DISCUSSION QUESTIONS FOR
EUCLID'S ELEMENTS BOOK I

Viktor Blåsjö

§ E0. General

E0.1. Argue that geometry, as conceived in Euclid's *Elements*, is distinct from the physical world; that it is purely logical, abstract, Platonic.

E0.2. Argue that geometry, as conceived in Euclid's *Elements*, concerns physical space and in anchored in concrete and hands-on operations with real-world tools like ruler and compass.

E0.3. What are some examples of "assumption minimalism" in Euclid? That is, situations where it would have been natural to assume a bit more but Euclid would rather start from only the least possible assumption even if this means more work.

§ E1. Defs.–Prop. 3: foundations; ruler and compass

E1.1. Each of Euclid's definitions may be considered a characterisation of a certain class of entities X. But in what sense? Match each of the following types of definitions with the best example from Euclid.

- ☐ "Test-condition" definition that enable us to answer, for any given object(s), the question "is this an instance of X?" in an unambiguous way that guarantees us to reach a yes or no answer by following a practically executable set of steps.
- ☐ "Exclusion" definition that specifies X negatively, in terms of what it is not.
- ☐ "Fiat property" definition that characterise X in terms of its properties without being straightforwardly testable. In other words, it allows us to infer that if something is an X then it has such-and-such properties.
- ☐ "Maker's" definition that implies a way of producing an X.
- ☐ "Shorthand" definition that give a convenient label to an entire set of properties or conditions.
- ☐ "Psychological" definition that give you a hunch what it is about without being mathematically exact.
- ☐ Definitions 1, 4, 10, 15, 22, 23

E1.2. Match the concepts with how it would most naturally be defined.

- ☐ Test-condition, Fiat property, Maker's, Shorthand, Psychological, Exclusion
- ☐ prime number, magnetic, cousin, stack [of pancakes, for example], intuitive, unicorn

E1.3. Euclid's definition of a straight line is vague. Can we interpret it to say: a straight line is a line such that any piece of it fits anywhere else on the line?

- ☐ Yes
- ☐ No

E1.4. Euclid's Proposition 1 has a logical gap or hidden assumption in it, namely that:

- ☐ C exists
- ☐ AC, BC are straight
- ☐ the two circles have the same radius
- ☐ the triangle produced is unique when in fact there are two

E1.5. The gap in Euclid's Proposition 1 could be resolved or addressed at least in part by an argument based on:

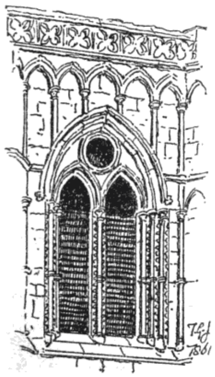
- ☐ Carrying out the constructions with ruler and compass.
- ☐ Allowing conclusions based on what is visually clear in the diagram, as long as it concerns not exact properties but only properties that would still hold even if the diagram was imperfectly drawn.
- ☐ The notions of "inside" and boundary in Definitions 13-15.

E1.6. Propositions 2 and 3 suggest that:

- ☐ Postulate 3 is more restricted than one might think.
- ☐ Euclid's compass is "collapsible": when lifted from the paper, it collapses and forgets the radius it was set to.
- ☐ Euclid's compass is "rusty": once set, it's stuck at a particular opening.
- ☐ Euclid's ruler is unmarked, just a "straightedge," that can be used to draw lines but not to measure lengths.
- ☐ Euclid's ruler can be used to produce equal line segments (as if you can put a single mark on it corresponding to the segment you have), but not to produce segments of other sizes (as if you had a full numerical scale on it).

E1.7. In the Declaration of Independence of the United States, the first part of the second sentence has a very Euclidean ring to it. It is reminiscent of one of Euclid's postulates in particular—which one? Postulate ☐. There are in fact several further allusions to Euclidean rhetoric in this document.

