is more perfect is not produced by an efficient and total cause which is less perfect; and moreover that there cannot be in us the idea or image of anything, of which there does not exist somewhere (either in us or outside us), some Original, which truly contains all its perfections. And because we in no way find in ourselves those supreme perfections of which we have the idea; from that fact alone we rightly conclude that they exist, or certainly once existed, in something different from us; that is, in God." (I.18)

"It follows from this that all the things which we clearly perceive are true, and that the doubts previously listed are removed" (I.30), since "God is not the cause of errors," owing to his perfection, seeing as "the will to deceive certainly never proceeds from anything other than malice, or fear, or weakness; and, consequently, cannot occur in God." (I.29) "Thus, Mathematical truths must no longer be mistrusted by us, since they are most manifest." (I.30)

In the same way we can be sure that material objects exist, since otherwise "it would be impossible to devise any reason for not thinking Him a deceiver" (II.1). But the argument forces upon us the restriction "that the nature of body does not consist in weight, hardness, color, or other similar properties; but in extension alone" (II.4), since a body can easily be conceived to be deprived of its secondary properties (cf. also II.11), but not its extension.

Physics, therefore, must be based on a theory of extended matter and nothing else. Two key characteristics of Cartesian physics follow quite naturally from this starting point, and are indeed introduced almost immediately: relativity of space (II.13-14) and contact mechanics (II.36-52).

The first is a quite unavoidable corollary of Descartes's starting point, since his perspective does not admit the possibility of space as a concept separate from body. Thus he is compelled to argue that "the names 'place' or 'space' do not signify a thing different from the body which is said to be in the place; but only designate its size, shape and situation among other bodies" (II.13). "So when we say that a thing is in a certain place, we understand only that it is in a certain situation in relation to other things" (II.14).

A second rather straightforward consequence of Descartes's starting point is that contact mechanics is the fundamental phenomena in terms of which all other physics must be construed. And indeed Descartes offers a detailed account of contact mechanics almost at once, in II.36-52.

RENÉ DESCARTES, *The Geometry*, translated by D. E. Smith & M. L. Latham, Dover, 1954.

The main theme of Descartes's geometry is the justification of algebraic methods in terms of the standards of classical, construction-based geometry.

"To treat all the curves I mean to introduce here [i.e., all algebraic curves], only one additional assumption [beyond ruler and compasses] is necessary, namely, [that] two or more lines can be moved, one [by] the other, determining by their intersection other curves. This seems to me in no way more difficult [than the classical constructions]." (43)

"I could give here several other ways of tracing and conceiving a series of curved lines, each curve more complex than any preceding one, but I think the best way to group together all such curves and then classify them in order, is by recognizing the fact that all points of those curves which we may call 'geometric', that is, those which admit of precise and exact measurement, must bear a definite relation to all points of a straight line, and that this relation must be expressed by means of a single equation" (48). In other words, the legitimate curves of exact geometry are precisely those representable by algebraic equations in rectilinear coordinates.

The converse implication, that all algebraic curves are traceable in these ways, is apparently seen by Descartes as a consequence of the fact arbitrary points of an algebraic curve can be constructed (by fixing some x-value and constructing the corresponding y; this assumes the general construction procedure for algebraic equations discussed below): "this method of tracing a curve by determining a number of its points taken at random applies only to curves than can be generated by a regular and continuous motion" (91), he asserts without proof.

In contrast to algebraic curves, "the spiral, the quadratrix, and similar curves ... are not among those curves that I think should be included here, since they must be conceived of as described by two separate movements whose relation does not admit of exact determination" (44), "since the ratios between straight and curved lines are not known, and I believe cannot be discovered by human minds, and therefore no conclusion based upon such rations can be accepted as rigorous and exact" (91). In other words, these kinds of curves involve independent linear and circular motions of coordinated speed, and thus essentially depend on π, which is not an algebraic number and thus unknowable by Cartesian standards. Similarly, one cannot construct arbitrary points on these curves (a given y-coordinate is generally not an algebraic function of the corresponding x-coordinate).

The above establishes algebraic curves as legitimate construction tools on par with ruler and compasses. It remains to carry out the actual constructions themselves, that is, for a given problem to construct the required points as intersections of various algebraic curves. Thus Descartes shows how to find the roots of any third or forth degree equation by intersecting a circle and a parabola (192-205), and the roots of any fifth or sixth degree equation by intersecting a circle and a Cartesian parabola (220-237). Constructions for higher degrees are "intentionally omitted so as to leave to others the pleasure of discovery," "for in the case of a mathematical progression, whenever the first two or three terms are given, it is easy to find the rest" (240).

These constructions mean that a problem can be considered solved according to classical construction standards (as enlarged by Descartes) whenever it is reduced to an algebraic equation. Thus, to Descartes, algebraic geometry does not replace classical construction-based geometry, but is rather subsumed by it. A concluding remark shows that this was no mere theoretical point but that the constructions were indeed intended to be carried out in concrete cases: "in many of these problems it may happen that the circle cuts the [Cartesian parabola] so obliquely that it is hard to determine the exact point of intersection. In such cases this construction is not of practical value. The difficulty could easily be overcome by forming other rules analogous to these, which might be done in a thousand different ways." (239)

"While it is true that every curve which can be described by a continuous motion should be recognized in geometry, this does not mean that we should use at random the first one that we meet in the construction of a given problem. We should always choose with care the simplest curve that can be used in the solution of a problem, but it should be noted that the simplest means not merely the one most easily described, not the one that leads to the easiest demonstration or construction of the problem, but rather the one of the simplest class [i.e., degree] that can be used to determine the required quantity." (152, 155)

