Selections from
Euclid's Elements Book I

based on the translation by Richard Fitzpatrick
`farside.ph.utexas.edu/books/Euclid`

edited and illustrated by Viktor Blåsjö
`intellectualmathematics.com`

Definitions

1. A *point* is that of which there is no part.

2. And a *line* is a length without breadth.

3. And the extremities of a line are points.

4. A *straight line* is any one which lies evenly with points on itself.

5. And a *surface* is that which has length and breadth only.

6. And the extremities of a surface are lines.

7. A *plane surface* is any one which lies evenly with the straight lines on itself.

8. And a *plane angle* is the inclination of the lines to one another, when two lines in a plane meet one another, and are not lying in a straight line.

9. And when the lines containing the angle are straight then the angle is called *rectilinear*.

10. And when a straight line stood upon another straight line makes adjacent angles which are equal to one another, each of the equal angles is a *right angle*, and the former straight line is called a *perpendicular* to that upon which it stands.

13. A *boundary* is that which is the extremity of something.

14. A *figure* is that which is contained by some boundary or boundaries.

15. A *circle* is a plane figure contained by a single line [which is called a circumference], such that all of the straight lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.

16. And the point is called the *center* of the circle.

19. *Rectilinear figures* are those figures contained by straight lines: trilateral figures being those contained by three straight lines, quadrilateral by four, and multilateral by more than four.

20. And of the trilateral figures: an *equilateral* triangle is that having three equal sides, an *isosceles* triangle that having only two equal sides.

21. And further of the trilateral figures: a *right angled triangle* is that having a right angle.

22. And of the quadrilateral figures: a *square* is that which is right angled and equilateral.

23. *Parallel lines* are straight lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither of these directions.

Postulates

1. Let it have been postulated to draw a straight line from any point to any point.

2. And to produce a finite straight line continuously in a straight line.

3. And to draw a circle with any center and radius.

4. And that all right angles are equal to one another.

5. And that if a straight line falling across two other straight lines makes internal angles on the same side of itself whose sum is less than two right angles, then the two other straight lines, being produced to infinity, meet on that side of the original straight line that the sum of the internal angles is less than two right angles and do not meet on the other side.

Common Notions

1. Things equal to the same thing are also equal to one another.

2. And if equal things are added to equal things then the wholes are equal.

3. And if equal things are subtracted from equal things then the remainders are equal.

4. And things coinciding with one another are equal to one another.

5. And the whole is greater than the part.

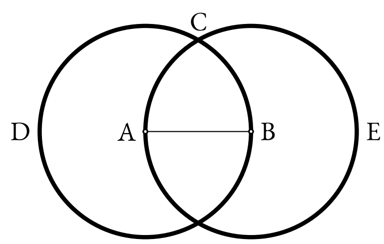
Proposition 1

To construct an equilateral triangle on a given finite straight line.

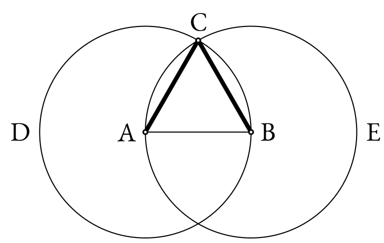
Let AB be the given finite straight line.

So it is required to construct an equilateral triangle on the straight line AB.

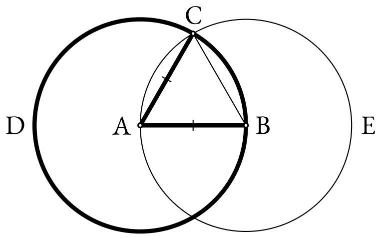
Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3].



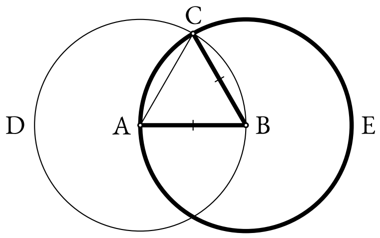
And let the straight lines CA and CB have been joined from the point C, where the circles cut one another, to the points A and B respectively [Post. 1].



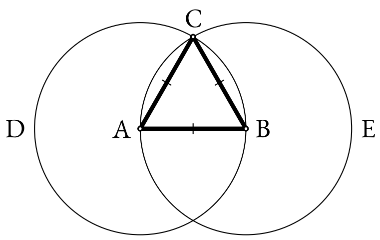
And since the point A is the center of the circle CDB, AC is equal to AB [Def. 1.15].



Again, since the point B is the center of the circle CAE, BC is equal to BA [Def. 1.15].



But CA was also shown to be equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, CA is also equal to CB. Thus, the three straight lines CA, AB, and BC are equal to one another.

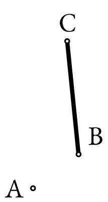


Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight line AB. Which is the very thing it was required to do.

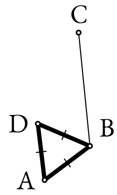
Proposition 2

To place a straight line equal to a given straight line at a given point as an extremity.

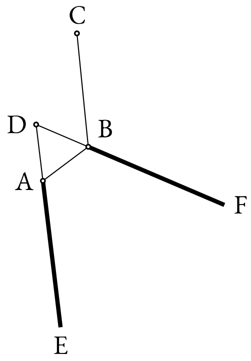
Let A be the given point, and BC the given straight line. So it is required to place a straight line at point A equal to the given straight line BC.



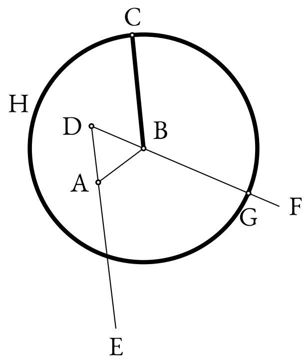
For let the straight line AB have been joined from point A to point B [Post. 1], and let the equilateral triangle DAB have been constructed upon it [Prop. 1].



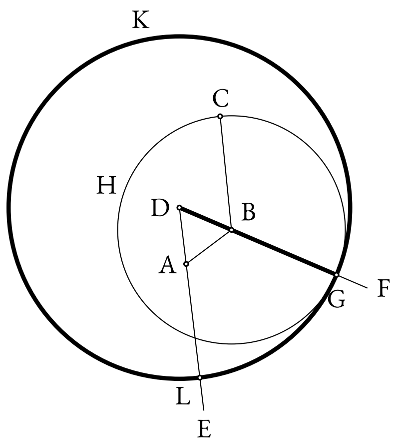
And let the straight lines AE and BF have been produced in a straight line with DA and DB respectively [Post. 2].



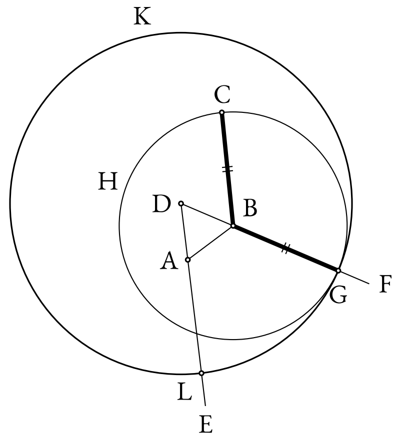
And let the circle CGH with center B and radius BC have been drawn [Post. 3],



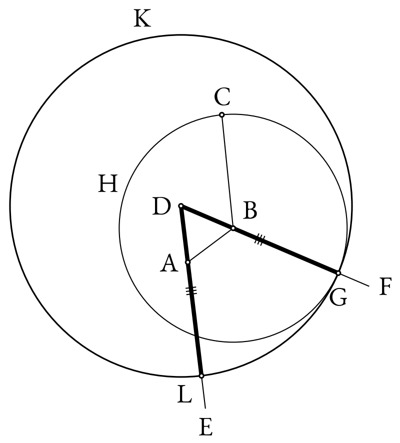
and again let the circle GKL with center D and radius DG have been drawn [Post. 3].



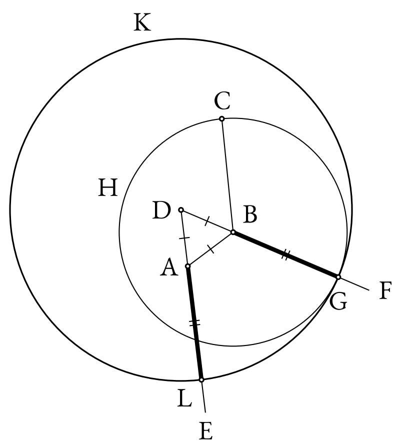
Therefore, since the point B is the center of the circle CGH, BC is equal to BG [Def. 1.15].



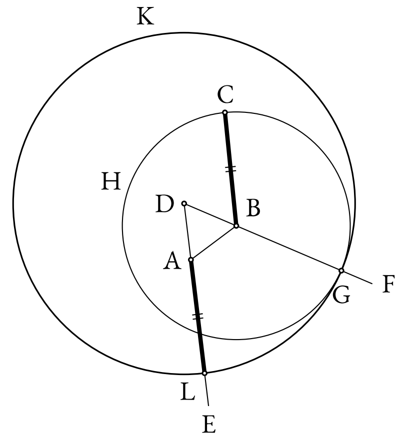
Again, since the point D is the center of the circle GKL, DL is equal to DG [Def. 1.15].



And within these, DA is equal to DB. Thus, the remainder AL is equal to the remainder BG [C.N. 3].



But BC was also shown to be equal to BG. Thus, AL and BC are each equal to BG. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, AL is also equal to BC.

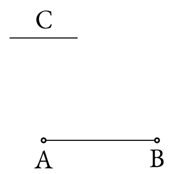


Thus, the straight line AL, equal to the given straight line BC, has been placed at the given point A. Which is the very thing it was required to do.