E1.8. This is a typical window design in Gothic architecture:



This design is reminiscent of Euclid's Proposition ☐. The Gothic style of architecture arose in the early 12th century, within a decade or two of the first Latin translation of Euclid's *Elements*.

§ E2. Props. 4–8: triangle congruence

E2.1. The proof of Proposition 4 arguably assumes that:

- ☐ Triangles are "motion invariant": they don't change whether you put them in one place or another.
- ☐ The straight line of Postulate 1 is unique.
- ☐ Propositions 2 and 3 enable us to reconstruct a given triangle in a new position.

E2.2. Propositions 1-3 were "problems" (showing how to do or make something) while Proposition 4 is a "theorem" (showing that a property or relation holds for certain objects). The difference is signalled by fixed stock phrases, notably in the ☐ paragraph(s).

- ☐ First
- ☐ Second
- ☐ Third
- ☐ Last

E2.3. Proposition 5 consists of two claims: (a) that regarding the angles of the triangle, and (b) that regarding the angles under the base. What is the relation between the two claims as they are proved?

- ☐ (a) is used to prove (b).
- ☐ (b) is used to prove (a).
- ☐ None of the above.

E2.4. Proposition 5 (cont.). Can I infer (a) directly by applying Proposition 4 to the "two" triangles ABC and ACB? (Corresponding to the intuitive idea that if I flip ABC over it fits on top of itself, so the base angles must be equal.)

- ☐ Yes.
- ☐ No, because the proof of Proposition 4 only works for distinct triangles.
- ☐ No, because we are proving Proposition 5 generally, without assuming anything about the magnitude of angle BAC.

E2.5. Proposition 5 (cont.). Suppose I know (a). Can I then infer (b) immediately from there on the grounds that it is "the rest" of angles ABD and ACE?

- ☐ Yes, using C.N. 3.
- ☐ No, because we don't know that "angles" ABD and ACE are equal.
- ☐ No, because that wouldn't show that the two angles are equal to one another.

E2.6. Proposition 5. There is a superfluous remark in the proof that serves no logical function in the argument. Find it. The points mentioned in this remark are:

- ☐ A
- ☐ B
- ☐ C
- ☐ D
- ☐ E
- ☐ F
- ☐ G

E2.7. Proposition 7. The proof doesn't work if:

- ☐ AC=CB and AD=DB.
- ☐ C is on AD.
- ☐ C is inside triangle ADB.

E2.8. Proposition 8. How can we conclude that the remaining pairs of angles are equal?

- ☐ It follows from the proof of Proposition 8 itself.
- ☐ It follows by applying Proposition 8 anew to the same triangles but with a different choice of "base".
- ☐ It can be obtained by applying Proposition 4.
- ☐ We can't; they may not be equal.

§ E3. Props. 9–11: further basic constructions

E3.1. Why does Euclid bisect an angle (Proposition 9) before a length (Proposition 10)?

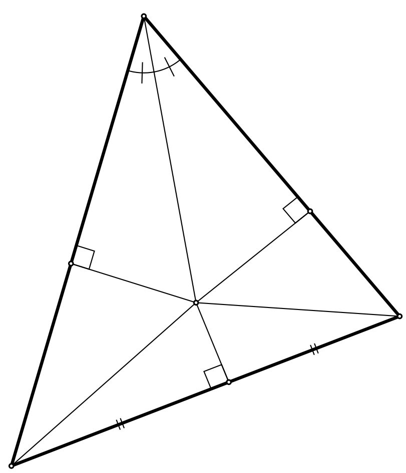
- ☐ Because the ruler-and-compass steps of the corresponding constructions proceed in this order.
- ☐ Because the only previous proposition that has equalities of sides in its conclusion is SAS.
- ☐ Because we have a postulate about equal angles (Postulate 4) but none about equal lengths.
- ☐ The order is an arbitrary choice since the two propositions are independent.

E3.2. Proposition 9. Suppose I follow Euclid's construction up to the point when DE has been drawn. If I divided DE into three equal parts (assuming for the moment that I can do this somehow), and then connected the dividing points up to A, will I have cut the angle into three equal parts?