Three main applications of the new geometry are discussed by Descartes. The first and by far the most substantial is Pappus's problem, which asks for the locus of all points that have particular distance relations to a set of given lines. This problem is a showcase for algebraic geometry, as it is an extremely general problem with great classical prestige, yet eminently treatable by algebraic means. In fact, Descartes claims (mistakenly) that the set of all possible solutions to Pappus's problem is exactly the set of all algebraic curves (59). A second application is "a discussion of certain ovals which you will find very useful in the theory of catoptrics and dioptrics" (p, 115), which is interesting as Descartes gives pointwise constructions only of these ovals in place of equations (114-149). The third main application is Descartes's double-root method for finding normals (and thereby tangents).

RENÉ DESCARTES, *Rules for the direction of the mind*, Liberal Arts Press, 1961.

Mathematics is the model for all knowledge. "In seeking the correct path to truth we should be concerned with nothing about which we cannot have a certainty equal to that of the demonstrations of arithmetic and geometry." (II, p. 367) "And although I speak a good deal here of figures and numbers ... nevertheless anyone who pays close attention to my meaning will easily observe that I am not thinking at all of common mathematics, but I am setting forth a certain new discipline ... broad enough to bring out the truths of any subject whatsoever." (IV, p. 375) The power of mathematics stems from "certain basic roots of truth implanted in the human mind by nature, which we extinguish in ourselves daily by reading and hearing many varied errors" (IV, p. 377).

Proofs should be intuited as wholes. Deductions "may sometimes be accomplished through such a long chain of inferences that when we have arrived at the conclusions we do not easily remember the whole procedure which led us to them ... Because of this, I have learned to consider each of these steps by a certain continuous process of the imagination ... Thus I go from first to last so quickly that by entrusting almost no parts of the process to the memory, I seem to grasp the whole series at once." (VII, pp. 388-389) "In this way our knowledge is made much more certain and the capacity of our minds is increased as much as possible." (XI, p. 408)

To achieve this end "we must make use of every assistance of the intellect, the imagination, the senses, and the memory" (XII, p. 411). "By the aid of each faculty ... human efforts can serve to repair the deficiencies of the mind." (XII, p. 417). For example, "if the intellect proposes to examine something which can be related to the body it should produce in the imagination the most distinct idea of it possible; and in order to do this more readily, the object which this idea represents should be exhibited to the external senses." (XII, pp. 417-418). "We are to do nothing from this point on without the aid of the imagination." (XIV, p. 444)

Algebra and analytic geometry is intended to be precisely such an aid to the intuition. For having recorded the steps of a proof in algebraic terms, "we can run through all of them in a very rapid movement of thought and grasp as many as possible at the same time" (XVI, p. 456). Intuiting the whole in this way is important to us "who are seeking evident and distinct knowledge of things; but not the arithmeticians, who are satisfied if the have discovered the number sought even though they have not noticed how it depends upon the given facts, although this latter is the only point in which science truly lies" (XVI, p. 459).

Another benefit of algebra in this regard. By algebra "we translate what we understand to be affirmed about magnitudes in general into that particular magnitude that we can most easily and distinctly picture in our imagination" (XIV, p. 442); specifically, "a magnitude should never be regarded in the imagination otherwise than as a line or a surface, even though it may be called a 'cube' or a 'biquadratic'" (XVI, p. 457).

A quip on why we should denote the answers we seek by a letter such as x. "It frequently happens that individuals are so eager to investigate problems that they apply their capricious intelligence to finding a solution before they have determined by what signs they will recognize the object of their search, if they should stumble upon it by accident; these persons are no less foolish that would be a boy, sent somewhere by his master, who was so eager to obey that he started to run without waiting for instructions, and without knowing where he was ordered to go." (XIII, p. 435)

JOHN AUBREY, *Brief Lives*, late 17$^{\text{th}}$ century.

"[Descartes] was so eminently learned that all learned men made visits to him, and many of them would desire him to show them his store of instruments. He would draw out a little drawer under his table and show them a pair of compasses with one of the legs broken; and then for his ruler, he used a sheet of paper folded double."

§ R18. Leibniz

VIKTOR BLÅSJÖ, *Transcendental curves in the Leibnizian calculus*, Ph.D. dissertation, Utrecht University, 2016.

"Leibniz ... made it his mission in mathematics to do Descartes one better. While Descartes had pushed the boundaries of geometry to include all algebraic curves, Leibniz would push them further still and include also the curves that went beyond, or transcended, algebra—the transcendental curves. Thus Leibniz faced the problem of providing these curves with a Euclidean-style, construction-based foundation.