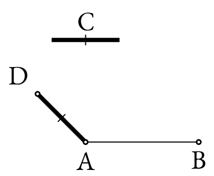
Proposition 3

For two given unequal straight lines, to cut off from the greater a straight line equal to the lesser.

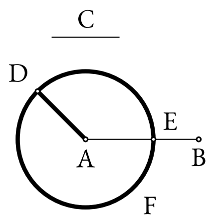
Let AB and C be the two given unequal straight lines, of which let the greater be AB. So it is required to cut off a straight line equal to the lesser C from the greater AB.



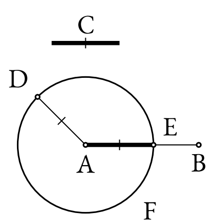
Let the line AD, equal to the straight line C, have been placed at point A [Prop. 2].



And let the circle DEF have been drawn with center A and radius AD [Post. 3].



And since point A is the center of circle DEF, AE is equal to AD [Def. 1.15]. But, C is also equal to AD. Thus, AE and C are each equal to AD. So AE is also equal to C [C.N. 1].

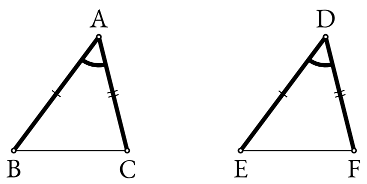


Thus, for two given unequal straight lines, AB and C, the straight line AE, equal to the lesser C, has been cut off from the greater AB. Which is the very thing it was required to do.

Proposition 4

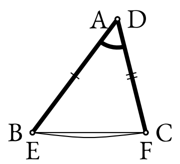
If two triangles have two sides equal to two sides, respectively, and have the angles enclosed by the equal straight lines equal, then they will also have the base equal to the base, and the triangle will be equal to the triangle, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles.

Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF, respectively. That is AB to DE, and AC to DF. And let the angle BAC be equal to the angle EDF.

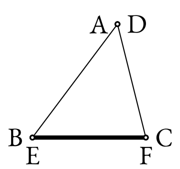


I say that the base BC is also equal to the base EF, and triangle ABC will be equal to triangle DEF, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. That is ABC to DEF, and ACB to DFE.

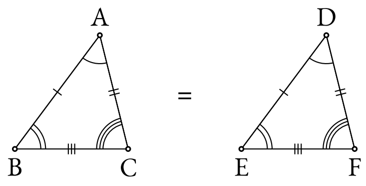
For if triangle ABC is applied to triangle DEF,



the point A being placed on the point D, and the straight line AB on DE, then the point B will also coincide with E, on account of AB being equal to DE. So because of AB coinciding with DE, the straight line AC will also coincide with DF, on account of the angle BAC being equal to EDF. So the point C will also coincide with the point F, again on account of AC being equal to DF. But, point B certainly also coincided with point E, so that the base BC will coincide with the base EF. For if B coincides with E, and C with F, and the base BC does not coincide with EF, then two straight lines will encompass an area. The very thing is impossible [Post. 1]. Thus, the base BC will coincide with EF, and will be equal to it [C.N. 4].



So the whole triangle ABC will coincide with the whole triangle DEF, and will be equal to it [C.N. 4]. And the remaining angles will coincide with the remaining angles, and will be equal to them [C.N. 4]. That is ABC to DEF, and ACB to DFE [C.N. 4].

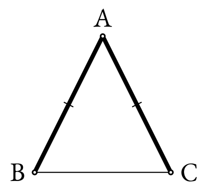


Thus, if two triangles have two sides equal to two sides, respectively, and have the angles enclosed by the equal straight line equal, then they will also have the base equal to the base, and the triangle will be equal to the triangle, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. Which is the very thing it was required to show.

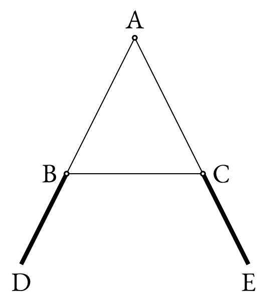
Proposition 5

For isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another.

Let ABC be an isosceles triangle having the side AB equal to the side AC,

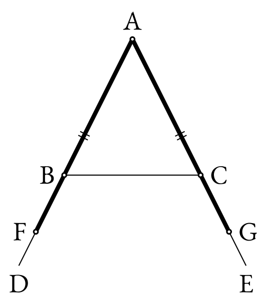


and let the straight lines BD and CE have been produced in a straight line with AB and AC respectively [Post. 2].

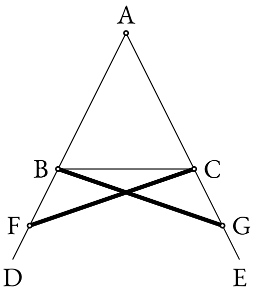


I say that the angle ABC is equal to ACB, and angle CBD to BCE.

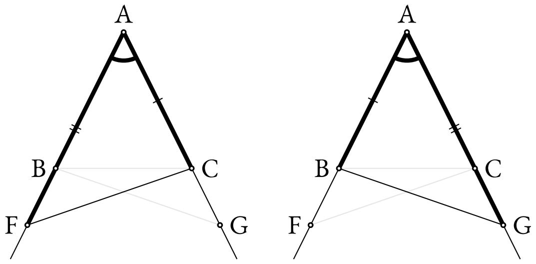
For let the point F have been taken at random on BD, and let AG have been cut off from the greater AE, equal to the lesser AF [Prop. 3].



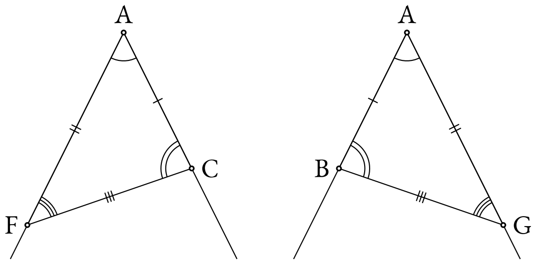
Also, let the straight lines FC and GB have been joined [Post. 1].



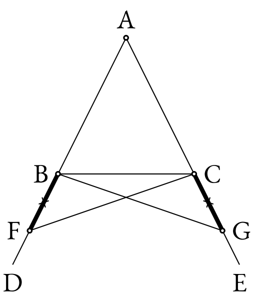
In fact, since AF is equal to AG, and AB to AC, the two straight lines FA, AC are equal to the two straight lines GA, AB, respectively. They also encompass a common angle, FAG.



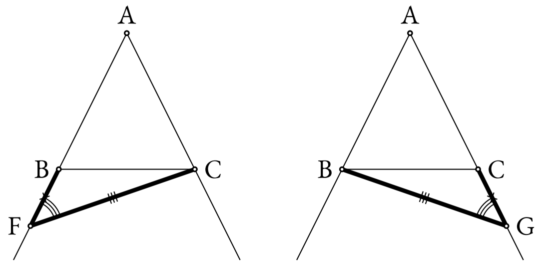
Thus, the base FC is equal to the base GB, and the triangle AFC will be equal to the triangle AGB, and the remaining angles subtendend by the equal sides will be equal to the corresponding remaining angles [Prop. 4]. That is ACF to ABG, and AFC to AGB.



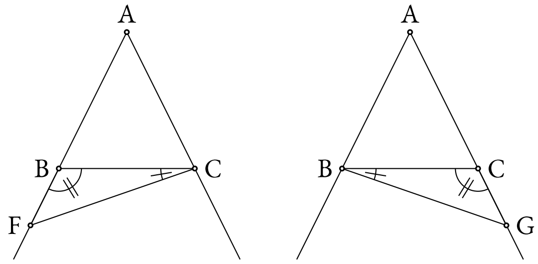
And since the whole of AF is equal to the whole of AG, within which AB is equal to AC, the remainder BF is thus equal to the remainder CG [C.N. 3]



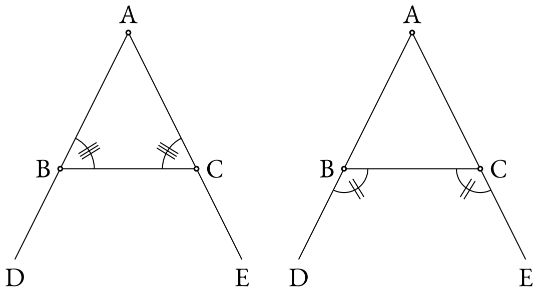
But FC was also shown to be equal to GB. So the two straight lines BF, FC are equal to the two straight lines CG, GB, respectively, and the angle BFC is equal to the angle CGB, and the base BC is common to them.



Thus, the triangle BFC will be equal to the triangle CGB, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 4]. Thus, FBC is equal to GCB, and BCF to CBG.



Therefore, since the whole angle ABG was shown to be equal to the whole angle ACF, within which CBG is equal to BCF, the remainder ABC is thus equal to the remainder ACB [C.N. 3]. And they are at the base of triangle ABC. And FBC was also shown to be equal to GCB. And they are under the base.

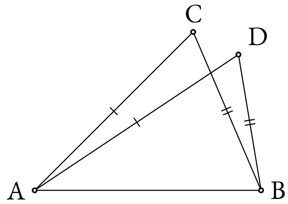


Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. Which is the very thing it was required to show.

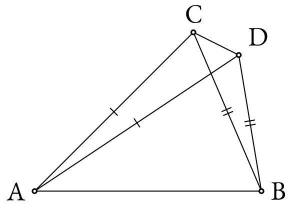
Proposition 7

On the same straight line, two other straight lines equal, respectively, to two given straight lines which meet cannot be constructed meeting at a different point on the same side of the straight line, but having the same ends as the given straight lines.

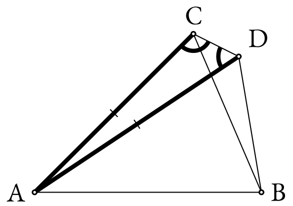
For, if possible, let the two straight lines AC, CB, equal to two other straight lines AD, DB, respectively, have been constructed on the same straight line AB, meeting at different points, C and D, on the same side of AB, and having the same ends on AB. So CA is equal to DA, having the same end A as it, and CB is equal to DB, having the same end B as it.