- ◻ Yes
- ◻ No

E3.3. Consider the figure below. The point in the middle is defined as the intersection of the bisector of the top angle and the perpendicular bisector of the base. By construction, the top two triangles are congruent, and the two base triangles are congruent. It follows that the remaining two triangles are congruent. Which triangle congruence principles were needed to establish these three congruences?

- ◻ AAS
- ◻ ASA
- ◻ SAS
- ◻ SSS
- ◻ SSRA

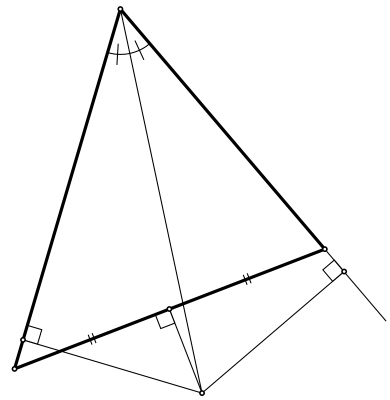


E3.4. (cont.) Since we started with an arbitrary triangle, it follows that all triangles are:

- ◻ right-angled
- ◻ isosceles
- ◻ bisected in area by perpendicular bisector of each side
- ◻ decomposable into six triangles with equal area

E3.5. (cont.) Actually the figure was incorrectly drawn. Below is the correct figure. On the basis of this example one could argue that:

- ◻ It can be dangerous to rely on diagrams when writing geometrical proofs.
- ◻ Exact constructions are important to safeguard rigour in geometrical reasoning.



§ E4. Props. 13–16: angles

E4.1. Postulate 4 is implicitly used in:

- ◻ Proposition 11
- ◻ Proposition 13
- ◻ Proposition 14
- ◻ Proposition 15
- ◻ None of the above

E4.2. (cont.) Is one or more of those propositions false on a 450 degree cone?

- ◻ Yes
- ◻ No

E4.3. Proposition 13. Could Euclid have gone directly from DBA+ABC=DBE+EBA+ABC [this is Euclid's second five-term equation] to DBE+CBE by using CBE=EBA+ABC [Euclid's first equation]?

- ◻ Yes, and this would have shortened his proof.
- ◻ Yes, but this would not have shortened his proof since it needs to apply to both cases mentioned in the statement of the proposition.
- ◻ No, because the proof needs to apply to both cases mentioned in the statement of the proposition.
- ◻ No, because there is no Common Notion legitimating such a step.

E4.4. Proposition 14. What does the phrase "Similarly, we can show that ..." refer to?

- ◻ The proof so far has only proved a limited statement, not the full theorem. But the remaining steps can be handled in a similar manner to what has already been explained. So Euclid omits them, expecting that the reader will be able to supply them if desired.
- ◻ The proof so far has ruled out one possible straight line. Theoretically, we would need to rule out all possible straight lines other than CBD. Since there are infinitely many there is no way of doing this

explicitly. But the argument given for CBE applies equally well to any other case.

- ☐ The proof so far has shown that CBD is a straight line but not that there is only one unique straight-line continuation of CB. Euclid omits a full proof of this, but it could be supplied by appeal to the implicit assumption of uniqueness in Postulate 1.

E4.5. (cont.) Isn't it precisely Euclid's strength that he proves everything systematically? So isn't it strange to see him speaking of what "we can show" instead of actually showing? This seems to undermine any claim that Euclid is offering:

- ☐ Structured investigation of the precise foundations of geometry.
- ☐ Precise logical exposition.
- ☐ Psychologically convincing arguments.

E4.6. Proposition 16. It is assumed that F falls within angle ABC, rather than below (the extension of) BC or above BA. This follows from:

- ☐ BE being an angle bisector.
- ☐ Two lines cannot enclose a space.
- ☐ None of the above.

E4.7. Proposition 16 does not hold generally on a sphere. Consider the case where B is the south pole, F is the north pole, and AEC is on the equator. If we try to apply Euclid's proof to this configuration, what goes wrong?