Leibniz considered this *the* foundational problem of the day, and he did so with good reason. Transcendental curves and the quantities constructible with their aid were at this time being found indispensable in numerous branches of mathematics and physics, such as the brachistochrone in dynamics, the catenary in statics, the cycloidal path of the optimal pendulum clock in horology, the loxodrome in navigation, caustics in optics, arc lengths of ellipses in astronomy, and logarithms in computational mathematics. ... But these new [curves] were profoundly incompatible with the norms of mathematical rigour of their day, as the very epithet 'transcendental' attests: though the literal meaning of this term, coined by Leibniz himself, is that these curves 'transcend all algebraic equations,' this meant by extension that they transcended geometry itself as far as the authoritative vision of Descartes was concerned. In this way these new transcendental curves exerted a profound strain on the foundations of the subject. Simply letting all transcendental curves through the gates of geometry *en masse* would be an unthinkable betrayal of what geometry had always stood for. Geometry was defined by its foundational stringency, minimalism and constructivism; this was the source of all its credibility. So to suddenly open the floodgates for transcendental curves would be much more than a bold extension of geometry: it would be, arguably, to stop doing geometry altogether in any meaningful sense of the term." (14–15)

Against this background we can easily understand why Leibniz was so eager to stress that: "I do not in the least pretend to the glory of being an innovator … On the contrary I normally find that the oldest and commonly received opinions are the best. And I do not think one can be accused of being an innovator when on produces only a few new truths, without overturning established opinions. For this is what geometers do and all who penetrate more deeply." (27)

But at the same time "Leibniz was certainly very impressed by the recent triumphs of analytical methods, which 'reduce everything from imagination to analysis'. Indeed he envisioned this as a model for stringent reasoning in general. … It was in these kinds of terms that Leibniz saw the greatness of his infinitesimal calculus: 'As far as the differential calculus is concerned, I admit that there is much in common between it and the things which were explored by both you [Wallis] and Fermat and others, indeed already by Archimedes himself. Yet now the matter is perhaps carried much further, so that now those things can be accomplished which in the past seemed closed even to the greatest geometers as Huygens himself recognised. The matter is almost the same in the analytical calculus applied to conical curves or higher: Who does not consider Apollonius and other ancients to have had theorems which present matters for the equations by which Descartes later preferred to designate curves. In the meantime the matter has been reduced to calculation by the method of Descartes, so that now conveniently and without trouble that can be done which formerly required much effort of contemplation and imagination. In the same way, by our differential calculus, transcendentals too, which Descartes himself excluded in the past, are subjected to analytical operations.'

Or more succinctly: 'For what I love most in this calculus is that it gives us the same advantage over the ancients in the geometry of Archimedes as Viete and Descartes gave us in the geometry of Euclid and Apollonius; and it dispenses with the efforts of the imagination.'

In sum, there can be no doubt that Leibniz attributed the utmost importance to the analytical side of mathematics. To him it was absolutely essential that whatever solution of the problem of transcendental curves one may come up with, it must in any case be accompanied by a successful analytical method comparable to that of Descartes.

However, despite this—despite analytic expressions being 'what I love most'—Leibniz would not let this displace the construction paradigm as the foundations of geometry. To him, as to Descartes, curves were properly defined and made geometrical only by construction; their analytic representations were but a welcome bonus. Thus when Leibniz needs to justify the inclusion of transcendental curve in geometry he falls back on their construction by motion. For example…: '[Certain problems] transcend all algebraic equations. Yet since these problems can nevertheless actually be proposed in geometry, nay should even be considered among the foremost ones, … it is therefore certainly necessary to receive such curves into geometry, by which alone [such problems] can be constructed. And since they can be drawn exactly by a continuous motion, as is clear for the cycloid and similar [curves], they are to be considered not mechanical but geometrical, especially since by their usefulness they leave the curves of ordinary geometry (if you except the line and the circle) far behind, and have properties of the greatest importance, which are entirely capable of geometrical demonstrations.'

[Or again:] 'Descartes, in order to maintain the universality and sufficiency of his method, found it appropriate to exclude from geometry all the problems and all the curves which could not be subjected to this method, under the pretext that these things were only mechanical. Since, however, these problems and lines can be constructed or conceived by means of certain exact motions, and have important properties, and nature often uses them, one may say that he commits the same error as one who criticises some ancients for restricting themselves to constructions for which one needs nothing but ruler and compass, as if all the rest was mechanical.'

In short, transcendental curves are ultimately justified in terms of their construction, not in terms of their analytical representations. This insistence on retaining both the analytic and construction-based paradigms leads to a fundamental conflict acknowledged, somewhat reluctantly, by Leibniz: 'And I must admit that, other things being equal, I like constructions by motion better than pointwise ones, and when the motion is of proper simplicity I consider it not as mechanical but as geometrical. The pointwise construction does indeed lend itself more conveniently to analytical calculation. But properly speaking one is not concerned about this in geometry.'

The point here is that an equation of the form $y = f(x)$ is effectively a recipe for pointwise construction: pick some point x on the axis, raise a perpendicular above it, and mark off the height $f(x)$ on this perpendicular. Though no one minds this anymore, it is still true today: the y-values of the graphs of, say, a trigonometric function are defined not in terms of a single generation of this graph but in terms of separate circle-measurements for each x-value. We may have a difficult time seeing this as a drawback today but Huygens makes a compelling case:

'One cannot say that the description of a curved line through found points is geometrical, that is to say complete, or that lines so described can serve as a geometrical construction for some problems, because for this, in my opinion, no curved lines can serve except those that can subsequently be described by some instrument, as the circle by a pair of compasses; and the conic sections, conchoids and others by the instruments invented thereto. For the lines drawn by hand from point to point can only give the sought quantity approximately and consequently not according to geometrical perfection. For what does it help to find as many points as one wishes, in case one does not find the one point that is sought?'