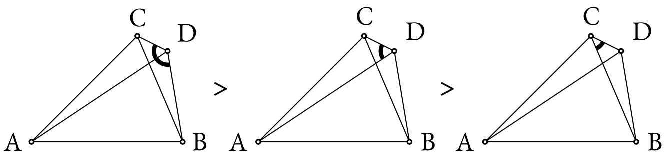
And let CD have been joined [Post. 1].



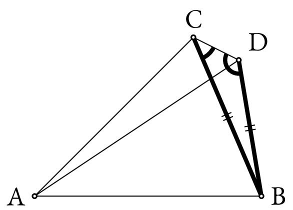
Therefore, since AC is equal to AD, the angle ACD is also equal to angle ADC [Prop. 5].



Thus, ADC is greater than DCB [C.N. 5]. Thus, CDB is much greater than DCB [C.N. 5].



Again, since CB is equal to DB, the angle CDB is also equal to angle DCB [Prop. 5].



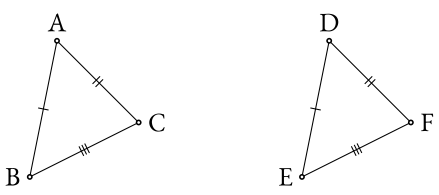
But it was shown that the former angle is also much greater than the latter. The very thing is impossible.

Thus, on the same straight line, two other straight lines equal, respectively, to two given straight lines which meet cannot be constructed meeting at a different point on the same side of the straight line, but having the same ends as the given straight lines. Which is the very thing it was required to show.

Proposition 8

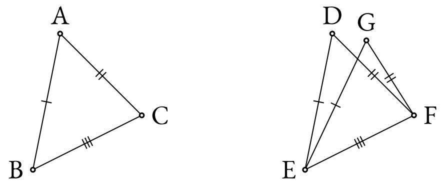
If two triangles have two sides equal to two sides, respectively, and also have the base equal to the base, then they will also have equal the angles encompassed by the equal straight lines.

Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF, respectively. That is AB to DE, and AC to DF. Let them also have the base BC equal to the base EF.

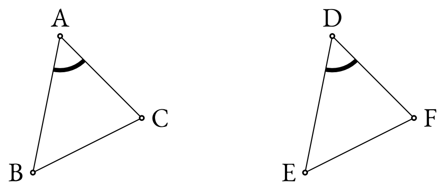


I say that the angle BAC is also equal to the angle EDF.

For if triangle ABC is applied to triangle DEF, the point B being placed on point E, and the straight line BC on EF, then point C will also coincide with F, on account of BC being equal to EF. So because of BC coinciding with EF, the sides BA and CA will also coincide with ED and DF respectively. For if base BC coincides with base EF, but the sides AB and AC do not coincide with ED and DF respectively, but miss like EG and GF in the above figure,



then we will have constructed upon the same straight line, two other straight lines equal, respectively, to two given straight lines, and meeting at a different point on the same side of the straight line, but having the same ends. But such straight lines cannot be constructed [Prop. 7]. Thus, the base BC being applied to the base EF, the sides BA and AC cannot not coincide with ED and DF respectively. Thus, they will coincide. So the angle BAC will also coincide with angle EDF, and will be equal to it [C.N. 4].

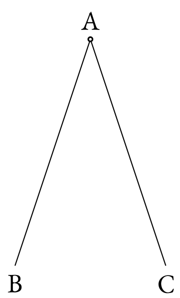


Thus, if two triangles have two sides equal to two side, respectively, and have the base equal to the base, then they will also have equal the angles encompassed by the equal straight lines. Which is the very thing it was required to show.

Proposition 9

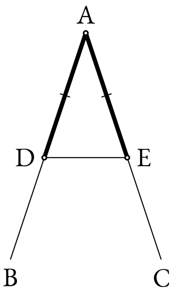
To cut a given rectilinear angle in half.

Let BAC be the given rectilinear angle.

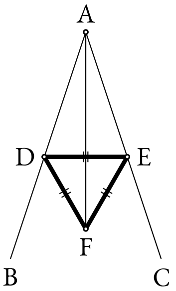


So it is required to cut it in half.

Let the point D have been taken at random on AB, and let AE, equal to AD, have been cut off from AC [Prop. 3], and let DE have been joined.

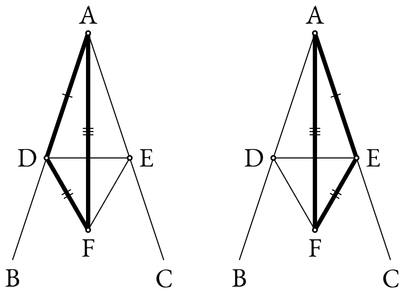


And let the equilateral triangle DEF have been constructed upon DE [Prop. 1], and let AF have been joined.

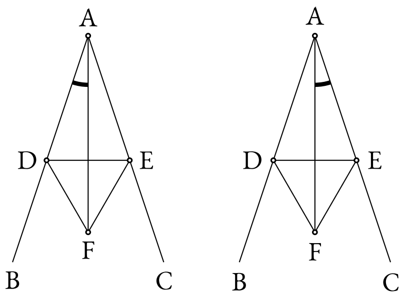


I say that the angle BAC has been cut in half by the straight line AF.

For since AD is equal to AE, and AF is common, the two straight lines DA, AF are equal to the two straight lines EA, AF, respectively. And the base DF is equal to the base EF.



Thus, angle DAF is equal to angle EAF [Prop. 8].



Thus, the given rectilinear angle BAC has been cut in half by the straight line AF. Which is the very thing it was required to do.

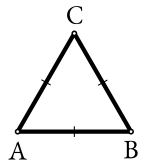
Proposition 10

To cut a given finite straight line in half.

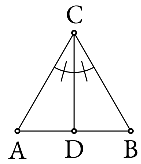
Let AB be the given finite straight line.

So it is required to cut the finite straight line AB in half.

Let the equilateral triangle ABC have been constructed upon AB [Prop. 1],

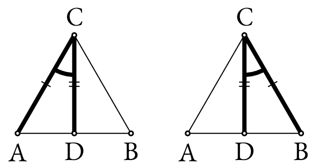


and let the angle ACB have been cut in half by the straight line CD [Prop. 9].

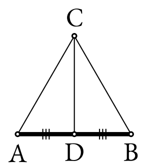


I say that the straight line AB has been cut in half at point D.

For since AC is equal to CB, and CD is common, the two straight lines AC, CD are equal to the two straight lines BC, CD, respectively. And the angle ACD is equal to the angle BCD.



Thus, the base AD is equal to the base BD [Prop. 4].

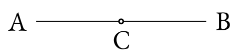


Thus, the given finite straight line AB has been cut in half at point D. Which is the very thing it was required to do.

Proposition 11

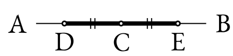
To draw a straight line at right angles to a given straight line from a given point on it.

Let AB be the given straight line, and C the given point on it.

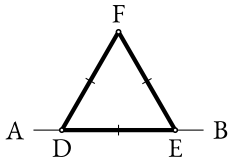


So it is required to draw a straight line from the point C at right angles to the straight line AB.

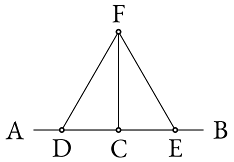
Let the point D be have been taken at random on AC, and let CE be made equal to CD [Prop. 3],



and let the equilateral triangle FDE have been constructed on DE [Prop. 1],

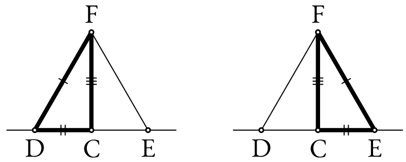


and let FC have been joined.

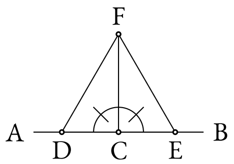


I say that the straight line FC has been drawn at right angles to the given straight line AB from the given point C on it.

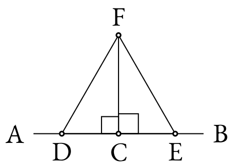
For since DC is equal to CE, and CF is common, the two straight lines DC, CF are equal to the two straight lines, EC, CF, respectively. And the base DF is equal to the base FE.



Thus, the angle DCF is equal to the angle ECF [Prop. 8], and they are adjacent.



But when a straight line stood on another straight line makes the adjacent angles equal to one another, each of the equal angles is a right angle [Def. 1.10]. Thus, each of the angles DCF and FCE is a right angle.

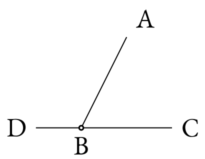


Thus, the straight line CF has been drawn at right angles to the given straight line AB from the given point C on it. Which is the very thing it was required to do.

Proposition 13

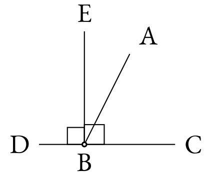
If a straight line stood on another straight line makes angles, it will certainly either make two right angles, or angles whose sum is equal to two right angles.

For let some straight line AB stood on the straight line CD make the angles CBA and ABD.



I say that the angles CBA and ABD are certainly either two right angles, or have a sum equal to two right angles.

In fact, if CBA is equal to ABD then they are two right angles [Def. 1.10]. But, if not, let BE have been drawn from the point B at right angles to [the straight line] CD [Prop. 11].

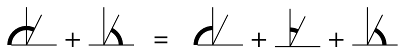


Thus, CBE and EBD are two right angles. And since CBE is equal to the two angles CBA and ABE,

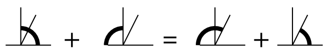
let EBD have been added to both. Thus, the sum of the angles CBE and EBD is equal to the sum of the three angles CBA, ABE, and EBD [C.N. 2].