- ☐ Nothing. The proof works and the theorem is true for configurations of this type (though not necessarily for all spherical triangles).
- ☐ The inference based on Proposition 15 fails.
- ☐ The inference based on Proposition 4 fails.
- ☐ ECD is not greater than ECF.

§ E5. Props. 22–26: more triangle congruence

E5.1. Proposition 22. If the condition regarding the lengths of the sides is not satisfied, it is always impossible to construct a triangle from the given sides.

- ☐ Yes
- ☐ No

E5.2. Proposition 22 can be seen as a generalisation of Proposition 1. In our discussion of Proposition 1, we considered some ways of addressing a gap in the reasoning. Which of those ways are also applicable to the analogous gap in the proof of Proposition 22?

- ☐ Carrying out the constructions with ruler and compass.
- ☐ Allowing conclusions based on what is visually clear in the diagram, as long as it concerns not exact properties but only properties that would still hold even if the diagram was imperfectly drawn.
- ☐ The notions of "inside" and boundary in Definitions 13-15.

E5.3. Proposition 23 shows how to move an angle. But didn't we already assume the ability to move angles in our proof of Proposition 4? (In fact, Proposition 23 is logically dependent on Proposition 4, so we cannot make it the foundation for that part of the proof.)

E5.4. Proposition 26. This proof does not "apply" one triangle to the other, as the proofs of Propositions 4 and 8 did. Could this method have been used?

- ☐ Yes, but Euclid's proof is shorter.
- ☐ Yes, and it would have shortened the proof.
- ☐ No, it wouldn't work here because we would have to move an angle one of whose associated sides is not given.

E5.5. Proposition 26. Euclid has now proved SAS, SSS, and ASA triangle congruence. (Actually his Proposition 26 also includes SAA but I left this out in my edition.) Is there a triangle congruence theorem for every such letter combination? That is, if two triangles have "three things in common" then they are the same?

- ☐ Yes
- ☐ No

E5.6. You are at point A and you want to know the distance to point B. However, point B is inaccessible (it is on the other side of a river, for example), so you cannot measure AB directly. Instead you proceed as follows. From A measure along a straight line at right angles to AB a length AC and bisect it at D. From C draw CE at right angles to CA on the side of it remote from B, and let E be the point on it which is in a straight line with B and D. The sought distance AB is equal to the measurable distance:

- ☐ DC
- ☐ DE
- ☐ CE
- ☐ AC

E5.7. (cont.) The Euclidean propositions directly involved in setting up and justifying this procedure are:

- ☐ 1
- ☐ 2
- ☐ 4
- ☐ 5
- ☐ 7
- ☐ 8

- ☐ 9
- ☐ 10
- ☐ 11
- ☐ 13
- ☐ 14
- ☐ 15
- ☐ 16
- ☐ 22
- ☐ 23
- ☐ 26

E5.8. Simple bookshelves (such as the IKEA Ivar) consist of two vertical and some horizontal planks. A problem is that they could tip askew, so that when we look at the bookshelf from the front we see, instead of a rectangle with vertical sides, a parallelogram with its longer sides inclined a few degrees with respect to the floor. Two ways of addressing this are often implemented in such bookshelves. The X method is to nail one or two long metal fortifiers diagonally across the back of the shelf. This forces the distance between diagonally opposite points to be fixed. The L method is to nail L-shaped metal fortifiers where the shelves meet the sides. This forces the angle between them to be fixed at 90 degrees. Match each method with a proposition from Euclid that can be used to explain it.

- ☐ X method
- ☐ L method
- ☐ Neither
- ☐ Proposition 4 (SAS), Proposition 8 (SSS), Proposition 26 (ASA)

§ E6. Props. 27–32: parallels

E6.1. Proposition 27. Could Euclid have proved this by using ASA triangle congruence to infer that if the lines meet on one side they would also meet on the other and hence enclose a space, which is a contradiction?