By extension, then, this is a case against accepting formulas such as $\cos(x)$, $\arcsin(x)$, $\log(x)$, e^x, etc., as legitimate solutions of geometrical problems. In the 18th century these kinds of expressions were increasingly seen as self-sufficient, but Leibniz's generation would accept nothing of the sort, since doing so would mean giving up the construction-based paradigm and with it all the accumulated credibility of classical geometry. It is indeed the irony of history that the arsenal of analytic expressions that are the de facto ontology of the calculus today was once a set of geometrical entities carefully selected for the very opposite purpose, namely to ensure that no reliance was made on analytical expressions in the interpretation of the results of the calculus, by instead giving them geometrical meaning as arcs of circles, areas under hyperbolas, etc." (16–18)

§ R19. Newton

ZEV BECHLER (ED.), *Contemporary Newtonian Research*, Reidel, 1982.

Newton was "a mathematician to his toe-tips"; in physics, however, he "took a Cartesian track ... which all too soon ended in a boggy vortical swirl" (Whiteside, p. 116).

In 1679 Hooke wrote to Newton for help with the mathematical aspects of his hypothesis "of compounding the celestiall motions of the planetts of a direct motion by the tangent & an attractive motion towards the centrall body" (36). But "Newton was still mired in very confusing older notions" (35) and wrote a reply with a rather basic error in it. To get Newton going Hooke had to explicitly suggest the inverse square law and plead that "I doubt not but that by your excellent method you will easily find out what that Curve [the orbit] must be" (37). Only then "Newton quickly broke through to dynamical enlightenment ... following [Hooke's] signposted track" (Whiteside, p. 117).

The experimental aspect of Newton's work on mechanics did not go so well, and sometimes hampered rather than aided the development of his theory. The famous moon test, for example, first came out negatively (owing to a bad value for the radius of the earth), which "made Sir Isaac suspect that this Power was partly that of Cartesius's Vortices" (Whiston, p. 34).

OFER GAL, *Meanest Foundations and Nobler Superstructures: Hooke, Newton and "the Compounding of the Celestiall Motions of the Planetts"*, Springer, 2002.

Hooke's programme of "compounding the celestiall motions of the planetts of a direct motion by the tangent & an attractive motion towards the central body" (2) "occasioned my findings" (17) on planetary motion, Newton admitted. "It is difficult to overstate the novelty of Hooke's Programme." (ix) "For Kepler as well as Galileo, for Descartes himself, as well as for Grassendi and the Cartesians Mersenne and Huygens, for that venerable departed genious Horrox as well as for Newton's own favorite Borelli, the explication of the planetary motions had always included rotation as a primary cause." (2; see pp. 24-29) The inverse square relation between gravity and distance, however, "was rather common" (9).

Hooke used the term "inflection" to describe the planet's deviation from inertial motion. This is the same term he had earlier used to describe the bending of light rays in the atmosphere: "This inflection (if I may so call it) I imagine to be nothing else, but a multiplicate refraction, caused by the unequal density of the constituent parts of the medium, whereby the motion ... of the Ray of light is hindered from proceeding in a streight line" (35). Indeed, the thinning of the air at high altitudes is "clear enough evinc'd" from experiments "tryed at the tops and feet of Mountains" (36), but there is "no Experiment yet known to prove a saltus, or skipping from one degree of rarity [of the atmosphere] to another much differing from it" (34).

Hooke realised that planetary motion might be explained analogously, "if the aether be somewhat of the nature of air" (37), Thus "if we suppose, that part of the medium, which is farthest from the center, or sun, to be more dense outward, than that which is more near, it will follow, that the direct motion will always be deflected inwards, by the easier yielding of the inward, and the greater resistance of the outward part of the medium." (36-37)

But Hooke immediately dismisses this theory owing to "improbabilities, that attend to this supposition, which being nothing to my present purpose I shall omit" (37; presumably the moon's motion is one of the "improbabilities" in question). Therefore he discards the medium aspect of the theory, but nevertheless retains the forces suggested by it: his goal thus being only to "shew, that circular motion is compounded of an endeavour by a direct motion by the tangent, and of another endeavour tending to the center'" which he "endeavour[s] to explicate" experimentally with the aid of a "pendulous body" (37), i.e. a conical pendulum.

Hooke's first use of the inverse square law also occurred in the context of atmospheric investigations—in 1665, "much earlier than usually noted." The context is the idea that the pressure of the air is the weight of "a Cylinder [of air] indefinitely extended upwards": "I say Cylinder, not a piece of a cone, because, as I may elsewhere shew in the Explication of Gravity, that triplicate proportion of the shels of a Sphere, to their respective diameters, I suppose to be removed by the decrease of the power of Gravity." (169) In other words, while the base area of a cone with its vertex at the surface of the earth is as the height squared, gravity is as the inverse height squared, meaning that the weight is equivalent to that of a cylinder with constant gravity.

§ R20. Analytical mathematics

PAPPUS, *Collection*, c. 340. Translation quoted from Heath, *A History of Greek Mathematics*, Volume 2, p. 400.

"Analysis, then, takes that which is sought as if it were admitted and passes from it through its successive consequences to something which is admitted as the result of synthesis: for in analysis we admit that which is sought as if it were already done and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until by so retracing our steps we come upon something already known or belonging to the class of first principles, and such a method we call analysis as being solution backwards.

But in synthesis, reversing the process, we take as already done that which was last arrived at in the analysis and, by arranging in their natural order as consequences what before were antecedents, and successively connecting them one with another, we arrive finally at the construction of what was sought; and this we call synthesis."