Again, since DBA is equal to the two angles DBE and EBA,

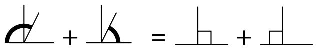
let ABC have been added to both. Thus, the sum of the angles DBA and ABC is equal to the sum of the three angles DBE, EBA, and ABC [C.N. 2].



But the sum of CBE and EBD was also shown to be equal to the sum of the same three angles. And things equal to the same thing are also equal to one another [C.N. 1]. Therefore, the sum of CBE and EBD is also equal to the sum of DBA and ABC.



But, the sum of CBE and EBD is two right angles. Thus, the sum of ABD and ABC is also equal to two right angles.

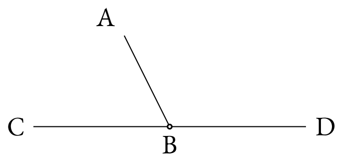


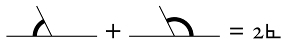
Thus, if a straight line stood on another straight line makes angles, it will certainly either make two right angles, or angles whose sum is equal to two right angles. Which is the very thing it was required to show.

Proposition 14

If two straight lines, not lying on the same side, make adjacent angles whose sum is equal to two right angles with some straight line, at a point on it, then the two straight lines will be straight on with respect to one another.

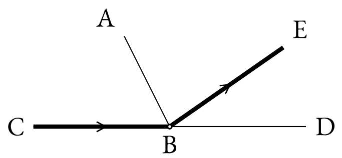
For let two straight lines BC and BD, not lying on the same side, make adjacent angles ABC and ABD whose sum is equal to two right angles with some straight line AB, at the point B on it.



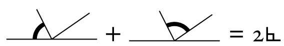


I say that BD is straight on with respect to CB.

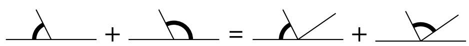
For if BD is not straight on to BC then let BE be straight on to CB.



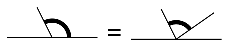
Therefore, since the straight line AB stands on the straight line CBE, the sum of the angles ABC and ABE is thus equal to two right angles [Prop. 13].



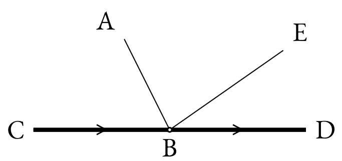
But the sum of ABC and ABD is also equal to two right angles. Thus, the sum of angles CBA and ABE is equal to the sum of angles CBA and ABD [C.N. 1].



Let angle CBA have been subtracted from both. Thus, the remainder ABE is equal to the remainder ABD [C.N. 3],



the lesser to the greater. The very thing is impossible. Thus, BE is not straight on with respect to CB. Similarly, we can show that neither is any other straight line than BD. Thus, CB is straight on with respect to BD.

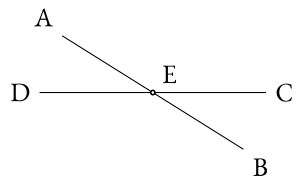


Thus, if two straight lines, not lying on the same side, make adjacent angles whose sum is equal to two right angles with some straight line, at a point on it, then the two straight lines will be straight on with respect to one another. Which is the very thing it was required to show.

Proposition 15

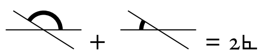
If two straight lines cut one another then they make the vertically opposite angles equal to one another.

For let the two straight lines AB and CD cut one another at the point E.

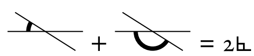


I say that angle AEC is equal to angle DEB, and angle CEB to angle AED.

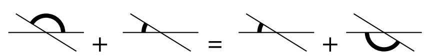
For since the straight line AE stands on the straight line CD, making the angles CEA and AED, the sum of the angles CEA and AED is thus equal to two right angles [Prop. 13].



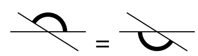
Again, since the straight line DE stands on the straight line AB, making the angles AED and DEB, the sum of the angles AED and DEB is thus equal to two right angles [Prop. 13].



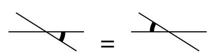
But the sum of CEA and AED was also shown to be equal to two right angles. Thus, the sum of CEA and AED is equal to the sum of AED and DEB [C.N. 1].



Let AED have been subtracted from both. Thus, the remainder CEA is equal to the remainder BED [C.N. 3].



Similarly, it can be shown that CEB and DEA are also equal.

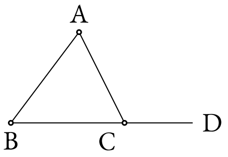


Thus, if two straight lines cut one another then they make the vertically opposite angles equal to one another. Which is the very thing it was required to show.

Proposition 16

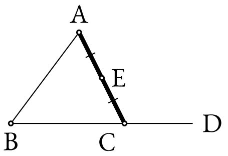
For any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles.

Let ABC be a triangle, and let one of its sides BC have been produced to D.

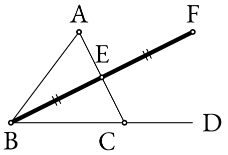


I say that the external angle ACD is greater than each of the internal and opposite angles, CBA and BAC.

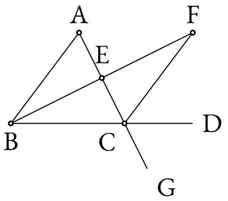
Let the straight line AC have been cut in half at point E [Prop. 10].



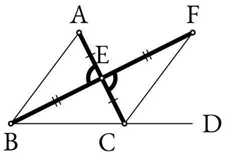
And BE being joined, let it have been produced in a straight line to point F. And let EF be made equal to BE [Prop. 3],



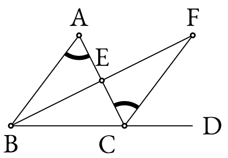
and let FC have been joined, and let AC have been drawn through to point G.



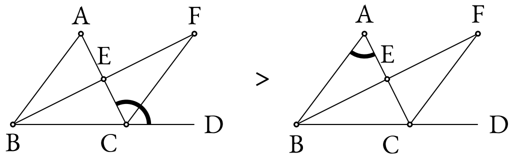
Therefore, since AE is equal to EC, and BE to EF, the two straight lines AE, EB are equal to the two straight lines CE, EF, respectively. Also, angle AEB is equal to angle FEC, for they are vertically opposite [Prop. 15].



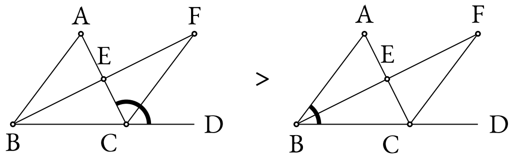
Thus, the base AB is equal to the base FC, and the triangle ABE is equal to the triangle FEC, and the remaining angles subtended by the equal sides are equal to the corresponding remaining angles [Prop. 4]. Thus, BAE is equal to ECF.



But ECD is greater than ECF. Thus, ACD is greater than BAE.



Similarly, by having cut BC in half, it can be shown that BCG—that is to say, ACD—is also greater than ABC.

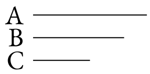


Thus, for any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles. Which is the very thing it was required to show.

Proposition 22

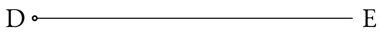
To construct a triangle from three straight lines which are equal to three given [straight lines] [such that] the sum of two of the straight lines taken together in any possible way to be greater than the remaining one.

Let A, B, and C be the three given straight lines, of which let the sum of two taken together in any possible way be greater than the remaining one. Thus, the sum of A and B is greater than C, the sum of A and C than B, and also the sum of B and C than A.

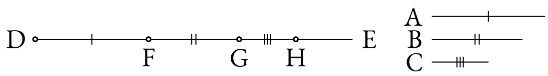


So it is required to construct a triangle from straight lines equal to A, B, and C.

Let some straight line DE be set out, terminated at D, and infinite in the direction of E.

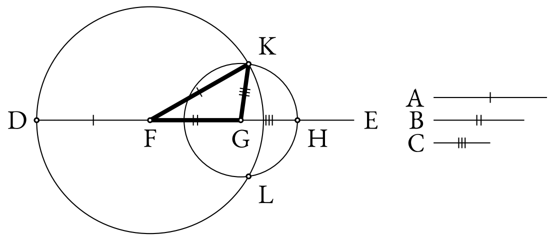


And let DF made equal to A, and FG equal to B, and GH equal to C [Prop. 3].



And let the circle DKL have been drawn with center F and radius FD. Again, let the circle KLH have been drawn with center G and radius GH. And let KF and KG have been joined. I say that the triangle KFG has been constructed from three straight lines equal to A, B, and C.

For since point F is the center of the circle DKL, FD is equal to FK. But, FD is equal to A. Thus, KF is also equal to A. Again, since point G is the center of the circle LKH, GH is equal to GK. But, GH is equal to C. Thus, KG is also equal to C. And FG is also equal to B. Thus, the three straight lines KF, FG, and GK are equal to A, B, and C respectively.

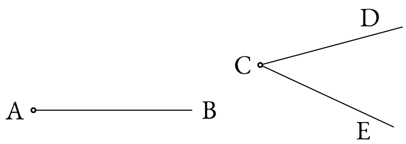


Thus, the triangle KFG has been constructed from the three straight lines KF, FG, and GK, which are equal to the three given straight lines A, B, and C respectively. Which is the very thing it was required to do.

Proposition 23

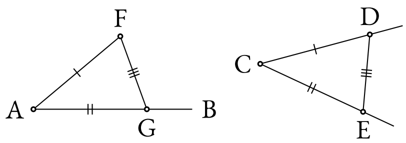
To construct a rectilinear angle equal to a given rectilinear angle at a given point on a given straight line.

Let AB be the given straight line, A the given point on it, and DCE the given rectilinear angle.