- ☐ No, because the impossibility of the double-meeting configuration does not necessarily imply that the lines are parallel.
- ☐ No, because this would require theorems about angles not yet proved.
- ☐ Sort of, but not really, because Proposition 26 is not set up in the way needed for this argument.
- ☐ Yes, but Euclid's proof is preferable because it is shorter and uses fewer assumptions.

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E6.2. Euclid's theory of parallels (Propositions 27-31) studies whether two given lines are parallel or not by cutting them with a third line (which may be called the "test line"), and then looking at the angles produced. One can look at either the "alternate angles" (the pair of angles involved in Proposition 27; also called "Z-angles" because of their configuration) or the "internal angles" (the pair of angles involved in Postulate 5).

- ☐ Euclid uses alternate angles/Proposition 16 ...
- ☐ Euclid uses internal angles/Postulate 5 ...
- ☐ ... when he wants to prove that given lines are parallel.
- ☐ ... to derive properties any parallel lines must have.

E6.3. Which of the following does Euclid establish without relying on the parallel postulate in any way?

- ☐ Parallel lines exist.
- ☐ There is precisely one unique parallel to a given line through a given point.
- ☐ Alternate angles (Z-angles) are equal.
- ☐ Vertical angles (opposite pair in an X configuration) are equal.
- ☐ Angle sum of triangle is two right angles.
- ☐ Proposition 30.
- ☐ None of the above.

E6.4. Proposition 31 uses nothing later than Proposition 27, so Euclid could have placed it earlier. Do the intermediate propositions add some illumination?

- ☐ Yes, they imply that the parallel constructed in Proposition 31 is unique.
- ☐ Yes, they imply that the parallel constructed in Proposition 31 is not unique.
- ☐ Yes, they justify the implicit assumption in Proposition 31 that a parallel line exists.
- ☐ None of the above.

E6.5. Parallel lines can also be characterised as lines that always have the same distance between them. But the notion of not crossing is arguably more fundamental. As Postulate 2 says, lines are fundamentally things that can be extended indefinitely. Asking whether they will cross seems quite natural already from this standpoint alone, whereas talking about the distance between lines requires us to involve a lot more secondary concepts to specify what it even means.

The geometry of other surfaces also problematise the equivalence of parallelism and equidistance in interesting ways. For example, it can happen that the curve equidistant to a straight line is not a straight line. In which of the following geometries does this occur?

- ☐ sphere

- ☐ 450 degree cone with stretched-string definition of straight line
- ☐ 450 degree cone with half-turn-symmetry definition of straight line

If we wanted to base the theory of parallels on the notion of equidistance we would have to prove that these kinds of situations do not occur.

E6.6. Think about how Proposition 32 is related to Proposition 16. Why didn't Euclid prove the more powerful 32 first and then derive 16 very easily from there?

- ☐ Euclid's proof of Proposition 32 is logically dependent on Proposition 16, so 16 was a necessary stepping-stone along the way; without 16 we could not have reached 32.
- ☐ Although Euclid's proof of Proposition 32 is logically independent of Proposition 16, Euclid's way of proving Proposition 16 shows that it is independent of the parallel postulate, which would not have been show if it was derived as a corollary of Proposition 32.

E6.7. Proposition 32. Suppose I stand on one side of a triangle with my nose pointing in the direction of the side. I walk once around the triangle, turning accordingly, returning eventually to my original position. How many degrees did I turn? Does this correspond to Proposition 32?

- ☐ Yes, there is a simple relationship between the amount of turning and the angle sum of the triangle.
- ☐ No, because the turning amount would be the same if you walked around a four-sided figure, a five-sided figure, etc., so any correspondence with Proposition 32 is a coincidence.
- ☐ No, because the total amount of turning depends on weather the triangle has an obtuse angle or not, while Proposition 32 holds for any triangle.

E6.8. Why did Euclid wait so long (until Proposition 29) to use the parallel postulate? Does this tell us something about the status of the postulate?