G. W. LEIBNIZ, *La vraie méthode*, 1677. Translation quoted from Wiener, *Leibniz selections*, p. 15.

"Whence it is manifest that if we could find characters or signs appropriate for expressing all our thoughts as definitely and as exactly as arithmetic expresses numbers or geometric analysis expresses lines, we could in all subjects in so far as they are amenable to reasoning accomplish what is done in Arithmetic and Geometry. For all inquiries which depend on reasoning would be performed by the transposition of characters and by a kind of calculus, which would immediately facilitate the discovery of beautiful results. For we should not have to break our heads as much as is necessary today, and yet we should be sure of accomplishing everything the given facts allow. Moreover, we should be able to convince the world what we should have found or concluded, since it would be easy to verify the calculation either by doing it over or by trying tests similar to that of casting out nines in arithmetic. And if someone would doubt my results, I should say to him: 'Let us calculate, Sir' and thus by taking to pen and ink, we should soon settle the question."

LEONHARD EULER, *Mechanica sive motus scientia analytice exposita*, 1736. Translation based on I. Bruce and M. Mahoney.

"What distracts the reader the most [in previous works on mechanics], is the fact that everything is carried out synthetically, with the demonstrations presented in the manner of the old geometry, and the analysis hidden ... I always have the same trouble, when I might chance to glance through Newton's Principia ... Whenever the solutions of problems seem to be sufficiently well understood by me, yet by making only a small change, I might not be able to solve the new problem using this method. Thus I have endeavoured for a long time now, to get at the analysis behind those synthetic method in order to draw out the same propositions." (Preface)

PIERRE-SIMON LAPLACE, *Exposition du système du monde*, 6th ed., Paris, 1835 = *OEuvres*, 6 (Paris, 1884). Translation quoted from Hawkins, *Emergence of the Theory of Lie Groups*, 2000, p. 108.

"By abandoning oneself to the operations of Analysis ... one is led, by the generality of this method and by the inestimable advantage of transforming the reasoning into mechanical procedures, to results often inaccessible to synthesis. Such is the fruitfulness of Analysis that it suffices to translate particular truths into this language in order to see emerge from their very expression a multitude of new and unexpected truths." (465)

LEONHARD EULER, *Foundations of Differential Calculus*, 1755, translated by J. D. Blanton, Springer, 2000.

"103. ... The general infinite series that originates from the fraction

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$$

seems to labor under most serious difficulties. If for x we successively substitute the numbers 1, 2, 3, 4, ..., we obtain the following series with their sums:

A. $1+1+1+1+1+\cdots = \frac{1}{1-1} = \infty$

B. $1+2+4+8+16+\cdots = \frac{1}{1-2} = -1$

C. $1+3+9+27+81+\cdots = \frac{1}{1-3} = -\frac{1}{2}$

D. $1+4+16+64+256+\cdots = \frac{1}{1-4} = -\frac{1}{3}$

and so forth. Since each term of series B, except for the first, is greater than the corresponding term of series A, the sum of series B must be much more than the sum of series A. Nevertheless, this calculation shows that series A has an infinite sum, while series B has a negative sum, which is less than zero, and this is beyond comprehension. Even less can we reconcile with ordinary ideas the results of this and the following series C, D, and so forth, which have negative sums while all of the terms are positive."

"109. From this we conclude that series of this kind, which are called divergent, have no fixed sums, since the partial sums do not approach any limit that would be the sum for the infinite series. This is certainly a true conclusion, since we have shown the error in neglecting the final remainder. However, it is possible, with considerable justice, to object that these sums, even though they seem not to be true, never lead to error. Indeed, if we allow them, then we can discover many excellent results that we would not have if we rejected them out of hand. Furthermore, if these sums were really false, they would not consistently lead to true results; rather, since they differ from the true sum not just by a small difference, but by infinity, they should mislead us by an infinite amount. Since this does not happen, we are left with a most difficult knot to unravel."

"111. These inconveniences and apparent contradictions can be avoided if we give the word *sum* a meaning different from the usual. Let us say that the sum of any infinite series is a finite expression from which the series can be derived. In this sense, the true sum of the infinite series $1+x+x^2+x^3+x^4+x^5+\cdots$ is $\frac{1}{1-x}$, since this series is derived from the fraction, no matter what value is substituted for x. With this understanding, if the series is convergent, the new definition of sum agrees with the usual definition. Since divergent series do not have a sum, properly speaking, there is no real difficulty which arises from this new meaning. Finally, with the aid of this definition we can keep the usefulness of divergent series and preserve their reputations."

§ R21. Foundations of the calculus

DOUGLAS M. JESSEPH, Leibniz on the Foundations of the Calculus, *Perspectives on Science*, 6 (1998), pp. 6-40.