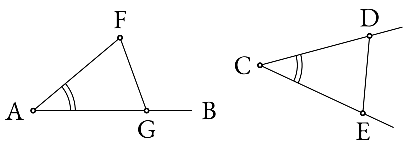


So it is required to construct a rectilinear angle equal to the given rectilinear angle DCE at the given point A on the given straight line AB.

Let the points D and E have been taken at random on each of the straight lines CD and CE respectively, and let DE have been joined. And let the triangle AFG have been constructed from three straight lines which are equal to CD, DE, and CE, such that CD is equal to AF, CE to AG, and further DE to FG [Prop. 22].



Therefore, since the two straight lines DC, CE are equal to the two straight lines FA, AG, respectively, and the base DE is equal to the base FG, the angle DCE is thus equal to the angle FAG [Prop. 8].

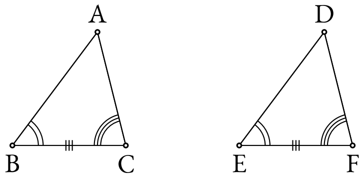


Thus, the rectilinear angle FAG, equal to the given rectilinear angle DCE, has been constructed at the given point A on the given straight line AB. Which is the very thing it was required to do.

Proposition 26

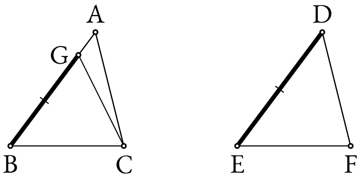
If two triangles have two angles equal to two angles, respectively, and one side equal to one side—. . . that by the equal angles . . . —then the triangles will also have the remaining sides equal to the [corresponding] remaining sides, and the remaining angle equal to the remaining angle.

Let ABC and DEF be two triangles having the two angles ABC and BCA equal to the two angles DEF and EFD, respectively. That is ABC equal to DEF, and BCA to EFD. And let them also have one side equal to one side. [Specifically], the side by the equal angles. That is BC equal to EF.

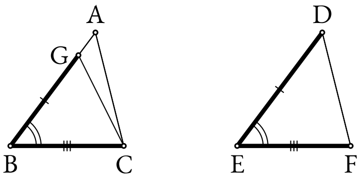


I say that they will have the remaining sides equal to the corresponding remaining sides. That is AB equal to DE, and AC to DF. And they will have the remaining angle equal to the remaining angle. That is BAC equal to EDF.

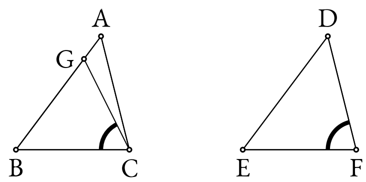
For if AB is unequal to DE then one of them is greater. Let AB be greater, and let BG be made equal to DE [Prop. 3], and let GC have been joined.



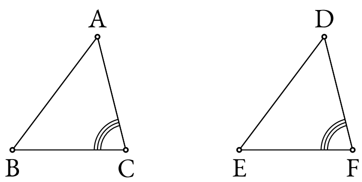
Therefore, since BG is equal to DE, and BC to EF, the two straight lines GB, BC are equal to the two straight lines DE, EF, respectively. And angle GBC is equal to angle DEF.



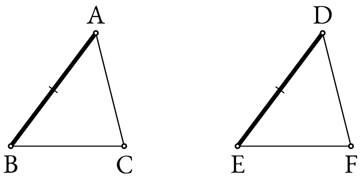
Thus, the base GC is equal to the base DF, and triangle GBC is equal to triangle DEF, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 4]. Thus, GCB is equal to DFE.



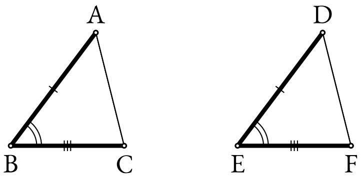
But, DFE was assumed to be equal to BCA.



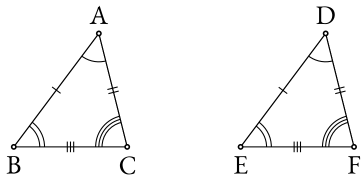
Thus, BCG is also equal to BCA, the lesser to the greater. The very thing is impossible. Thus, AB is not unequal to DE. Thus, it is equal.



And BC is also equal to EF. So the two straight lines AB, BC are equal to the two straight lines DE, EF, respectively. And angle ABC is equal to angle DEF.



Thus, the base AC is equal to the base DF, and the remaining angle BAC is equal to the remaining angle EDF [Prop. 4].

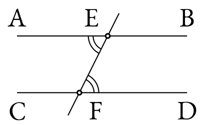


Thus, if two triangles have two angles equal to two angles, respectively, and one side equal to one side—. . . that by the equal angles . . .—then the triangles will also have the remaining sides equal to the corresponding remaining sides, and the remaining angle equal to the remaining angle. Which is the very thing it was required to show.

Proposition 27

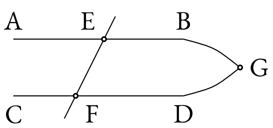
If a straight line falling across two straight lines makes the alternate angles equal to one another then the two straight lines will be parallel to one another.

For let the straight line EF, falling across the two straight lines AB and CD, make the alternate angles AEF and EFD equal to one another.

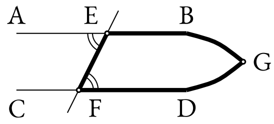


I say that AB and CD are parallel.

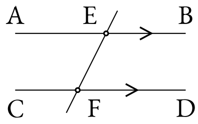
For if not, being produced, AB and CD will certainly meet together: either in the direction of B and D, or in the direction of A and C [Def. 1.23]. Let them have been produced, and let them meet together in the direction of B and D at point G.



So, for the triangle GEF, the external angle AEF is equal to the interior and opposite angle EFG.



The very thing is impossible [Prop. 16]. Thus, being produced, AB and CD will not meet together in the direction of B and D. Similarly, it can be shown that neither will they meet together in the direction of A and C. But straight lines meeting in neither direction are parallel [Def. 1.23]. Thus, AB and CD are parallel.

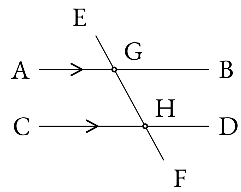


Thus, if a straight line falling across two straight lines makes the alternate angles equal to one another then the two straight lines will be parallel to one another. Which is the very thing it was required to show.

Proposition 29

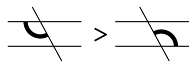
A straight line falling across parallel straight lines makes the alternate angles equal to one another, the external angle equal to the internal and opposite angle, and the sum of the internal angles on the same side equal to two right angles.

For let the straight line EF fall across the parallel straight lines AB and CD.

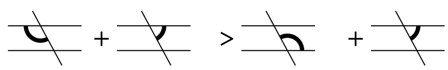


I say that it makes the alternate angles, AGH and GHD, equal, the external angle EGB equal to the internal and opposite angle GHD, and the sum of the internal angles on the same side, BGH and GHD, equal to two right angles.

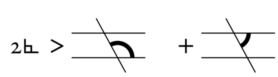
For if AGH is unequal to GHD then one of them is greater. Let AGH be greater.



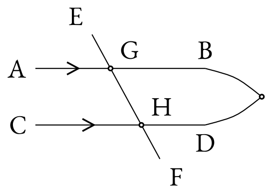
Let BGH have been added to both. Thus, the sum of AGH and BGH is greater than the sum of BGH and GHD.



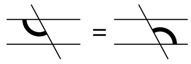
But, the sum of AGH and BGH is equal to two right angles [Prop 1.13]. Thus, the sum of BGH and GHD is [also] less than two right angles.



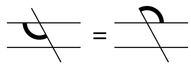
But straight lines being produced to infinity from internal angles whose sum is less than two right angles meet together [Post. 5]. Thus, AB and CD, being produced to infinity, will meet together.



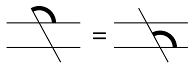
But they do not meet, on account of them initially being assumed parallel to one another [Def. 1.23]. Thus, AGH is not unequal to GHD. Thus, it is equal.



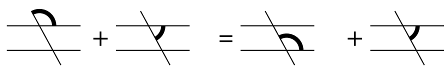
But, AGH is equal to EGB [Prop. 15].



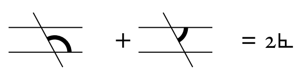
And EGB is thus also equal to GHD.



Let BGH be added to both. Thus, the sum of EGB and BGH is equal to the sum of BGH and GHD.



But, the sum of EGB and BGH is equal to two right angles [Prop. 13]. Thus, the sum of BGH and GHD is also equal to two right angles.

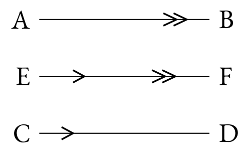


Thus, a straight line falling across parallel straight lines makes the alternate angles equal to one another, the external angle equal to the internal and opposite angle, and the sum of the internal angles on the same side equal to two right angles. Which is the very thing it was required to show.

Proposition 30

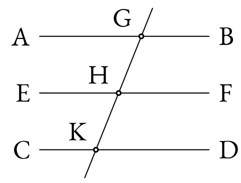
Straight lines parallel to the same straight line are also parallel to one another.

Let each of the straight lines AB and CD be parallel to EF.

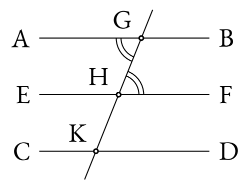


I say that AB is also parallel to CD.

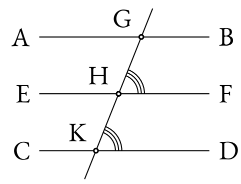
For let the straight line GK fall across AB, CD, and EF.