E6.9. The parallel postulate is considerably more convoluted than the other postulates. But it transpires from the *Elements* itself that Euclid could have used a simpler, equivalent statement in place of it, such as: given any line and any point not on this line, there is no more than one parallel to the line through that point. Why did Euclid choose his formulation of the postulate?

§ E7. Props. 34–41: parallelograms; area

E7.1. The simplest sense in which two figures can have the same area is that of superposition: one "fits" on top of the other. But figures can also have the same area without being capable of such alignment; that is, without having the same shape. The first time Euclid talks about equality of area in this sense is in Proposition:

- ☐ 34
- ☐ 35
- ☐ 37
- ☐ 41

E7.2. In Propositions 34-41, Euclid's strategy for proving areas equal is to:

- ☐ Transform one area into another with an area-preserving transformation.
- ☐ Cut them into pieces and use triangle congruence.
- ☐ First establish area of three-sided figures, then use this to establish area of four-sided figures, and so on upwards.

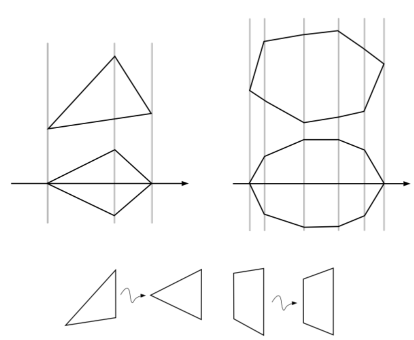
E7.3. Proposition 35. If AD and EF overlap:

- ☐ Euclid's proof can be considered to still apply, with G being above the parallelograms.
- ☐ A longer proof is needed.
- ☐ A shorter proof is possible.
- ☐ The proposition no longer holds.

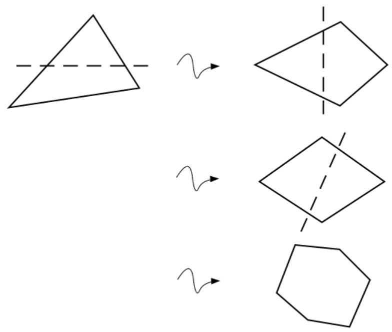
E7.4. The enemy is advancing toward your capital. They have currently set up camp not far from you. In the night, you send a spy to estimate the size of their army. The spy reports back that the enemy camp has the shape of a parallelogram and is 4000 paces all the way around. It is estimated that each soldier uses about 10 square paces of area for their night camp. Approximately how many soldiers does the enemy have in the camp?

- ☐ 4000
- ☐ 40000
- ☐ 400000
- ☐ 1000
- ☐ 10000
- ☐ 100000
- ☐ Cannot be determined from the information given.

E7.5. This process is called "symmetrisation":



In particular, it turns parallelograms into rectangles, which can be likened to straightening out a stack of books that has been knocked askew. Here is the symmetrisation process applied repeatedly to a specific figure:



The symmetrisation process is akin to packing a snowball; both these processes lead to a round shape for the same reason.

Among all figures with the same area, the circle has the least perimeter. I can prove this using symmetrisation if I first assume or prove that:

- ☐ Symmetrisation preserves area but decreases perimeter for any non-circular figure.
- ☐ Symmetrisation has no effect on circles.
- ☐ Among all figures with the same area, there exists one that has the least perimeter.

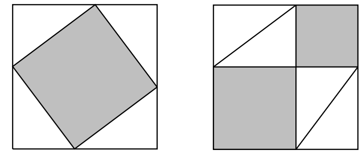
§ E8. Props. 46–48: Pythagorean Theorem

E8.1. Proposition 47. An odd phrase occurs: "either … or". This phrasing in effect amounts to an implicit application of Proposition ☐.

E8.2. Match the additional, implicitly assumed Common Notions used by Euclid with the proof in which they are first needed.