"I have assumed in the demonstrations incomparably small quantities … If someone does not want to employ infinitely small quantities, he can take them to be as small as he judges sufficient to be incomparable, so that they produce an error of no importance and even smaller than any given [error]." (20; Leibniz, *Tentamen de motuum coelestium causis*)

"In the end, I do not dispute whether these inassignable quantities are true or fictive; it suffices that they serve for the abbreviation of thought, and they always bring with them a demonstration in a different style; and so I observed that if someone substitutes the incomparably small or that which is sufficiently small for the infinitely small, I would not oppose it." (28; Leibniz to Wallis, 30 March 1699)

"There is no need to take the infinite here rigorously, but only as when we say in optics that the rays of the sun come from a point infinitely distant, and thus are regarded as parallel. And when there are more degrees of infinity, or infinitely small, it is as the sphere of the earth is regarded as a point in respect to the distance of the sphere of the fixed stars, and a ball which we hold in the hand is also a point in comparison with the semidiameter of the sphere of the earth. And then the distance to the fixed stars is infinitely infinite or an infinity of infinities in relation to the diameter of the ball. For in place of the infinite or the infinitely small we can take quantities as great or as small as is necessary in order that the error will be less than any given error. In this way we only differ from the style of Archimedes in the expressions, which are more direct in our method and better adapted to the art of discovery." (30; Leibniz, 1701)

"Philosophically speaking, I no more admit magnitudes infinitely small than infinitely great. … I take both for mental fictions, as more convenient ways of speaking, and adapted to calculation, just like imaginary roots are in algebra. I once demonstrated that these expressions have a great use both in abbreviating thought and aiding discovery, and that they cannot lead to error, since in place of the infinitely small one may substitute [a quantity] as small as one wishes, and since any error will always be less than this, it follows that no error can be given." (34; Leibniz to Des Bosses, 11 March 1706)

GEORGE BERKELEY, *The Analyst*, London, 1734.

"A Discourse Addressed to an Infidel Mathematician." "I am not, Sir, a stranger to the reputation you have acquired in that branch of learning which hath been your peculiar study [i.e., mathematics]; nor to the authority that you therefore assume in things foreign to your profession, nor to the abuse that you, and too many more of the like character, are known to make of such undue authority, to the misleading of unwary persons in matters of the highest concernment, and whereof your mathematical knowledge can by no means qualify you to be a competent judge. Equity indeed and good sense would incline one to disregard the judgment of men, in points which they have not considered or examined. But several who make the loudest claim to those qualities do nevertheless the very thing they would seem to despise, clothing themselves in the livery of other men's opinions, and putting on a general deference for the judgment of you, Gentlemen, who are presumed to be of all men the greatest masters of reason, to be most conversant about distinct ideas, and never to take things upon trust, but always clearly to see your way, as men whose constant employment is the deducing truth by the justest inference from the most evident principles. With this bias on their minds, they submit to your decisions where you have no right to decide. And that this is one short way of making Infidels, I am credibly informed." (§1)

"The Method of Fluxions [i.e., the calculus] is the general key by help whereof the modern mathematicians unlock the secrets of Geometry, and consequently of Nature. And, as it is that which hath enabled them so remarkably to outgo the ancients in discovering theorems and solving problems, the exercise and application thereof is become the main if not sole employment of all those who in this age pass for profound geometers. But whether this method be clear or obscure, consistent or repugnant, demonstrative or precarious, as I shall inquire with the utmost impartiality." (§3)

"As our sense is strained and puzzled with the perception of objects extremely minute, even so the imagination, which faculty derives from sense, is very much strained and puzzled to frame clear ideas of the least particles of time, or the least increments generated therein: and much more so to comprehend the moments, or those increments of the flowing quantities in statu nascenti, in their very first origin or beginning to exist, before they become finite particles. And it seems still more difficult to conceive the abstracted velocities of such nascent imperfect entities. But the velocities of the velocities, the second, third, fourth, and fifth velocities, &c., exceed, if I mistake not, all human understanding." (§4)

"All these points, I say, are supposed and believed by certain rigorous exactors of evidence in religion, men who pretend to believe no further than they can see." (§7) "It must indeed be acknowledged the modern mathematicians do not consider these points as mysteries, but as clearly conceived and mastered by their comprehensive minds. … But if we remove the veil and look underneath … we shall discover much emptiness, darkness, and confusion." (§8)

"If a man, by methods not geometrical or demonstrative, shall have satisfied himself of the usefulness of certain rules; which he afterwards shall propose to his disciples for undoubted truths; which he undertakes to demonstrate in a subtile manner, and by the help of nice and intricate notions; it is not hard to conceive that such his disciples may, to save themselves the trouble of thinking, be inclined to confound the usefulness of a rule with the certainty of a truth, and accept the one for the other; especially if they are men accustomed rather to compute than to think." (§10)

Critique of how derivatives are computed. The derivative of $y = x^2$ is traditionally found as follows. Let x increase by dx. Then y increases by $dy = (x+dx)^2 - x^2 = 2x\,dx + dx^2$. Therefore $dy/dx = 2x + dx$. But dx is infinitely small, so it can be discarded. Thus the final result is that the derivative is $2x$. Before the final step, "I have supposed that … [dx] is something. And I have proceeded all along on that supposition, without which I should not have been able to have made so much as one single step." But in the final step "I now beg leave to make a new supposition contrary to the first, i.e. I will suppose that there is no increment of x, or that [dx] is nothing; which second supposition destroys my first, and is inconsistent with it, and therefore with every thing that supposeth it. I do nevertheless beg leave to retain [the expression for dy], which is an expression obtained in virtue of my first supposition, which necessarily presupposeth such supposition, and which could not be obtained without it: All which seems a most inconsistent way of arguing, and such as would not be allowed of in Divinity." (§14)

"And what are these fluxions? The velocities of evanescent increments? And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?" (§35)