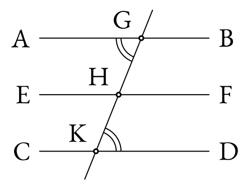
And since the straight line GK has fallen across the parallel straight lines AB and EF, angle AGK is thus equal to GHF [Prop. 29].



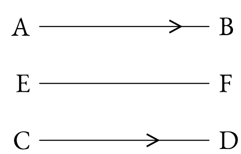
Again, since the straight line GK has fallen across the parallel straight lines EF and CD, angle GHF is equal to GKD [Prop. 29].



But AGK was also shown to be equal to GHF. Thus, AGK is also equal to GKD.



And they are alternate angles. Thus, AB is parallel to CD [Prop. 27].

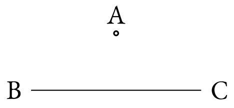


[Thus, straight lines parallel to the same straight line are also parallel to one another.] Which is the very thing it was required to show.

Proposition 31

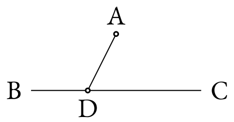
To draw a straight line parallel to a given straight line, through a given point.

Let A be the given point, and BC the given straight line.

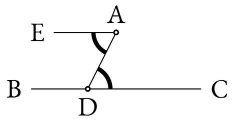


So it is required to draw a straight line parallel to the straight line BC, through the point A.

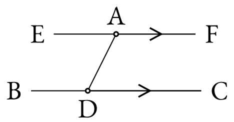
Let the point D have been taken a random on BC, and let AD have been joined.



And let angle DAE, equal to angle ADC, have been constructed on the straight line DA at the point A on it [Prop. 23].



And let the straight line AF have been produced in a straight line with EA. And since the straight line AD, in falling across the two straight lines BC and EF, has made the alternate angles EAD and ADC equal to one another, EAF is thus parallel to BC [Prop. 27].

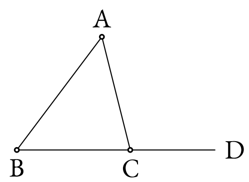


Thus, the straight line EAF has been drawn parallel to the given straight line BC, through the given point A. Which is the very thing it was required to do.

Proposition 32

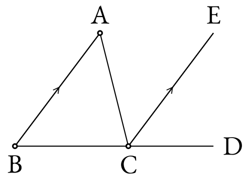
In any triangle, if one of the sides is produced then the external angle is equal to the sum of the two internal and opposite angles, and the sum of the three internal angles of the triangle is equal to two right angles.

Let ABC be a triangle, and let one of its sides BC have been produced to D.

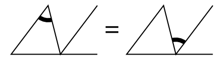


I say that the external angle ACD is equal to the sum of the two internal and opposite angles CAB and ABC, and the sum of the three internal angles of the triangle—ABC, BCA, and CAB—is equal to two right angles.

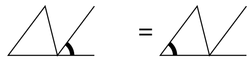
For let CE have been drawn through point C parallel to the straight line AB [Prop. 31].



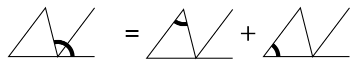
And since AB is parallel to CE, and AC has fallen across them, the alternate angles BAC and ACE are equal to one another [Prop. 29].



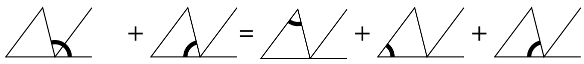
Again, since AB is parallel to CE, and the straight line BD has fallen across them, the external angle ECD is equal to the internal and opposite angle ABC [Prop. 29].



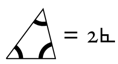
But ACE was also shown to be equal to BAC. Thus, the whole angle ACD is equal to the sum of the two internal and opposite angles BAC and ABC.



Let ACB have been added to both. Thus, the sum of ACD and ACB is equal to the sum of the three angles ABC, BCA, and CAB.



But, the sum of ACD and ACB is equal to two right angles [Prop. 13]. Thus, the sum of ACB, CBA, and CAB is also equal to two right angles.

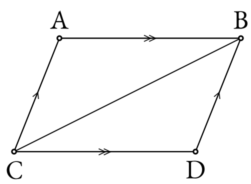


Thus, in any triangle, if one of the sides is produced then the external angle is equal to the sum of the two internal and opposite angles, and the sum of the three internal angles of the triangle is equal to two right angles. Which is the very thing it was required to show.

Proposition 34

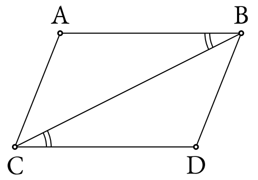
In parallelogrammic figures the opposite sides and angles are equal to one another, and a diagonal cuts them in half.

Let ACDB be a parallelogrammic figure, and BC its diagonal.

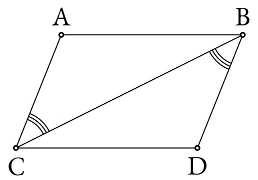


I say that for parallelogram ACDB, the opposite sides and angles are equal to one another, and the diagonal BC cuts it in half.

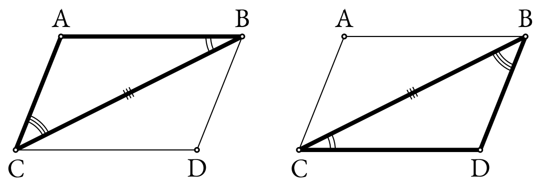
For since AB is parallel to CD, and the straight line BC has fallen across them, the alternate angles ABC and BCD are equal to one another [Prop. 29].



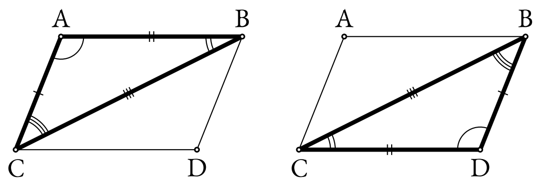
Again, since AC is parallel to BD, and BC has fallen across them, the alternate angles ACB and CBD are equal to one another [Prop. 29].



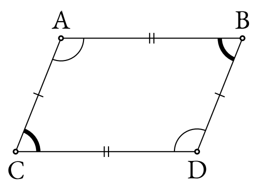
So ABC and BCD are two triangles having the two angles ABC and BCA equal to the two angles BCD and CBD, respectively, and one side equal to one side—the one by the equal angles and common to them, namely BC.



Thus, they will also have the remaining sides equal to the corresponding remaining sides, and the remaining angle equal to the remaining angle [Prop. 26].

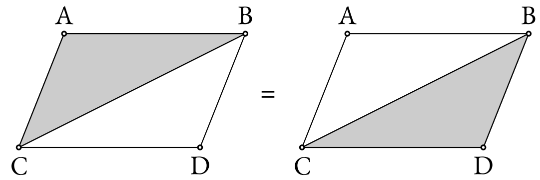


Thus, side AB is equal to CD, and AC to BD. Furthermore, angle BAC is equal to CDB. And since angle ABC is equal to BCD, and CBD to ACB, the whole angle ABD is thus equal to the whole angle ACD. And BAC was also shown to be equal to CDB.



Thus, in parallelogrammic figures the opposite sides and angles are equal to one another.

And, I also say that a diagonal cuts them in half. For since AB is equal to CD, and BC is common, the two straight lines AB, BC are equal to the two straight lines DC, CB, respectively. And angle ABC is equal to angle BCD. Thus, the base AC is also equal to DB, and triangle ABC is equal to triangle BCD [Prop. 4].

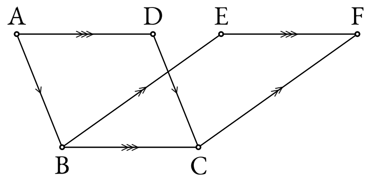


Thus, the diagonal BC cuts the parallelogram ACDB in half. Which is the very thing it was required to show.

Proposition 35

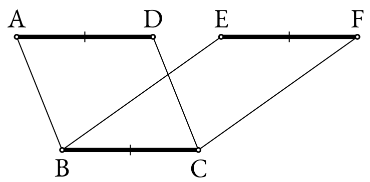
Parallelograms which are on the same base and between the same parallels are equal to one another.

Let ABCD and EBCF be parallelograms on the same base BC, and between the same parallels AF and BC.

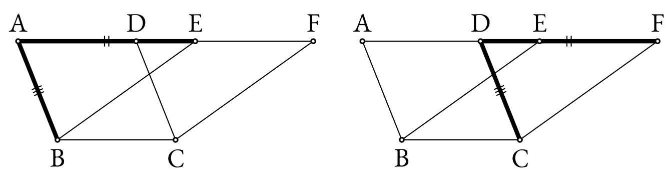


I say that ABCD is equal to parallelogram EBCF.

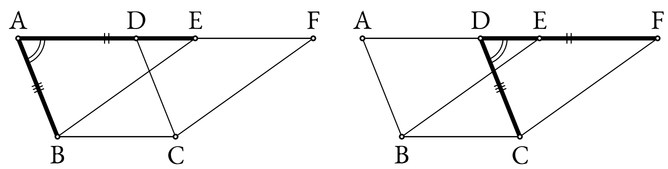
For since ABCD is a parallelogram, AD is equal to BC [Prop. 34]. So, for the same reasons, EF is also equal to BC. So AD is also equal to EF.



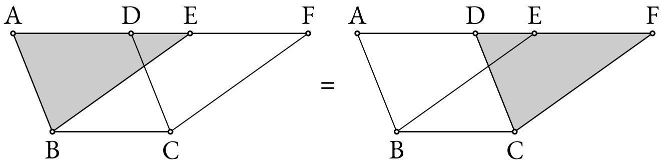
And DE is common. Thus, the whole straight line AE is equal to the whole straight line DF. And AB is also equal to DC. So the two straight lines EA, AB are equal to the two straight lines FD, DC, respectively.