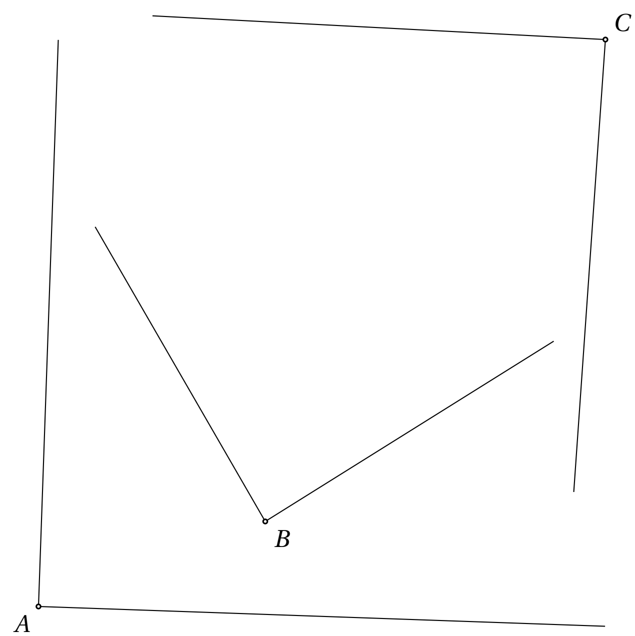
- ☐ The doubles of equal things are equal to one another.
- ☐ The squares on equal straight lines are equal to one another.
- ☐ Proposition 47
- ☐ Proposition 48

E8.3. Explain how the figure below proves the Pythagorean Theorem. Why did Euclid prefer his "windmill" proof?



E8.4. Which is a right angle? Use Proposition 48 to find out.

- ☐ A
- ☐ B
- ☐ C



E8.5. There are some indications that the ancient Egyptians knew that a triangle with sides 3, 4, 5 has a right angle. Is it realistic that they used this to construct right angles for practical purposes, such as when building the pyramids?

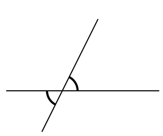
- ☐ Yes
- ☐ No

E8.6. In `https://youtu.be/I4tKOvDKhE8`, what is presented as the most likely motivation for the discovery of the Pythagorean Theorem?

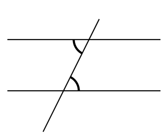
- ☐ Determination of the sizes of fields.
- ☐ Determination of geographical distances.
- ☐ Practical engineering questions such as what size ladder is needed for a certain task.
- ☐ Astronomical applications having to do with eclipses.
- ☐ No practical motivation, rather general interest in mathematical challenges/play/curiosity.

§ E9. Reference table

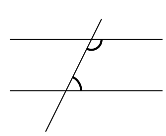
Prop.	Construction	Theorem
1	make equilateral △	
2	move segment	
3	cut off given length	
4		SAS △ congruence
5		isosceles △ ⇒ base angles equal
7		SSS uniqueness
8		SSS △ congruence
9	bisect angle	
10	bisect segment	
11	draw perpendicular	
13		angle on one side of straight line = $2\llcorner$
14		angle on one side = $2\llcorner$ ⇒ straight line
15		vertical angles equal
16		△ external angle > each opposite internal angle
22	make △ from three segments	
23	move angle	
26		ASA △ congruence
27		alternate angles equal ⇒ parallel
29		parallel ⇒ alternate angles equal, … internal angles = $2\llcorner$
30		parallel to same ⇒ parallel to each other
31	draw parallel through point	
32		△ angle sum = $2\llcorner$
34		▱ ⇒ opposite sides, angles equal; diagonal bisects
35		▱ w same base, height ⇒ equal area
37		△ w same base, height ⇒ equal area
41		▱ area = 2× corresponding △ area
46	draw a square	
47		right-angle △ ⇒ $a^2 + b^2 = c^2$
48		$a^2 + b^2 = c^2$ ⇒ right-angle △



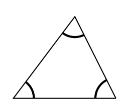
vertical angles
X angles



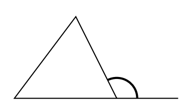
alternate angles
Z angles



internal angles
U angles



internal angles



external angle