"It is with the method of fluxions as with all other methods, which presuppose their respective principles and are grounded thereon; although the rules may be practised by men who neither attend to, nor perhaps know the principles. In like manner, therefore, … as any ordinary man may solve divers numerical questions, by the vulgar rules and operations of arithmetic, which he performs and applies without knowing the reasons of them: Even so it cannot be denied that you may apply the rules of the fluxionary method: … You may operate and compute and solve problems thereby, not only without an actual attention to, or an actual knowledge of, the grounds of that method, and the principles whereon it depends, and whence it is deduced, but even without having ever considered or comprehended them." (§32) "But then it must be remembered that in such case although you may pass for an artist, computist, or analyst, yet you may not be justly esteemed a man of science and demonstration. Nor should any man, in virtue of being conversant in such obscure analytics, imagine his rational faculties to be more improved than those of other men which have been exercised in a different manner and on different subjects; much less erect himself into a judge and an oracle concerning matters that have no sort of connexion with or dependence on those species, symbols or signs, in the management whereof he is so conversant and expert." (§33)

COLIN MACLAURIN, , Reaction to Berkeley, quoted from *Collected Letters of Colin Maclaurin*, Birkhäuser, 1982.

It is often said that Maclaurin's *Treatise of Fluxions* (1742) was written in reply to Berkeley's critique. In his letters, however, Maclaurin is very dismissive of Berkeley.

Berkeley's critique is "groundless" (427) and the alleged flaws that he "pretends to discover" (427) are all due to him having "not understood" (427) the mathematics in question. "[Newton's] notion of fluxions has nothing obscure, mysterious, unintelligible or absurd in it." (428) "What this writer [Berkeley] advances against the foundations of the methods of Fluxions serves only to shew that he has not considered or understood what its great Author [Newton] said in their defence when he first published them; for if he had, he would have found the most material of his objections prevented & answered there." (425)

Berkeley's critique was religiously motivated but: "I am satisfied that the interests of true Science and true Religion are united, & that they do real prejudice to Mankind who endeavour to represent them as opposite in any measure." (427) "I believe it will be easily granted by all who are acquainted with the History of Learning that there is no other order or Class of Learned Men that has produced fewer writers on the side of Infidelity, or fewer adversarys to natural or revealed Religion than that of the Mathematicians. The greatest Men among them have distinguished themselves as firm in the belief, and ornaments to the practice of Christianity, and particularly these men who invented or promoted the parts which this Author has so warmly attack'd." (426)

LEONHARD EULER, *Foundations of Differential Calculus*, 1755, translated by J. D. Blanton, Springer, 2000.

"83. This theory of the infinite will be further illustrated if we discuss that which mathematicians call the infinitely small. There is no doubt that any quantity can be diminished until it all but vanishes and then goes to nothing. But an infinitely small quantity is nothing but a vanishing quantity, and so it is really equal to 0. There is also a definition of the infinitely small quantity as that which is less than any assignable quantity. If a quantity is so small that it is less than any assignable quantity, then it cannot not be 0, since unless it is equal to 0 a quantity can be assigned equal to it, and this contradicts our hypothesis. To anyone who asks what an infinitely small quantity in mathematics is, we can respond that it really is equal to 0. There is really not such a great mystery lurking in this idea as some commonly think and thus have rendered the calculus of the infinitely small suspect to so many."

"85. These things are very clear, even in ordinary arithmetic. Everyone knows that when zero is multiplied by any number, the product is zero and that $n \cdot 0 = 0$, so that $n : 1 = 0 : 0$. Hence, it is clear that any two zeros can be in a geometric ratio, although from the perspective of arithmetic, the ratio is always of equals. Since between zeros any ratio is possible, in order to indicate this diversity we use different notations on purpose, especially when a geometric ratio between two zeros is being investigated. In the calculus of the infinitely small, we deal precisely with geometric ratios of infinitely small quantities."

"86. If we accept the notation used in the analysis of the infinite, then dx indicates a quantity that is infinitely small, so that both $dx = 0$ and $a\,dx = 0$, where a is any finite quantity. Despite this, the geometric ratio $a\,dx : dx$ is finite, namely $a : 1$. For this reason these two infinitely small quantities dx and $a\,dx$, both being equal to 0, cannot be confused when we consider their ratio. In a similar way, we will deal with infinitely small quantities dx and dy. Although these are both equal to 0, still their ratio is not that of equals. Indeed, the whole force of differential calculus is concerned with the investigation of the ratios of any two infinitely small quantities of this kind."

"87. … From this we obtain the well-known rule that the infinitely small vanishes in comparison with the finite and hence can be neglected. For this reason the objection brought up against the analysis of the infinite, that it lacks geometric rigor, falls to the ground under its own weight, since nothing is neglected except that which is actually nothing. Hence with perfect justice we can affirm that in this sublime science we keep the same perfect geometric rigor that is found in the books of the ancients."

§ R22. Non-Euclidean geometry

IMMANUEL KANT, *Inaugural Dissertation*, 1770.

"Space is not something objective and real, nor a substance, nor an accident, nor a relation; instead, it is subjective and ideal, and originates from the mind's nature in accord with a stable law as a scheme, as it were, for coordinating everything sensed externally." (Ak 2: 403)

IMMANUEL KANT, *The Critique of Pure Reason*, preface to the second edition, 1787, translated by J. M. D. Meiklejohn.

"It has hitherto been assumed that our cognition must conform to the objects; but all attempts to ascertain anything about these objects a priori, by means of conceptions, and thus to extend the range of our knowledge, have been rendered abortive by this assumption. Let us then make the experiment whether we may not be more successful in metaphysics, if we assume that the objects must conform to our cognition. This appears, at all events, to accord better with the possibility of our gaining the end we have in view, that is to say, of arriving at the cognition of objects a priori, of determining something with respect to these objects, before they are given to us. … If the intuition must conform to the nature of the objects, I do not see how we can know anything of them a priori. If, on the other hand, the object conforms to the nature of our faculty of intuition, I can then easily conceive the possibility of such an a priori knowledge."