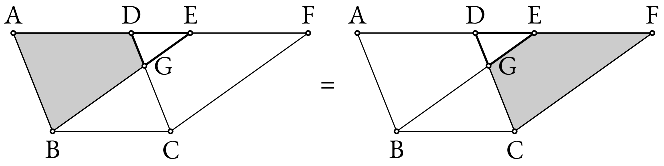
And angle FDC is equal to angle EAB, the external to the internal [Prop. 29].



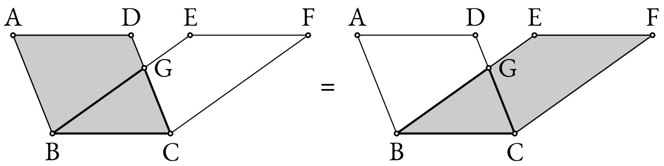
Thus, the base EB is equal to the base FC, and triangle EAB will be equal to triangle DFC [Prop. 4].



Let DGE have been taken away from both. Thus, the remaining trapezium ABGD is equal to the remaining trapezium EGCF.



Let triangle GBC have been added to both. Thus, the whole parallelogram ABCD is equal to the whole parallelogram EBCF.

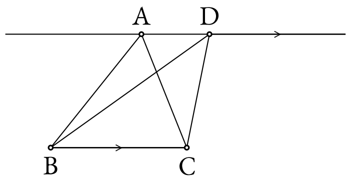


Thus, parallelograms which are on the same base and between the same parallels are equal to one another. Which is the very thing it was required to show.

Proposition 37

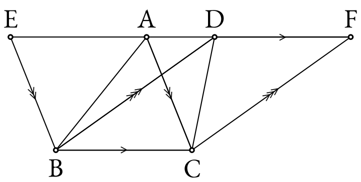
Triangles which are on the same base and between the same parallels are equal to one another.

Let ABC and DBC be triangles on the same base BC, and between the same parallels AD and BC.

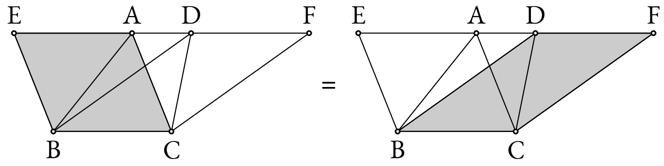


I say that triangle ABC is equal to triangle DBC.

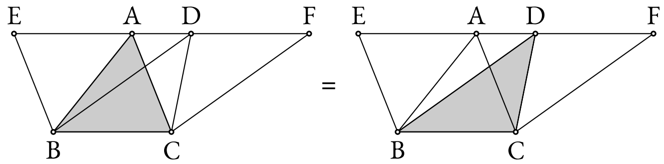
Let AD have been produced in both directions to E and F, and let the straight line BE have been drawn through B parallel to CA [Prop. 31], and let the straight line CF have been drawn through C parallel to BD [Prop. 31].



Thus, EBCA and DBCF are both parallelograms, and are equal. For they are on the same base BC, and between the same parallels BC and EF [Prop. 35].



And the triangle ABC is half of the parallelogram EBCA. For the diagonal AB cuts the latter in half [Prop. 34]. And the triangle DBC is half of the parallelogram DBCF. For the diagonal DC cuts the latter in half [Prop. 34]. Thus, triangle ABC is equal to triangle DBC.

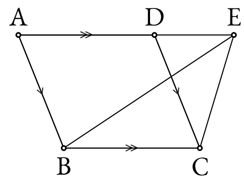


Thus, triangles which are on the same base and between the same parallels are equal to one another. Which is the very thing it was required to show.

Proposition 41

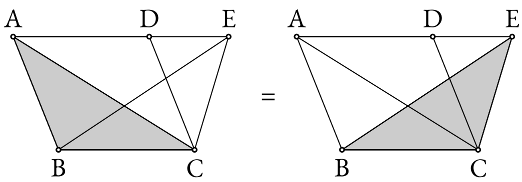
If a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double the area of the triangle.

For let parallelogram ABCD have the same base BC as triangle EBC, and let it be between the same parallels, BC and AE.

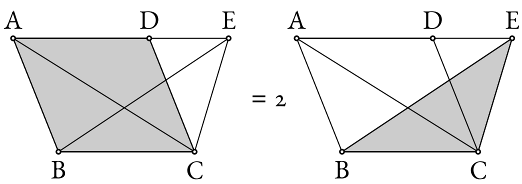


I say that parallelogram ABCD is double the area of triangle BEC.

For let AC have been joined. So triangle ABC is equal to triangle EBC. For it is on the same base, BC, as EBC, and between the same parallels, BC and AE [Prop. 37].



But, parallelogram ABCD is double the area of triangle ABC. For the diagonal AC cuts the former in half [Prop. 34]. So parallelogram ABCD is also double the area of triangle EBC.

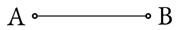


Thus, if a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double the area of the triangle. Which is the very thing it was required to show.

Proposition 46

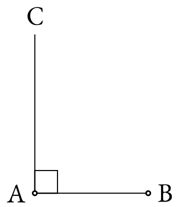
To describe a square on a given straight line.

Let AB be the given straight line.

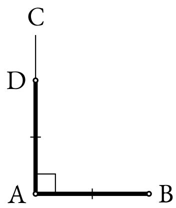


So it is required to describe a square on the straight line AB.

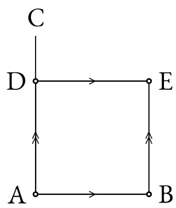
Let AC have been drawn at right angles to the straight line AB from the point A on it [Prop. 11],



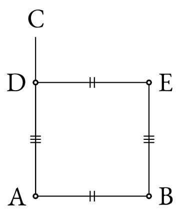
and let AD have been made equal to AB [Prop. 3].



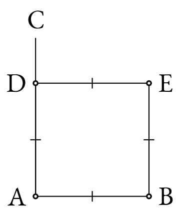
And let DE have been drawn through point D parallel to AB [Prop. 31], and let BE have been drawn through point B parallel to AD [Prop. 31].



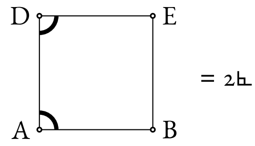
Thus, ADEB is a parallelogram. Therefore, AB is equal to DE, and AD to BE [Prop. 34].



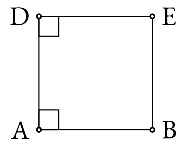
But, AB is equal to AD. Thus, the four sides BA, AD, DE, and EB are equal to one another. Thus, the parallelogram ADEB is equilateral.



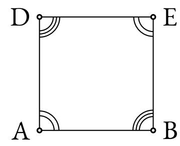
So I say that it is also right angled. For since the straight line AD falls across the parallels AB and DE, the sum of the angles BAD and ADE is equal to two right angles [Prop. 29].



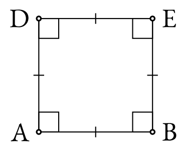
But BAD is a right angle. Thus, ADE is also a right angle.



And for parallelogrammic figures, the opposite sides and angles are equal to one another [Prop. 34].



Thus, each of the opposite angles ABE and BED are also right angles. Thus, ADEB is right angled. And it was also shown to be equilateral.

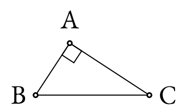


Thus, ADEB is a square [Def. 1.22]. And it is described on the straight line AB. Which is the very thing it was required to do.

Proposition 47

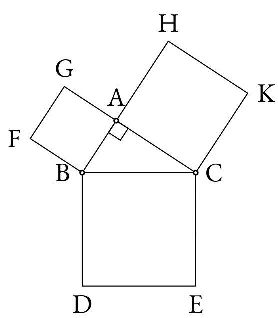
In right angled triangles, the square on the side subtending the right angle is equal to the sum of the squares on the sides containing the right angle.

Let ABC be a right angled triangle having the angle BAC a right angle.

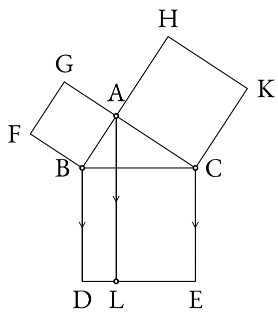


I say that the square on BC is equal to the sum of the squares on BA and AC.

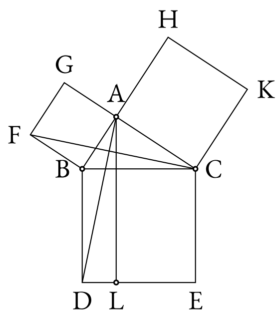
For let the square BDEC have been described on BC, and the squares GB and HC on AB and AC respectively [Prop. 46].



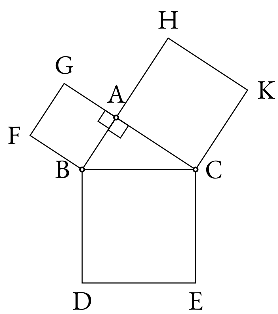
And let AL have been drawn through point A parallel to either of BD or CE [Prop. 31].



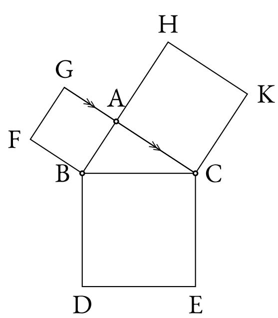
And let AD and FC have been joined.



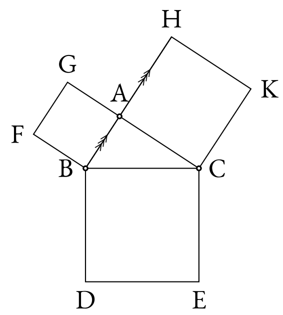
And since angles BAC and BAG are each right angles,



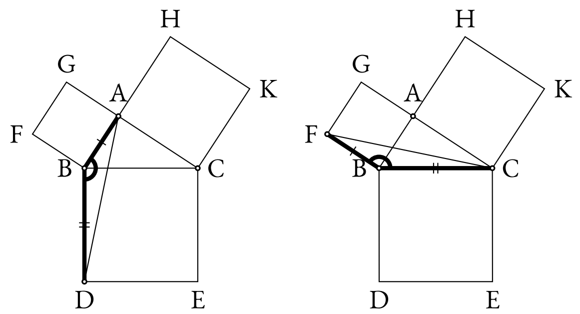
then two straight lines AC and AG, not lying on the same side, make the adjacent angles with some straight line BA, at the point A on it, whose sum is equal to two right angles. Thus, CA is straight on to AG [Prop. 14].



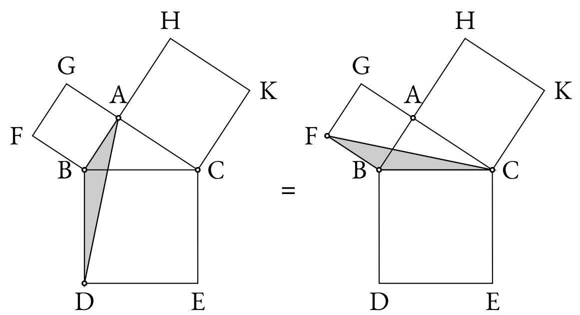
So, for the same reasons, BA is also straight on to AH.



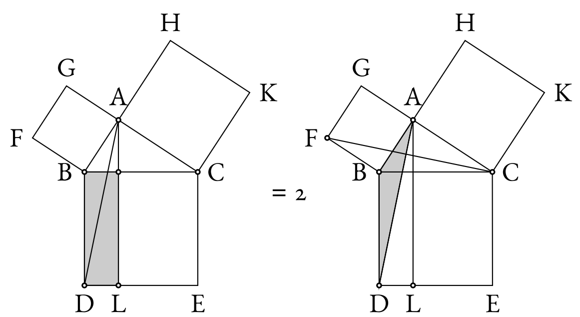
And since angle DBC is equal to FBA, for they are both right angles, let ABC have been added to both. Thus, the whole angle DBA is equal to the whole angle FBC. And since DB is equal to BC, and FB to BA, the two straight lines DB, BA are equal to the two straight lines CB, BF, respectively. And angle DBA is equal to angle FBC.



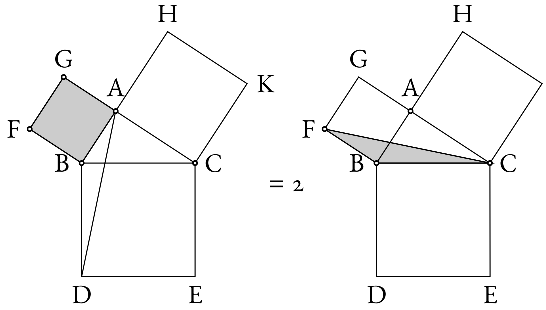
Thus, the base AD is equal to the base FC, and the triangle ABD is equal to the triangle FBC [Prop. 4].



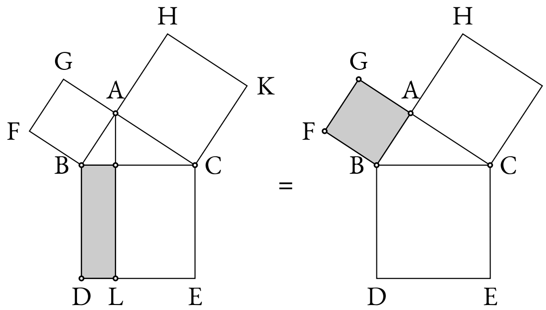
And parallelogram BL is double the area of triangle ABD. For they have the same base, BD, and are between the same parallels, BD and AL [Prop. 41].



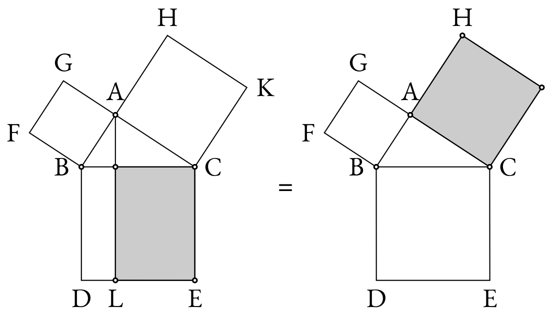
And square GB is double the area of triangle FBC. For again they have the same base, FB, and are between the same parallels, FB and GC [Prop. 41].



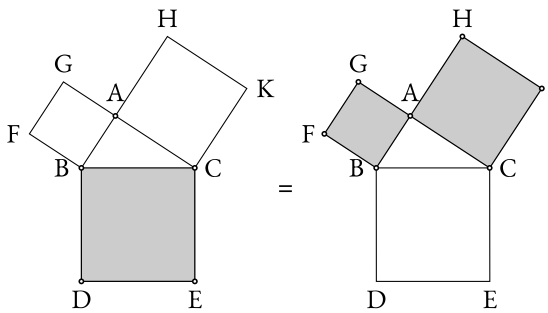
Thus, the parallelogram BL is also equal to the square GB.



So, similarly, AE and BK being joined, the parallelogram CL can be shown to be equal to the square HC.



Thus, the whole square BDEC is equal to the sum of the two squares GB and HC. And the square BDEC is described on BC, and the squares GB and HC on BA and AC respectively. Thus, the square on the side BC is equal to the sum of the squares on the sides BA and AC.

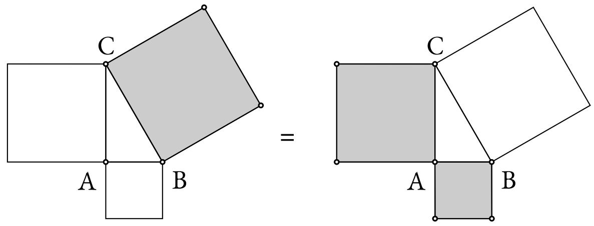


Thus, in right angled triangles, the square on the side subtending the right angle is equal to the sum of the squares on the sides surrounding the right [angle]. Which is the very thing it was required to show.

Proposition 48

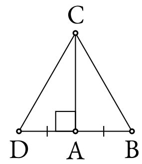
If the square on one of the sides of a triangle is equal to the sum of the squares on the two remaining sides of the triangle then the angle contained by the two remaining sides of the triangle is a right angle.

For let the square on one of the sides, BC, of triangle ABC be equal to the sum of the squares on the sides BA and AC.

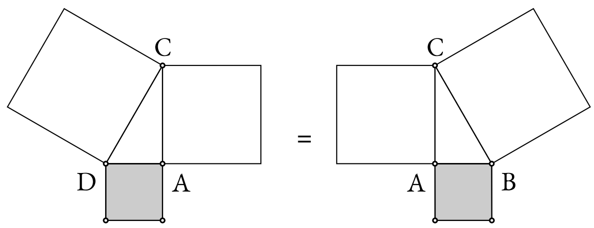


I say that angle BAC is a right angle.

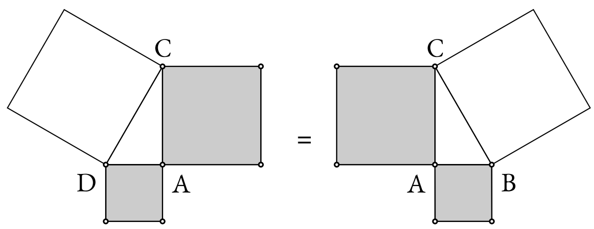
For let AD have been drawn from point A at right angles to the straight line AC [Prop. 11], and let AD have been made equal to BA [Prop. 3], and let DC have been joined.



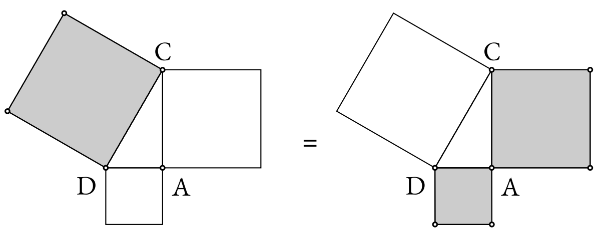
Since DA is equal to AB, the square on DA is thus also equal to the square on AB.



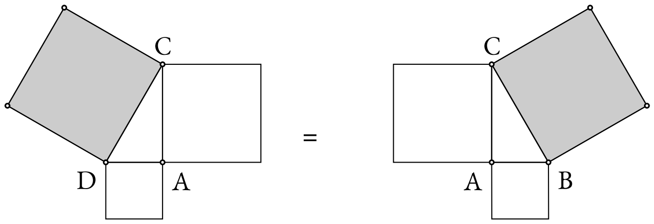
Let the square on AC have been added to both. Thus, the sum of the squares on DA and AC is equal to the sum of the squares on BA and AC.



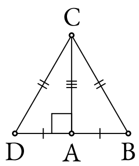
But, the square on DC is equal to the sum of the squares on DA and AC. For angle DAC is a right angle [Prop. 47].



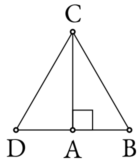
But, the square on BC is equal to sum of the squares on BA and AC. For that was assumed. Thus, the square on DC is equal to the square on BC.



So side DC is also equal to side BC. And since DA is equal to AB, and AC is common, the two straight lines DA, AC are equal to the two straight lines BA, AC. And the base DC is equal to the base BC.



Thus, angle DAC is equal to angle BAC [Prop. 8]. But DAC is a right angle. Thus, BAC is also a right angle.



Thus, if the square on one of the sides of a triangle is equal to the sum of the squares on the remaining two sides of the triangle then the angle contained by the remaining two sides of the triangle is a right angle. Which is the very thing it was required to show.