MORRIS KLINE, *Mathematics: The Loss of Certainty*, Oxford University Press, 1980.

"The assumption that the angle sum [of a triangle] is less than 180° leads to a curious geometry, quite different from ours [Euclidean] but thoroughly consistent, which I have developed to my entire satisfaction. The theorems of this geometry appear to be paradoxical, and, to the uninitiated, absurd, but calm, steady reflection reveals that they contain nothing at all impossible." (Gauss, 1824; p. 82)

"I am becoming more and more convinced that the [physical] necessity of our [Euclidean] geometry cannot be proved, at least not by human reason nor for human reason. … We must not place geometry in the same class with arithmetic, which is purely a priori, but with mechanics." (Gauss, 1817; p. 87)

"No candid and intelligent person can doubt the truth of the chief properties of Parallel Lines, as set forth by Euclid in his Elements, two thousand years ago … The doctrine involves no obscurity nor confusion of thought, and leaves in the mind no reasonable ground for doubt." (Hamilton, 1837; p. 95)

"My own view is that Euclid's [parallel postulate] does not need demonstration, but is part of our notion of space, of the physical space of our experience—which one becomes acquainted with by experience, but which is the representation lying at the foundation of all external experience. … Not that the propositions of geometry are only approximately true, but that they remain absolutely true in regard to that Euclidean space which has been so long regarded as being the physical space of our experience." (Cayley, 1883; pp. 96–96)

"Insofar as the propositions of mathematics give an account of reality they are not certain; and insofar as they are certain they do not describe reality. … But it is, on the other hand, certain that mathematics in general and geometry in particular owe their existence to our need to learn something about the properties of real objects." (Einstein, 1921; p. 97)

NIKOLAI LOBACHEVSKY, *Pangeometry [1855]*, European Mathematical Society, 2010.

"Pangeometry [i.e., non-Euclidean geometry] … proves that the assumption that the value of the sum of the three angles of any right rectilinear triangle is constant, an assumption which is explicitly or implicitly adopted in ordinary geometry, is not a consequence of our notions of space. Only experience can confirm the truth of this assumption, for instance, by by effectively measuring the sum of three angles of a rectilinear triangle … One must give preference to triangles whose edges are very large, since according to Pangeometry, the difference between two right angles and the three angles of a rectilinear triangle increases as the edges increase." (75) "The distances between the celestial bodies provide us with a means for observing the angles of triangles whose edges are very large." (76)

HERMANN VON HELMHOLTZ, *Science and Culture: Popular and Philosophical Essays*, University Of Chicago Press, 1995.

The mind invents the categories in terms of which it perceives the world. "The same aether vibrations which the eye feels as light, the skin feels as heat. The same aerial vibrations which the skin feels as whirring motions, the ear feels as sound." (346). "Kant, however, went further ... [and] considered spatial determinations as little belonging to the real world, 'to the thing in itself,' as the colors which we see belonging to bodies in themselves, and which we rather brought into them through out eyes." (348). Be here there is a strong disanalogy: spatial intuition, according to Kant, contains definite content, namely the Euclidean axioms. The supposed proof of this is that we can all intuit Euclidean geometry and no other geometries. But this proof fails. By the same reasoning one could "prove" that the English language is innate while Swahili is not. Just as we are born with a general language capacity that quickly specialises in response to given environmental conditions, there is every reason to think that our spatial intuition is initially neutral with respect to Euclidean or non-Euclidean geometry and subsequently formed by empirical data. Thus Kant's claim that the Euclidean axioms are innate is: "1. an unproven hypothesis; 2. an unnecessary hypothesis, since it pretends to explain nothing in our factual world of representation that could not also be explained without its help; and 3. a completely unusable hypothesis for the explanation of our knowledge of the real world, since the theorems established by it may first be applied to the relations of the real world after its objective validity has been experimentally proven and determined" (380). Point 3 may be illustrated by the following example (373). Let ABC be an equilateral triangle. Extend AB and AC above A and mark the points b and c on these lines that have the same distance to A as do B and C. Now: does bc=BC? Euclidean geometry says yes; non-Eucliden no. It is not for the mind to decide the outcome: the mind invents the categories in terms of which it perceives the world, but equality or inequality of impressions must depend on equality or inequality in the underlying physical reality, whatever it may be. (From "The Facts in Perception," 1878)

HENRI POINCARÉ, *On the Foundations of Geometry*, 1898.

"[Geometry is not] imposed by experience. It is simply guided by experience. ... To ask whether the geometry of Euclid is true or that of Lobachevsky is false, is as absurd as to ask whether the metric system is true and that of the yard, foot, and inch, is false."

HENRI POINCARÉ, *Science and Hypothesis*, 1902, tr. 1905.

"If ... negative parallaxes were found [i.e., in effect, angle sums of less than 180° in astronomical triangles] ..., two courses would be open to us; we might either renounce Euclidean geometry, or else modify laws of optics and suppose that light does not travel rigorously in a straight line. It is needless to add that all the world would regard the latter solution as the more advantageous. The Euclidean geometry has, therefore, nothing to fear from fresh experiments." (ch. 5)